Bernard Helffer

Thierry Ramond

Semiclassical expansion for the thermodynamic limit of the ground state energy of Kac's operator

Semiclassical expansion for the thermodynamic limit of the ground state energy of Kac's operator

Bernard HELFFER   Thierry RAMOND

Abstract

We continue the study started by the first author of the semiclassical Kac Operator. This kind of operator has been obtained for example by M. Kac as he was studying a 2D spin lattice by the so-called "transfer operator method". We are interested here in the thermodynamical limit $\Lambda(h)$ of the ground state energy of this operator. For Kac's spin model, $\Lambda(h)$ is the free energy per spin, and the semiclassical regime corresponds to the mean-field approximation.

Under suitable assumptions, which are satisfied by the physical examples we have in mind, we construct a formal asymptotic expansion for $\Lambda(h)$ in powers of h, from which we derive precise estimates on $\Lambda(h)$.

We work in the settings of Standard Functions introduced by J. Sjöstrand for the study of similar questions in the case of Schrödinger operators.

1. Introduction

We continue the study started by the first author of spectral properties of a class of integral operators, often called Kac operators or transfer operators. These operators appear in particular in statistical physics, for example when studying gaussian-like measures $\exp(-\Phi(x))dx$ by the so-called transfer matrix method (see e.g. [He3, Section 6]). As already noticed in the case of the Ising spin model, it appears indeed that thermodynamical properties of some statistical systems can be described in terms of spectral quantities attached to a transfer operator (a matrix in the Ising model case). Let us describe briefly the example of a spin model introduced by M. Kac (see [Ka]).

We write $\Lambda_{m,n} = \{1, \ldots, n\} \times (\mathbb{Z}/m\mathbb{Z})$ and we consider the following hamiltonian $E_{n,m}$ for a configuration of spins $\sigma \in \{-1, 1\}^{\Lambda_{n,m}}$:

$$E_{n,m}(\sigma) = \sum_{P,P' \in \Lambda_{n,m}} w_{P,P'} \sigma_P \sigma_{P'},$$

where, with $P = (k, l)$, $P' = (k', l') \in \Lambda_{n,m}$,

$$w_{P,P'} = \begin{cases} 0 & \text{if } P = P', \\ \gamma e^{-\gamma|k-k'|}(\delta_{l,l'} + \frac{1}{2}(\delta_{l-1,l'} + \delta_{l+1,l'})) & \text{if } P \neq P'. \end{cases}$$

To make it short, we consider a nearest neighbour interaction between rows, with an exponential decay with respect to the distance between columns. The constant γ is positive, and measures the rate of decay of the interaction. The regime $\gamma \to 0$ corresponds to the so-called mean field approximation. Defining as usual (see for example [El]) the partition function $Z_{n,m}$ as

$$Z_{n,m} = \sum_{\sigma \in \{-1,1\}^{\Lambda_{n,m}}} \exp(-\beta E_{n,m}),$$

where $\beta > 0$ is the inverse of the temperature, the free energy per spin in the thermodynamical limit $-\beta\Psi$ is defined by

$$-\beta\Psi = \lim_{m \to \infty} \frac{1}{m} \lim_{n \to \infty} \frac{\ln Z_{n,m}}{n}.$$

M. Kac remarks that this free energy per spin can be computed through the following formula

$$-\beta\Psi = \ln 2 - \beta\gamma + \lim_{m \to \infty} \frac{\ln \mu_1^{(m)}}{m}.$$

Here $\mu_1^{(m)}$ is the first eigenvalue of the compact operator $K^{(m)}$ on $L^2(\mathbb{R}^m)$ given by

$$K = e^{-V/2}e^{-\gamma\Delta}e^{-V/2}, V(x) = \frac{1}{4}\sum_{j=1}^{m} x_j^2 - \sum_{j=1}^{m} \log\cosh(\sqrt{\beta/2}(x_j + x_{j+1})).$$

What is certainly more challenging is that phase transitions might also be seen looking at the behaviour of ratio $\mu_2^{(m)}/\mu_1^{(m)}$ as $m \to +\infty$. Indeed this quantity is directly related to the correlation function between two spins in the same row (see [He3]).

