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Blow-up for solutions of hyperbolic PDE and spacetime singularities.

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Abstract

An important question in mathematical relativity theory is that of the nature of spacetime singularities. The equations of general relativity, the Einstein equations, are essentially hyperbolic in nature and the study of spacetime singularities is naturally related to blow-up phenomena for nonlinear hyperbolic systems. These connections are explained and recent progress in applying the theory of hyperbolic equations in this field is presented. A direction which has turned out to be fruitful is that of constructing large families of solutions of the Einstein equations with singularities of a simple type by solving singular hyperbolic systems. Heuristic considerations indicate, however, that the generic case will be much more complicated and require different techniques.

1. Introduction

Anyone who has worked with nonlinear hyperbolic equations is familiar with the phenomenon that smooth solutions often cease to exist after a finite time. There are two common ways in which this happens. In the book of Alinhac[1] these have been called the 'ODE blow-up mechanism' and the 'geometric blow-up mechanism'. In the first case the solution itself is unbounded as the blow-up time is approached. In the second case the solution itself remains bounded while the first derivative is unbounded. Although the Einstein equations, which are the subject of the following, are arguably the most geometric of hyperbolic differential equations the blow-up which is typically encountered occurs by the ODE mechanism in the terminology of [1]. At the same time, the question of the sense in which the Einstein equations can be called hyperbolic is delicate. These points are related, as will now be explained

The basic unknown in the Einstein equations is a pseudoriemannian metric g. The equations are diffeomorphism invariant, in the sense that if a metric g on a manifold M satisfies the Einstein equations and if ϕ is a diffeomorphism of M onto

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itself, then the pull-back $\phi^* g$ is also a solution. In order to make contact with PDE theory as it is usually formulated, it is necessary to choose a local coordinate system x^α on M and treat the components $g_{\alpha\beta}$ of the metric in this coordinate system as the basic unknowns. The Einstein equations imply a system of second order partial differential equations for these components. This system cannot have a well-posed Cauchy problem because of diffeomorphism invariance. For suppose that $g_{\alpha\beta}$ is a solution of these coordinate equations with given initial data. The functions $g_{\alpha\beta}$ are the components of a metric g. Now let ϕ be a diffeomorphism which leaves a neighbourhood of the Cauchy surface invariant and let $\bar{g} = \phi^* g$. Then $\bar{g}_{\alpha\beta}$ is a solution with the same initial data as $g_{\alpha\beta}$. However it is in general not equal to $g_{\alpha\beta}$. Thus uniqueness in the Cauchy problem in the usual sense fails for this system. On the other hand, there is a different uniqueness statement which does hold. It says that if g and $\bar{g}$ are metrics with the same initial data on a suitable Cauchy surface then there exists a diffeomorphism ϕ such that $\bar{g} = \phi^* g$ which is the identity on the Cauchy surface. In the following the precise mathematical formulation of this statement will not be required, but the intuitive idea is important. It should be noted that this uniqueness property, often called geometric uniqueness, is exactly what is desired from the point of view of physics since metrics related by a diffeomorphism should be regarded as physically indistinguishable.

It has now been indicated that diffeomorphism invariance presents a difficulty when studying the Cauchy problem for the Einstein equations. How can this difficulty be overcome? In principle, one could attempt to construct a diffeomorphism-invariant version of PDE theory but this has apparently never been done. In practice, the procedure is to require some condition relating the coordinates to the metric. This breaks the diffeomorphism invariance and opens up the possibility that with this extra condition imposed the Einstein equations imply a system of hyperbolic equations called the reduced equations, and that conversely suitable solutions of the reduced equations give rise to solutions of the Einstein equations. Details on this procedure of hyperbolic reduction can be found in [6]. There are many different kinds of hyperbolic reduction and the task is to find reductions which are well-adapted to given problems.

The Einstein equations are the basic equations of general relativity, which is the best existing theory of the gravitational field. Within this theory spacetime singularities, such as the big bang, correspond to singularities of $g_{\alpha\beta}$ which cannot be removed by a diffeomorphism, even one which itself becomes singular where $g_{\alpha\beta}$ does.