Our concern in this paper is the semiclassical Kac operator

$$K^{(m)}(h) = \exp -\frac{V}{2} \cdot \exp h^2\Delta \cdot \exp -\frac{V}{2}, \tag{1.1}$$

where $V : \mathbb{R}^m \to \mathbb{R}$ is a C^∞ function like $V(x) = \sum_{j=1}^{m} v(x_j) + w(x_j, x_{j+1})$ with the convention that $x_{m+1} = x_1$. The function V is non-negative and convex at infinity. The function w will be called the *interaction potential*. Here we study the behaviour as $h \to 0$ of the thermodynamical limit $\Lambda(h)$, and more precisely we want to obtain an asymptotic expansion for $\Lambda(h)$ in powers of h. We would also like to mention a recent paper by J. Møller, where another asymptotical regime is studied (the low temperature limit) for this kind of operator. Both of these works have been initiated in [He4], following a strategy which has been succesfully used in [He-Sj] in the case of Schrödinger operators:

1. for each fixed m, perform a WKB-type construction for the first eigenfunction and the logarithm of the corresponding eigenvalue: $-\ln(\mu(h, m)) \sim F_0^{(m)} + F_1^{(m)}h + \dots$;

2. show that for each j the sequence $(F_j^{(m)}/m)_m$ converges towards a certain Λ_j;

3. show that $\Lambda(h)$ does indeed have $\Lambda_0 + \Lambda_1 h + \dots$ as an asymptotic expansion as $h \to 0$.

The first step has been completed in [He4], and it was also shown that the sequences $(F_j^{(m)}/m)_m$ are bounded. Our aim here is to complete the two last steps. As the reader will see, some information about the decay of the eigenfunction is still missing, and the third step is not yet completely done. However we will give in the last section a precise lower bound for $\Lambda(h)$.

Remark 1.1 *The question of existence of the thermodynamic limit has been already addressed in [He2] at least for fixed h. Former works were considering the same question for Laplace integrals ([Ru], [He1]), or for Schrödinger operators ([Sj1], [Sj2], [He-Sj]). Here we will not comment further on that problem.*

Our study will strongly rely on the notion of standard functions which has been introduced by J. Sjöstrand in [Sj3]. In particular in order to state our results, we have first to recall some notations and definitions.

2. Standard functions

For $x \in \mathbb{R}^m$ and for $p \in [1, +\infty[$, we shall write $|x|_p = (\sum_{j=1}^{p} |x_j|^p)^{1/p}$ and $|x|_\infty = \max_{j \in \{1,\dots,m\}} |x_j|$. If, for each $m \in \mathbb{N}\backslash\{0\}$, $\Omega^{(m)}$ is an open subset of $\mathbb{R}^m$, we write will $\Omega = (\Omega^{(m)})_m$. Most often we will be concerned with $B_\infty(0, R) = (B_\infty^{(m)}(0, R))_m$ where

$$B_\infty^{(m)}(0, R) = \{x \in \mathbb{R}^m, |x|_\infty < R\},$$

or more generally with an Ω such that $B_\infty(0, R_1) \subset \Omega \subset B_\infty(0, R_2)$ for some $0 < R_1 < R_2$. It will also be convenient to use tensorial notations for successive gradients of a $\mathcal{C}^\infty$ functions $f : \mathbb{R}^m \to \mathbb{R}$: for $k \geq 1$ we define by induction

$$\langle \nabla^k f(x), u_1 \otimes \dots \otimes u_k \rangle = \langle \nabla \langle \nabla^{k-1} f(x), u_1 \otimes \dots \otimes u_{k-1} \rangle, u_k \rangle$$

where $u_1, \dots, u_k$ are vectors in $\mathbb{R}^m$.

2.1. $\mathcal{S}^{k_0}$-functions

Definition 2.1 *A sequence $a = (a^{(m)})_m$ of $\mathcal{C}^\infty$ functions $a^{(m)} : \Omega^{(m)} \to \mathbb{R}$ is an $\mathcal{S}^0$-function on Ω (belongs to $\mathcal{S}^0(\Omega)$), if:*

$$\forall k \geq 1, \exists C_k > 0, \forall m \in \mathbb{N}\backslash\{0\}, \forall x \in \Omega^{(m)}, \forall(u_1, \dots, u_k) \in (\mathbb{R}^m)^k,$$

$$|\langle \nabla^k a^{(m)}(x), u_1 \otimes \dots \otimes u_k \rangle| \leq C_k |u_1|_{p_1} \dots |u_k|_{p_k} \qquad (2.1)$$

for any $(p_1, \dots p_k) \in [1, +\infty]^k$ such that $\dfrac{1}{p_1} + \dots + \dfrac{1}{p_k} = 1.$

Example 2.2 *Let* $a = (a^{(m)})$ *with* $a^{(m)}(x) = \sum_{j=1}^{m} f(x_j)$, *where* $f : \mathbb{R} \to \mathbb{R}$ *is a* C^∞ *function. We have*

$$\langle \nabla^k a^{(m)}(x), u_1 \otimes \ldots \otimes u_k \rangle = \sum_{j=1}^{m} f^{(k)}(x_j) u_1(j) \ldots u_k(j)$$

where $u(j)$ *is the* j-*th component of the vector* $u \in \mathbb{R}^m$. *With* $\Omega = B_\infty(0, R)$ *and for* $C_k = \sup_{|s| \leq R} |f^{(k)}(s)|$, *we get (2.1), using the multilinear Hölder inequality.*

Here and in what follows we write for short

$$|\underline{u}|_{\underline{p}} := |u_1|_{p_1} \ldots |u_k|_{p_k}.$$

We will also meet successive gradients of functions from $\mathbb{R}^{(m)}$ to $\mathbb{R}$ or directly functions from $\mathbb{R}^{(m)}$ to an $(\ell^p)^{k_0}$-space $(\ell^p = (\mathbb{R}^m, | \ |_p))$ for some k_0, in particular vector fields $(k_0 = 1)$. We will need the following definition.

Definition 2.3 *A sequence* $b = (b^{(m)})_m$, $b^{(m)} : \Omega^{(m)} \to ((\mathbb{R}^m)^{k_0})$, *belongs to* $\mathcal{S}_{\mathcal{R}}^{k_0}(\Omega)$ *if*

$$\forall k \geq 0, \exists C_k > 0, \forall m \in \mathbb{N} \backslash \{0\},$$

$$\forall x \in \Omega^{(m)}, \forall (u_1, \ldots, u_k) \in (\mathbb{R}^m)^k, \forall (v_1, \ldots, v_{k_0}) \in (\mathbb{R}^m)^{k_0},$$

$$|\langle \nabla^k \langle b^{(m)}(x), v_1 \otimes \ldots v_{k_0} \rangle, u_1 \otimes \ldots \otimes v_k \rangle| \leq C_k |\underline{v}|_{\underline{p}} |\underline{u}|_{\underline{p}'} \qquad (2.2)$$

for any $(\underline{p}, \underline{p}') \in [1, +\infty]^{k_0 + k}$ *such that* $\frac{1}{p_1} + \ldots + \frac{1}{p_k} + \frac{1}{p_1'} + \ldots + \frac{1}{p_k'} = 1$.

Notice that the estimates (2.2) are precisely those one gets for $\nabla^k b$ if $b = \nabla^{k_0} a$ for an $\mathcal{S}^0$-function. In particular $b = \nabla^{k_0} a$ is a $\mathcal{S}^{k_0}$-function as soon as $a \in \mathcal{S}^0$.