2. The Gowdy equations

A useful laboratory for studying questions concerning the global behaviour of solutions of the Einstein equations is the class of Gowdy solutions. These have high symmetry and after a suitable hyperbolic reduction give the following system of semilinear wave equations:

$$\begin{aligned} -\partial_t^2 X - t^{-1}\partial_t X + \partial_x^2 X &= 2(\partial_t X \partial_t Z - \partial_x X \partial_x Z) \\ -\partial_t^2 Z - t^{-1}\partial_t Z + \partial_x^2 Z &= -e^{-2Z}((\partial_t X)^2 - (\partial_x X)^2) \end{aligned} \tag{1}$$

Here $(t,x) \in (0,\infty) \times \mathbf{R}$ and X and Z are real-valued functions. Because of the symmetry only one space dimension remains. It will be useful to compare this with the equation

$$-\partial_t^2 u + \partial_x^2 u = -e^u \tag{2}$$

The equation (2) and its higher dimensional analogues have been studied in a series of papers by Kichenassamy and Littman[7, 8, 9, 10, 11]. Partial results of a similar type for (1) have been obtained in [12] and [15]. An obvious difference between the equations (1) and (2) is that while the singularities in solutions of (2) develop dynamically in the (t,x) plane there is a singularity in the coefficients of (1). Moreover it was proved by Moncrief[13] that smooth Cauchy data for (1) at $t = t_0 > 0$ develop into a smooth solution on the whole of the interval $(0,\infty)$. In fact this difference is more apparent than real. When proving theorems about (2) it is convenient to introduce a function which is constant on the singular hypersurface and rewrite the equations in terms of that. In reality, a corresponding procedure has been carried out when introducing the coordinates used in the Gowdy equation. This is so natural in the context of the Einstein equations, where a priori all time coordinates are equally valid, that it might pass unnoticed.

The nature of all these results is that they demonstrate the existence of large families of solutions of the equations whose singularities can be described precisely. This is achieved by using certain singular hyperbolic equations, whose singularity is of the type known as Fuchsian. The description of the singularity is given by an asymptotic expansion, which contains a number of free functions. These free functions may be called 'data on the singularity'. If these data are analytic then the expansion actually converges. The case of analytic data is easiest to handle. This was done for (2) in [7] and [8] and for (1) in [12]. The results for (2) were generalized to the case of smooth data (in fact data with sufficiently high order Sobolev regularity) in [9]. The theory developed there does not suffice to cover (1) with data which are smooth but not analytic. This difficulty was overcome in [15], where there is also a general discussion of the issue of extending results using Fuchsian techniques from the analytic to the smooth case and a survey of the literature where these techniques have been applied to problems in general relativity.

Now the asymptotic expansions for solutions of the Gowdy equations will be presented for illustration.

$$X(t,x) = X_0(x) + t^{2k(x)}(\psi(x) + v(t,x)) \tag{3}$$
$$Z(t,x) = k(x)\log t + \phi(x) + u(t,x) \tag{4}$$

Here the functions $u(t,x)$ and $v(t,x)$ are remainder terms which are $o(t)$ as $t \to 0$. Note the occurrence of powers of t with exponents depending on x in the expansion for X. It is these which cause additional technical difficulties in this case as compared to the case of (2). Solutions of the above form are obtained for smooth data X_0, k, ϕ and ψ which are arbitrary except for the restrictions that $0 < k(x) < 1$. The significance of the condition $k > 0$ will not be discussed here. The condition $k < 1$, known as the low velocity condition, will play a significant role in the following discussion. It turns out that it can be removed at the expense of requiring that X_0 should be constant. In this way a class of high velocity solutions can be obtained. They depend, however, on one less free function than the low velocity solutions.

For the physical interpretation of the results on singularities in Gowdy spacetimes it is important to know that the singularity in the metric coefficients at $t = 0$ cannot be removed by a diffeomorphism. One way of doing this is to consider scalar polynomial quantities in the curvature of the metric. These are real-valued functions which are defined in a coordinate invariant way. One of the simplest is the Kretschmann scalar K, which is the squared length (with respect to the given metric) of the Riemann curvature tensor. It is a function of X, Z and their first and second order partial derivatives whose explicit form is very complicated. Fortunately, for a solution with an asymptotic expansion of the type given above it is not too difficult to calculate the leading term in the expansion of K about $t = 0$. The result, discussed in [12], is

$$K \sim Q(k)t^{-3(k^2+1)/4} \tag{5}$$

for a certain quartic polynomial Q whose only positive root is at $k = 1$. Thus, provided $k \neq 1$ everywhere, K blows up uniformly as $t \to 0$.