2.2. $\mathcal{S}_{\mathcal{R}}^{k_0}$-functions

We will also need weighted standard functions. Let us first describe what we call weights. For any $m \in \mathbb{N}^*$ we denote by $\mathcal{R}^{(m)}$ a set of functions from $\{1, \ldots, m\}$ with values in $]0, +\infty[$ having the following properties:

1. If $\rho \in \mathcal{R}^{(m)}$, then $1/\rho$ belongs also to $\mathcal{R}^{(m)}$.

2. If $\rho_1, \rho_2 \in \mathcal{R}^{(m)}$, then for all $t \in [0, 1]$, $\rho_1^t \rho_2^{1-t}$ belongs to $\mathcal{R}^{(m)}$.

Example 2.4 *Let* α *be a positive number. The set* $\mathcal{R}_\alpha^{(m)}$ *of all functions* $\rho^{(m)}$ *such that*

$$e^{-\alpha} < \frac{\rho^{(m)}(j)}{\rho^{(m)}(j+1)} \leq e^\alpha, j \in \mathbb{Z}/m\mathbb{Z}$$

satisfies the above properties. We will consider below the class of functions $\rho^{(m)}$ *given by* $\rho^{(m)}(j) = e^{\alpha \min(j, m+1-j)}$ *for one fixed* $\alpha \in]0, 1]$.

Definition 2.5 *We denote by* $\mathcal{R}$ *the set of sequences* $\rho = (\rho^{(m)})_m$ *where* $\rho^{(m)} \in \mathcal{R}^{(m)}$, *and the elements of* $\mathcal{R}$ *will be called weights.*

From now on, we work with a fixed set of weights $\mathcal{R}$ and we will denote by $\ell_\rho^p(\mathbb{R}^m)$ the vector space $\mathbb{R}^m$ equipped with the norm $|\ \ |_{p,\rho}$, where $p \in [1, +\infty]$ and $\rho \in \mathcal{R}^{(m)}$, given by

$$|x|_{p,\rho} = \left(\sum_{j=1}^m |\rho(j)x_j|^p \right)^{1/p}.$$

The notion of $\mathcal{S}_\mathcal{R}^0$-function is only a slight modification of that of an $\mathcal{S}^0$-function.

Definition 2.6 *A sequence $a = (a^{(m)})_m$ of $\mathcal{C}^\infty$ functions $a^{(m)} : \Omega^{(m)} \to \mathbb{R}$ is an $\mathcal{S}_\mathcal{R}^0$-function on Ω (belongs to $\mathcal{S}_\mathcal{R}^0(\Omega)$), if:*

$$\forall k \geq 1, \exists C_k > 0, \forall m \in \mathbb{N}\backslash\{0\}, \forall x \in \Omega^{(m)}, \forall (u_1, \dots, u_k) \in (\mathbb{R}^m)^k$$

$$|\langle \nabla^k a^{(m)}(x), u_1 \otimes \dots \otimes u_k \rangle| \leq C_k |u_1|_{p_1,\rho_1} \dots |u_k|_{p_k,\rho_k} \tag{2.3}$$

for any $(p_1, \dots p_k) \in [1, +\infty]^k$ such that $\frac{1}{p_1} + \dots \frac{1}{p_k} = 1$, and for any $(\rho_1, \dots \rho_k) \in (\mathcal{R}^{(m)})^k$ such that

$$\rho_1 \dots \rho_k = 1.$$

The corresponding definition of a $\mathcal{S}_\mathcal{R}^{k_0}$-function is left to the reader.

2.3. One-parameter families of $\mathcal{S}_\mathcal{R}^{k_0}$-functions

We shall also meet one-parameter families of $\mathcal{S}_\mathcal{R}^0$-functions in the following sense.

Definition 2.7 *Let $\sigma = (\sigma^{(m)}) \in \mathcal{R}$ be such that*

$$\forall m \in \mathbb{N}^*, \forall j \in \{1, \dots, m\}, \sigma^{(m)}(j) \geq 1.$$

$\mathcal{S}_{\mathcal{R},\sigma}^0(\Omega \times [0,1])$ is the set of sequences $(a^{(m)})$ of $\mathcal{C}^\infty$ functions $a^{(m)} : \Omega^m \times [0,1] \to \mathbb{R}$ such that

$$\forall k \geq 1, \forall \ell \in \{0,1\}, \exists C_{k,\ell} > 0, \forall m \in \mathbb{N}\backslash\{0\},$$

$$\forall (x,\theta) \in \Omega^{(m)} \times [0,1], \forall \underline{u} = (u_1, \dots, u_k) \in (\mathbb{R}^m)^k,$$

$$|\langle \partial_\theta^\ell \nabla^k a^{(m)}(x,\theta), u_1 \otimes \dots \otimes u_k \rangle| \leq C_{k,\ell} |\underline{u}|_{\underline{p},\underline{\rho}} \tag{2.4}$$

for any $(p_1, \dots, p_k) \in [1, +\infty]^k$ and any $(\rho_1, \dots, \rho_k) \in (\mathcal{R}^{(m)})^k$ such that

$$\frac{1}{p_1} + \dots + \frac{1}{p_k} = 1, \rho_1 \dots \rho_k = \frac{1}{\sigma^\ell}.$$

To make it short, we impose that the functions of the family are $\mathcal{S}_\mathcal{R}^0$-functions uniformly with respect to θ, as well as their derivatives with respect to θ, for which moreover we ask for a gain of σ in the estimates. We also use one-parameter families of $\mathcal{S}_\mathcal{R}^{k_0}$-functions, but we omit the definition here.

2.4. Differential calculus for $\mathcal{S}$-functions

The standard function classes we have defined inherit a very natural calculus (see [Sj3]). We list here the properties of one-parameter families of $\mathcal{S}$-functions that we will need later on.

Proposition 2.8 *When u is a $\mathcal{S}_{\mathcal{R},\sigma}^{k_0}$-function in $\Omega \times [0,1]$, so is Δu.*

The proof of this property is based on the following (classical?) lemma which is also of independent interest.

Lemma 2.9 *If A belongs to $\mathcal{M}_m(\mathbb{R})$, then $|\operatorname{tr}(A)| \leq \|A\|_{\mathcal{L}(\ell^\infty, \ell^1)}$.*

Proposition 2.10 *If $u = (u^{(m)}) \in \mathcal{S}_{\mathcal{R}}^{k_0}(\Omega')$ and if $v = (v^{(m)}) \in \mathcal{S}_{\mathcal{R},\sigma}^1(\Omega \times [0,1], \Omega')$, the function $u \circ v = (u^{(m)} \circ v^{(m)})$ belongs to $\mathcal{S}_{\mathcal{R},\sigma}^{k_0}(\Omega \times [0,1])$.*

Proposition 2.11 *Suppose u and v are two $\mathcal{S}_{\mathcal{R},\sigma}^1$-functions in Ω. Then the map*

$$(x, \theta) \mapsto \langle u(x, \theta), v(x, \theta) \rangle$$

is a $\mathcal{S}_{\mathcal{R},\sigma}^0$-function. In particular, if $u, v \in \mathcal{S}_{\mathcal{R},\sigma}^0$, $(x, \theta) \mapsto \langle \nabla_x u(x, \theta), \nabla_x v(x, \theta) \rangle$ is also a $\mathcal{S}_{\mathcal{R},\sigma}^0$-function.