An important issue is that of the generality of the solutions constructed by the above techniques. They are general in the crude sense that they depend on the same number of free functions as data for the same equations on a regular Cauchy surface. In the case of (2) Kichenassamy proved in [10] using the Nash-Moser theorem that a non-empty open set of data on a regular Cauchy surface develop into solutions of the type constructed by Fuchsian techniques. Up to now no corresponding results have been proved in problems arising in general relativity, such as the Gowdy problem. By a more direct approach, Chruściel [5] was able to show that a large open set of data for the Gowdy equations lead to solutions where the Kretschmann scalar blows up as $t \to 0$. Note that while the methods used in [5] were special to one space dimension, Fuchsian techniques do not depend essentially on the dimension.

3. The Einstein-scalar field equations

As mentioned above, the unknown of central importance in the Einstein equations is a metric g. Physically, it contains the information about the gravitational field and hence about the mutual gravitational attraction of material bodies. Thus it is natural that in any model involving the Einstein equations a description of the matter present is also necessary. In fact, in general relativity the gravitational field has degrees of freedom which can propagate in the absence of matter. This leads to the phenomenon of gravitational waves. The equations in the absence of matter are known as the vacuum Einstein equations and have the simple geometric form $\mathrm{Ric}(g) = 0$, where $\mathrm{Ric}(g)$ is the Ricci curvature of g. In the presence of matter these equations acquire a non-vanishing right hand side. An example which will be important in the following is that of the Einstein-scalar field equations. In that case the Einstein equations take the form $\mathrm{Ric}(g) = \nabla\phi \otimes \nabla\phi$. In general the matter fields must satisfy some equations of motion and in the case of the scalar field the equation of motion is that ϕ should satisfy the linear wave equation defined by the metric g. In coordinate components this takes the form

$$g^{\alpha\beta}(\partial_\alpha\partial_\beta\phi - \Gamma^\gamma_{\alpha\beta}\partial_\gamma\phi) = 0 \tag{6}$$

where

$$\Gamma^\gamma_{\alpha\beta} = \frac{1}{2} g^{\gamma\delta}(\partial_\alpha g_{\beta\delta} + \partial_\beta g_{\alpha\delta} - \partial_\delta g_{\alpha\beta}) \tag{7}$$

are the Christoffel symbols of g. Here we use the summation convention and $g^{\alpha\beta}$ is the inverse matrix of $g_{\alpha\beta}$.

It is known that in some cases the singularities of solutions of the Einstein equations display complicated oscillatory behaviour and it is suspected that this occurs under very general circumstances. The best known example of complicated behaviour is the Mixmaster solution. This is a solution of the vacuum Einstein equations which is symmetric under the group $SU(2)$. The Einstein equations reduce to ordinary differential equations but the ODE solutions are very complicated. Until recently our knowledge of the Mixmaster solution was based on numerical calculations and heuristic arguments. Now a number of features of this picture have been proved rigorously by Ringström[16]. The generality of Mixmaster behaviour is still out of reach of present analytical results. There have, however, been interesting advances in numerical approaches. (See [17].)

General predictions about the nature of singularities in solutions of the Einstein equations come from heuristic work of Belinskii, Khalatnikov and Lifshitz [3]. According to their picture the singularities in solutions of the vacuum Einstein equations should look like a different Mixmaster solution at each spatial point near the singularity. In particular the evolution at different spatial points decouples and the solution of the PDE system can be approximated by solutions of an ODE system. (Here we are reminded of the ODE blow-up mechanism in the terminology of Alinhac.) The same picture suggests that for the Einstein-scalar field system the decoupling should still take place but the ODE solutions should now be monotone rather than oscillatory near the singularity. This gives rise to a hope that the Einstein-scalar field system should be easier to handle analytically than the Einstein vacuum equations. Following this lead, it was shown in [2] that a Fuchsian analysis could be carried out for the Einstein-scalar field system without any need for symmetry assumptions. Solutions are obtained which depend on the same number of free functions as there are in the Cauchy data on a regular Cauchy surface. In this crude sense the solutions are general. Only the analytic case was treated up to now.

The asymptotic expansions obtained will now be presented. This is only done for a certain diagonalizable case. This does not restrict the number of free functions although it is a restriction on the solutions and a large part of the work of [2] was devoted to showing that the restriction is unnecessary. On the other hand, the diagonalizable case is best suited to explain the results. For physical reasons we are interested in solutions on a four-dimensional manifold. The time coordinate is denoted by x^0 and the spatial coordinates by x^a with $a = 1, 2, 3$. The Gaussian coordinate conditions

$$g_{00} = -1 \tag{8}$$

$$g_{0a} = 0 \tag{9}$$

are imposed. The asymptotic expansions for the remaining unknowns are as follows:

$$g_{ab} = t^{2p_1} l_a l_b + t^{2p_2} m_a m_b + t^{2p_3} n_a n_b + o(t^{q_{ab}}) \tag{10}$$

$$\phi = A \log t + B + o(1) \tag{11}$$

Here p_1, p_2, p_3, l_a, m_a, n_a, A and B depend on the spatial coordinates x^a and $p_1 + p_2 + p_3 = 1$. The q_{ab} are suitably chosen real numbers, which will not be discussed in detail here. The important thing is that they represent higher order terms. The functions occurring in the expansion must satisfy certain algebraic and differential equations which will also not be discussed, except to say that there is a good understanding of the solution set of these equations. There is an equivalent of the condition $k < 1$ in the Gowdy case, namely that all p_a should be positive. This implies that all p_a are smaller than one.

The conclusion of this work is that it provides strong support for the ideas of Belinskii, Khalatnikov and Lifshitz in the case that a scalar field is present. The conceptual, geometric and calculational aspects of the proof are a lot more complicated than in the Gowdy case, but the analytic input is almost identical.

4. The nonlinear scalar field

From the foregoing discussion of the great difference in the nature of singularities in solutions of the Einstein equations caused by the presence of a scalar field the impression might arise that the whole question depends in a delicate way on the exact nature of the matter fields present. In fact this impression is probably false: there should probably only be a small number of universality classes which cover most types of matter. This will be illustrated by a class of matter models which will be referred to collectively as the nonlinear scalar field. It turns out that a wide variety of choices of the parameters which go into the definition all lead to the same asymptotic behaviour, which is that of the (linear) scalar field discussed in the previous section. This will also allow some of the aspects of the Fuchsian theory to be shown in action.

The main interest for physicists of the equations obtained by coupling the Einstein equations to a nonlinear scalar field has been the the phenomenon known as inflation. The mathematics of relevance to inflation concerns the behaviour of the solutions in the direction away from the singularity and so has no direct connection with the discussion below.

Consider first the following equation

$$-\partial_t^2\phi - t^{-1}\partial_t\phi + \Delta\phi = m^2\phi + V'(\phi) \tag{12}$$

Here ϕ is a real-valued function, Δ is the Laplacian on $\mathbf{R}^n$ (the case of $n = 3$ is of most interest for the following) and m is a constant (mass) which we may as well take to be non-negative. The function V (potential) is smooth and V' vanishes at least quadratically at the origin. The separation of the right hand side into the sum of a mass term and a potential term has no essential significance in this problem. It is convenient for making contact with the notation used in the physics literature. The aim is to determine whether this equation has solutions which near $t = 0$ have the leading order asymptotics $A(x)\log t + B(x)$ already encountered in the previous section. To this end, introduce $\psi = t^{-\epsilon}(\phi - A\log t - B)$, where ϵ is a positive constant, whose purpose will be seen later on. When reexpressed in terms of ψ the equation (12) becomes:

$$-t^2\partial_t^2\psi - (2\epsilon + 1)t\partial_t\psi \quad -\epsilon^2\psi + t^2\Delta\psi = -t^{2-\epsilon}\log t\Delta A - t^{2-\epsilon}\Delta B$$