Proposition 2.12 *Let $v = (v^{(m)})$, $v^{(m)} : B_\infty^{(m)}(0, R) \times [0, 1] \subset \mathbb{R}^m \to \mathbb{R}^m$ be a (one parameter family of) vector field in $\mathcal{S}_{\mathcal{R},\sigma}^1$. Then for any $0 < R' < R$, there exists a constant $T > 0$ independent of m such that the flow $t \mapsto \Phi_t^{(m)}(x)$ of $v^{(m)}$ is defined on $]-T, T[$. Moreover for any $t \in]-T, T[$, the map $\Phi_t = (\Phi_t^{(m)})$ belongs to $\mathcal{S}_{\mathcal{R},\sigma}^1(B_\infty(0, R'))$.*

3. WKB constructions

We describe now the formal WKB construction given by B. Helffer in [He4] for the ground state of Kac's operator. Our aim here is to get a semiclassical expansion for $-\ln \mu_1^{(m)}(h)$, where $\mu_1^{(m)}(h)$ is the highest eigenvalue of the compact operator (when suitable assumptions are made) $K_m(h)$ whose kernel is

$$K^{(m)}(x, y; h) = (4\pi h^2)^{-\frac{m}{2}} \exp -\frac{V(x)}{2} \, \exp -\frac{|x-y|^2}{4h^2} \, \exp -\frac{V(y)}{2}. \tag{3.1}$$

We suppose that V presents a non-degenerate minimum at 0, with say $V(0) = 0$, and we try to construct localy near 0 ("inside the well") a WKB expansion for the eigenfunction associated to $\mu_1^{(m)}(h)$. We insist that we work here at a formal level, and that we state only local assumptions. We will examine later on if the objects we build now are related to the true first eigenfunction and eigenvalue.

Since the highest eigenfunction can be chosen positive (by Krein-Rutman's Theorem), we put

$$u_1(x) = \exp\left(-\frac{\phi(x; h)}{h}\right), \phi(x, h) \sim \sum_{j=0}^\infty h^j \phi_j(x),$$

and we want

$$K^{(m)}(h)u_1(x, h) = \mu_1^{(m)}(h)u_1(x, h).$$

Writing $-\ln \mu_1^{(m)}(h) = F(h)$ with $F(h) \sim \sum_{j \geq 0} F_j h^j$, we obtain as a starting point the equation

$$e^{F(h)} \int_{\mathbb{R}^m} K^{(m)}(x, y; h) \exp\{-\frac{1}{h}(-\phi(x; h) + \phi(y; h))\} dy = 1. \tag{3.2}$$

Since V presents a minimum at 0, we treat the left hand side as a Laplace integral. Notice that at the level of formal expansions in h, its value depends only on the germs of V and ϕ at 0 (provided V is bounded from below by a positive constant ouside a neighborhood of 0, but again we work at a formal level here). Our way to do so is to look for a change of variable $y \mapsto z(x, y; h)$ which allows us to compute the left hand side of (3.2) as

$$\frac{2^m}{(4\pi h^2)^{m/2}} \int_{\mathbb{R}^m} e^{-z^2/h^2} dz.$$

It is convenient to search for a map z of the form $z(x, y; h) = \nabla_y f(x, y; h)$ where $f(x, y; h) \sim \sum_{j \geq 0} f_j(x, y) h^j$. Taking the jacobian into account, we obtain the following equation

$$-h^{-1}\phi(x; h) + \tfrac{1}{4}h^{-2}|x - y|^2 + h^{-1}\phi(y; h) + \frac{V(x)}{2} + \frac{V(y)}{2}$$
$$\sim h^{-2}|\nabla_y f(x, y; h)|^2 - \ln \det(\nabla_{yy}^2 f(x, y; h)) + F(h) - m \ln 2. \tag{3.3}$$

We may consider this equality between asymptotic expansions term by term. Indeed for the term $\ln \det(\nabla_{yy}^2 f(x, y; h))$ we have:

Lemma 3.1 *If $h \mapsto M(h)$ is a C^∞ function from $]0, h_0]$ to $GL_n(\mathbb{R})$ that has an asymptotic expansion $M(h) \sim \sum_{j \geq 0} M_j h^j$ as $h \to 0$, then $L(h) = \ln \det M(h)$ has an asymptotic expansion as $h \to 0$, and the coefficient L_k of h^k in this expansion only depends on the M_j's for $j \leq k$. More precisely one has $L_0 = \ln \det M_0$ and, for $k \geq 1$:*

$$L_k = \sum_{n=1}^{k} \frac{(-1)^{n-1}}{n} \sum_{j_1 + j_2 + \dots + j_n = k} \operatorname{tr} M_0^{-1} M_{j_1} M_0^{-1} M_{j_2} \dots M_0^{-1} M_{j_n}.$$

1. The h^{-2} term.

We need $|\nabla_y f_0(x, y)|^2 = \tfrac{1}{4}|x - y|^2$ (an x-dependent eikonal equation). There are of course many solutions, and we have to fix Cauchy data. The more convenient choice is to impose $f_0(x, x) = 0$. Then we get for f_0 exactly two solutions $(x, y) \mapsto \pm\tfrac{1}{4}|x - y|^2$ and we choose the only positive one

$$(T_0) \qquad\qquad f_0(x, y) = \frac{1}{4}|x - y|^2.$$

2. The h^{-1} term.

We must have $\phi_0(y) - \phi_0(x) = 2\langle \nabla_y f_0(x,y), \nabla_y f_1(x,y) \rangle$, that is, taking (T_0) into account,

(T_1) $$\phi_0(y) - \phi_0(x) = \langle (y - x), \nabla_y f_1(x,y) \rangle.$$

Notice that at this point neither ϕ_0 nor f_1 are known. However differentiating (T_1) with respect to y and choosing $y = x$ gives also

(T_1') $$\nabla \phi_0(x) = \nabla_y f_1(x,x).$$

3. The h^0 term.

The equation is

$$\phi_1(y) - \phi_1(x) + \frac{V(x)}{2} + \frac{V(y)}{2} = \langle (y - x), \nabla_y f_2(x,y) \rangle +$$
$$+ |\nabla_y f_1|^2(x,y) - L_0(x,y) + F_0 + m \ln \tfrac{1}{2}$$

But $L_0(x,y) = -m \ln 2$, so that

(T_2)
$$\phi_1(y) - \phi_1(x) + \tfrac{V(x)}{2} + \tfrac{V(y)}{2} = \langle (y - x), \nabla_y f_2(x,y) \rangle +$$
$$+ |\nabla_y f_1|^2(x,y) + F_0$$

Take now $y = x$ in (T_2). We get $V(x) = |\nabla_y f_1|^2(x,x) + F_0$, and with (T_1')

(T_2') $$V(x) = |\nabla \phi_0(x)|^2 + F_0.$$

The reader may recognize here again the usual eikonal equation that one obtains for example in the WKB construction of a solution for a Schrödinger equation. The solution of this equation within the class of $\mathcal{S}$-functions is due to J.Sjöstrand (see [Sj1,2], and also [He-Sj]). Here follow the assumptions:

(**A1**) $V = (V^{(m)})$ is a $\mathcal{S}^0$-function in $B_\infty(0,R)$ for some $R > 0$.

(**A2**) $V(0) = \nabla V(0) = 0$ and there exist two constants $r_0 > r_1 > 0$, a diagonal matrix D such that $D > r_0$, and such that, for any $p \in [1, +\infty]$:

$$\|\nabla^2 V(0) - D\|_{\mathcal{L}(\ell^p)} < r_1$$

A necessary condition for (T_2') to have a solution is then of course $F_0 = 0$, and this condition is also sufficient as can be seen from the following theorem.