$$+t^{2-\epsilon}m^2(A\log t+B+t^\epsilon\psi)+W(t,x,\psi) \qquad (13)$$

where $W(t,x,\psi)=t^{2-\epsilon}V'(A\log t+B+t^\epsilon\psi)$. The equation has been multiplied by t^2 so as to make the combination td/dt appear. Some choices of V to be found in the physics literature are $V_1(\phi)=V_0$, a constant, $V_2(\phi)=\phi^\alpha$, $V_3(\phi)=e^{\beta\phi}$ for positive constants α and β. The value of the constant V_0 has no effect on the equation (12) but does have an effect when the coupling to the Einstein equations is considered. The potential V_1 combined with a non-zero value of m occurs in a model known as chaotic inflation. The potential V_3 occurs in what is called power-law inflation. (It is not the potential itself which has power-law behaviour, but the long-time behaviour of the solution.) The functions W corresponding to V_1, V_2 and V_3 are $W_1(t,x,\psi)=0$, $W_2(t,x,\psi)=\alpha t^{2-\epsilon}(A\log t+B+t^\epsilon\psi)^{\alpha-1}$ and $W_3(t,x,\psi)=\beta e^{\beta B}t^{2-\epsilon+\beta A}\exp(\beta t^\epsilon\psi)$. The functions W_1, W_2 and W_3 are smooth for $t>0$. Suppose that the function A is nowhere vanishing. Then W_1 and W_2 extend continuously to $t=0$ by defining them to be zero there. This is also true of their derivatives with respect to the arguments x and ψ. This will be abbreviated by saying that the functions W_1 and W_2 are regular. As for W_3, can be made regular in this sense by choosing ϵ sufficiently small provided $A>-2\beta^{-1}$. This is a restriction on the data on the singularity. In particular, if β is negative then it bounds A away from zero. If V had been chosen to grow even faster at infinity, e.g. $V(\phi)=e^{\phi^2}$ then the corresponding W could not have been made regular by imposing inequalities on A. This type of potential with faster then exponential growth does not seem to occur in the physics literature and the Fuchsian theory does not seem to apply to it. It will not be considered further here.

The Fuchsian theory we will use concerns first order systems and thus the first step in the analysis is to reduce (13) to first order in a suitable way. Let $v=t\partial_t\psi+\epsilon\psi$ and $w=t\partial_x\psi$. Then the the following system is obtained:

$$\begin{aligned} t\partial_t\psi+\epsilon\psi-v &= 0 \\ t\partial_t w+(\epsilon-1)w &= t\partial_x v \\ t\partial_t v+\epsilon v &= t\partial_x w+\Delta A t^{2-\epsilon}\log t+t^{2-\epsilon}\Delta B \\ &\quad -t^{2-\epsilon}m^2(A\log t+B+\psi)+W(t,x,\psi) \end{aligned} \qquad (14)$$

This system is of the following general form for Fuchsian equations:

$$t\partial_t u+N(x)u=t^\eta f(t,x,u,\partial_x u) \qquad (15)$$

for some $\eta>0$. It is also symmetric hyperbolic, a luxury which is usually not easy to obtain in problems in this area. On the other hand, it does have a disadvantage. The theorem proved in [12] required the assumption that $t^{N(x)}$ be bounded for t near zero. To obtain this the eigenvalues of $N(x)$ should have non-negative real parts. Since ϵ has to be chosen small in general, depending on the data, the eigenvalue $\epsilon-1$ will be negative and this condition is not satisfied for the equation (14). An alternative form is obtained by substituting the definition of v back into (14) in one place. The result is:

$$t\partial_t\psi+\epsilon\psi-v = 0$$

$$
\begin{aligned}
t\partial_t w + \epsilon w &= t\partial_x v + t\partial_x \psi \\
t\partial_t v + \epsilon v &= t\partial_x w + \Delta A t^{2-\epsilon}\log t + t^{2-\epsilon}\Delta B \\
&\quad -t^{2-\epsilon}m^2(A\log t + B + \psi) + W(t,x,\psi)
\end{aligned}
\tag{16}
$$

The equation (16) is not symmetric hyperbolic. It does, however satisfy the conditions of the existence theorem for the case of analytic data proved in [12]. In order to ensure this it is important that ϵ is positive. If ϵ were chosen to be zero then all the eigenvalues of the matrix $N(x)$ would vanish but the matrix would not be diagonalizable. This would violate one of the conditions assumed in [12], namely that zero eigenvalues, if they occur at all, only occur through $N(x)$ being a direct sum of a matrix whose eigenvalues have positive real parts with a zero matrix.

The result of the previous discussion is that given analytic functions A and B with A nowhere vanishing then if the restrictions on V and A already obtained are satisfied there exists a unique solution, analytic in x and continuous in t of (12) with the desired asymptotics. This shows the existence of a family of solutions depending on two free analytic functions, which is the same as the number of functions which can be given as data on a regular Cauchy surface.

In [15] a method for going beyond the analytic case in Fuchsian problems was applied to the Gowdy equations. Since the matrix $N(x)$ occurring in (16) does not depend on x it should be possible to prove an existence theorem for that equation with smooth data given on the singularity by the methods of [9]. Here we would like to note that the method of [15] generalizes easily to the case of smooth data for (12). The main idea is to use both the systems (14) and (16). If the smooth data is approximated by a sequence of analytic data then a corresponding sequence of analytic solutions can be produced using (14). These define a sequence of analytic solutions of (16). Then the fact that the latter equation is symmetric hyperbolic allows the use of energy estimates to prove that the sequence of analytic solutions converges to the desired smooth solution. Details of the procedure can be found in [15].

5. Coupling the nonlinear scalar field to the Einstein equations

In this section it will be shown that the Einstein equations coupled to the nonlinear scalar field can be handled by the same method as that used for the case with $m = 0$ and $V = 0$ in [2]. The arguments will only be sketched since most of the steps are identical. The potential will be assumed to be one of the functions V_1, V_2 or V_3 considered in the last section. In addition it is assumed that $A \neq 0$ and that in the case of V_3 the inequality $A > -2\beta^{-1}$ holds. The Einstein equations take the form

$$
\mathrm{Ric}(g) = \nabla\phi \otimes \nabla\phi + [\frac{1}{2}m^2\phi^2 + V(\phi)]g \tag{17}
$$

Evidently this equation does change if V is changed by an additive constant. The equation for the scalar field is

$$
g^{\alpha\beta}(\partial_\alpha\partial_\beta\phi - \Gamma^\gamma_{\alpha\beta}\partial_\gamma\phi) = m^2\phi + V'(\phi) \tag{18}
$$

Applying the Gaussian coordinate conditions $g_{00} = -1$ and $g_{0a} = 0$ gives

$$g^{\alpha\beta}\partial_\alpha\partial_\beta\phi = -\partial_t^2\phi + g^{ab}\partial_a\partial_b\phi \tag{19}$$

and the close relation of equation (18) to equation (12) begins to become apparent. We are looking for solutions to the Einstein equations coupled to a nonlinear scalar field where the metric has the asymptotic form found in [2] for the case of a linear scalar field. This form implies that the quantities t^2g^{ab} all vanish like some positive power of t as $t \to 0$ and thus do not disturb the Fuchsian form of the wave equation. The same holds for the quantities $t\Gamma^a$ and the quantity $t\Gamma^0$ is equal to one up to higher order terms. These facts cannot be easily seen from the equations given in this paper. To check them it is necessary to use more detailed results from [2]. The conclusion is that up to higher order terms the equation (18) agrees with the model equation (12).

What remains to be done in order to check that the Einstein equations coupled to a nonlinear scalar field have solutions which behave asymptotically like those for a linear scalar field is to see that certain source terms in the Einstein equations are equal to those in the case with a linear massless scalar field (which has already been solved in [2]), up to terms of higher order in t. The essential thing is to estimate the quantity $\partial_t^2\phi + \frac{1}{2}m^2\phi^2 + V(\phi)$. More specifically it is necessary to estimate the deviation of this quantity from the corresponding quantity with $\psi = 0$ and show that it is $O(t^{-2+\alpha_0})$ where α_0 is a positive parameter introduced in [2]. In the course of the proofs in that paper it is permissible to reduce the size of the number α_0 if required. It is easy to check that, under the assumptions already made on V and A, the parameter α_0 can be chosen so as to ensure this estimate. This is the same kind of procedure as choosing ϵ small in the case of the nonlinear scalar field. The information obtained in this way must be used in two places. It must be used in the Einstein constraints in estimating the quantity $\bar{C}$ of [2]. It must also be used in evolution equations for $\kappa^a{}_b$ in [2], which constitute the essential part of the Einstein evolution equations. The additional contribution which arises when passing from the massless linear scalar field to the nonlinear scalar field does not change the equations for $a \neq b$. For $a = b$ the coefficient α_0 appears. It can be concluded from all this that solutions of the Einstein equations coupled to a nonlinear scalar field which have the same asymptotics as in the case of a massless linear scalar field and which are of the same degree of generality can be constructed by Fuchsian techniques. This has been shown under the assumption that the potential is of one of the forms V_1, V_2 or V_3 and that in the third case the initial datum A satisfies an additional restriction. In this proof the velocity-dominated system, which is the system of equations satisfied by the leading order approximation to the solution, is the same as in the case of the massless linear scalar field.

6. Formation of localized structure

It has already been indicated that the use of Fuchsian theory to construct spacetimes with singularities of a well-understood type is not likely to be sufficient to handle general singularities of solutions of the Einstein equations, due to the appearance of oscillatory behaviour in time as the singularity is approached. There is, however, another effect which complicates the situation in an essential way. Note that the solutions constructed by Fuchsian techniques are such that the solution blows up like $\log t$ as $t=0$ is approached and the spatial derivatives of the solutions of all orders are also $O(\log t)$. Taking spatial derivatives does not increase the rate of blow-up. It will now be shown that there are solutions for which this property does not hold. Consider the transformation:

$$\tilde{X} = X/(X^2+e^{2Z}) \tag{20}$$
$$\tilde{Z} = \log[e^Z/(X^2+e^{2Z})] \tag{21}$$

This transformation is equal to its own inverse and transforms solutions of the Gowdy equations into solutions. The idea is now to take a solution $(\tilde{X},\tilde{Z})$ with the low velocity asymptotic behaviour and transform it to a new solution (X,Z).

The case will be considered that $\tilde{X}_0(0)=0$, $\tilde{X}_0(x)\neq 0$ for $x\neq 0$ and $\partial_x\tilde{X}_0(0)\neq 0$. For the transformed solution X^2+e^{2Z} is of the form

$$(\tilde{X}_0)^2+(e^{2\tilde{\phi}}+2\tilde{X}_0\tilde{\psi})t^{2\tilde{k}}+o(t^{2\tilde{k}}) \tag{22}$$

From this it is not hard to see that X is bounded on an interval about $x=0$ uniformly in t. Similarly $Z=O(\log t)$. Thus the transformed solution (X,Z) does not blow up any faster than the original solution $(\tilde{X},\tilde{Z})$. However spatial derivatives blow up faster in general. To see this it suffices to evaluate $Z(t,0)$ and $Z(t,t^{\tilde{k}/2})$. The second of these is bounded for values of x close to $x=0$ as $t\to 0$ while the first behaves like $-\tilde{k}\log t$. It follows from the mean value theorem that the maximum value of $\partial_x Z$ in a small region about $x=0$ must blow up at least as fast as $t^{-\tilde{k}/2}$. Of course precise rates of blow-up could be calculated for this and higher order derivatives if desired. It is seen that these solutions of the Gowdy equations, unlike those constructed by the Fuchsian method, have spatial derivatives which blow up faster than the function itself. There is formation of localized structure.

In the explicit example given above the localized structures formed do not have an invariant geometrical meaning and are due to a bad choice of variables. On the other hand, there is convincing numerical evidence due to Berger, Moncrief and collaborators for a second kind of localized structure. In the latter case there is no reason to doubt that there is a real geometrical effect involved. For a survey of this work and some pictures of the structures, see [4]. There is also a lot of interesting work on the subject using heuristic methods. A rigorous treatment is still lacking.

Although this is still in the domain of speculation, there are arguments that in more general situations things will get yet more complicated. The basic idea will now be sketched. The oscillatory behaviour in the Mixmaster model is composed of a series of bounces where expansion turns into contraction and vice versa. In the inhomogeneous case bounces of this kind are supposed to happen independently at different spatial points. However this does not always happen coherently. There are spatial regions of coherent behaviour but these split into smaller regions as the singularity is approached. On the boundaries between regions spatial derivatives become large. The localized features observed in the Gowdy solutions are simple examples of boundaries of this kind. The heuristic models available suggest that in the Gowdy case this kind of formation of structure will only happen a finite number of times, at least for generic solutions. In the general case, on the other hand, it is expected to happen infinitely many times before the singularity is reached, creating structures on arbitrarily small spatial scales. This behaviour has been called 'spacetime turbulence'. Information on this can be found in [14]. It has also been suggested that this process which is predicted by the picture of Belinskii, Khalatnikov and Lifshitz actually shows that the picture must eventually break down in the generic case before the singularity is reached.

It seems clear that it will take a long time before the ideas mentioned in the last paragraph can be brought into the fold of well-understood mathematical phenomena. Nevertheless, the recent progress described in this paper is a ground for optimism that there will be significant mathematical developments in the area of singularities of solutions of the Einstein equations in the next few years.

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