Theorem 3.2 *Under the assumptions (A1) and (A2), and when $F_0 = 0$, there exist a positive constant $R' \leq R$ and a unique non-negative function ϕ_0 in $\mathcal{S}^0(B_\infty(0,R'))$ which satisfies the eikonal equation (T_2') with $\phi_0(0) = 0$.*

Moreover, the flow Φ_t of the vector field $\nabla \phi_0$ is defined on the whole negative real axis, and satisfies the following estimates of 1-standardness: for all $k \geq 0$ there is a constant $C_k > 0$ such that, for any m, any $(u_1, \dots, u_k) \in (\mathbb{R}^m)^k$ and any $p \in [1, +\infty]$,

$$|\langle \nabla^k \Phi_t^{(m)}(x), u_1 \otimes \dots u_k \rangle|_p \leq C_k e^{t/C} |\underline{u}|_{\underline{p}} \tag{3.4}$$

for some positive constant C independent of m.

We let ϕ_0 be the $\mathcal{S}^0$-function given by the above Theorem. Then (T_1) gives a unique f_1, modulo the normalization condition $f_1(x,x) = 0$. The procedure continues, giving the ϕ_k, the F_k and the f_k. At step k we have

$$2\langle \nabla \phi_0(x), \nabla \phi_k(x) \rangle = e_k(x) - F_k \tag{3.5}$$

where e_k is a function that depends only on the f_j's and the ϕ_j's that we have already computed:

$$
\begin{aligned}
e_k(x) = L_k(x,x) - \sum_{j=2}^{k} \langle \nabla_y f_j(x,y), \nabla_y f_{k+2-j}(x,y) \rangle \\
+ 2 \langle \nabla \phi_0(x), \nabla_y \left\{ \sum_{j=1}^{k} \nabla_y f_j . \nabla_y f_{k+1-j} - L_{k-1} \right\} (x,x) \rangle.
\end{aligned}
\tag{3.6}
$$

For $x = 0$ in particular, we obtain

$$F_k = e_k(0) = L_k(0,0) - \sum_{j=2}^{k} \langle \nabla_y f_j(0,0), \nabla_y f_{k+2-j}(0,0) \rangle. \tag{3.7}$$

At last we get the following transport equation for f_{k+1}:

$$
\begin{cases}
\langle (y-x), \nabla_y f_{k+1}(x,y) \rangle = g_k(x,y) \\
g_k(x,y) = \phi_k(y) - \phi_k(x) - 2\langle \nabla_y f_1, \nabla_y f_k \rangle (x,y) \\
\quad - \sum_{j=2}^{k-1} \langle \nabla_y f_j, \nabla_y f_{k+1-j} \rangle (x,y) + L_{k-1}(x,y) - F_{k-1}
\end{cases}
\tag{3.8}
$$

which can be solved uniquely under normalization condition $f_{k+1}(x,x) = 0$.

In order to get some control on the F_k's as we let the dimension m go to infinity, we have to prove that the ϕ_k's and the f_k's are $\mathcal{S}$-functions. Theorem 3.2 above shows that ϕ_0 is a $\mathcal{S}$-functions. We also have $f_0(x,y) = \frac{1}{4}|x-y|^2$, so that $(f_0^{(m)})$ is $\mathcal{S}_{\mathcal{R} \times \mathcal{R}}^0$-function when considered as a sequence of functions from $(\mathbb{R}^2)^m$ to $\mathbb{R}$.

Looking at (3.5) and (3.8) we see that all we need are the two following results.

Proposition 3.3 *Let $g = (g^{(m)})$ be a $\mathcal{S}_{\mathcal{R} \times \mathcal{R}}^0$-function in $B_\infty(0,R) \times B_\infty(0,R)$. Then for each m the equation*

$$
\begin{cases}
\langle (y-x), \nabla_y f^{(m)}(x,y) \rangle = g^{(m)}(x,y) \\
f^{(m)}(x,x) = 0
\end{cases}
$$

has a unique solution f^m, and $f = (f^{(m)})$ is a $\mathcal{S}_{\mathcal{R} \times \mathcal{R}}^0$-function in $B_\infty(0,R) \times B_\infty(0,R)$.

Proposition 3.4 *Let ϕ_0 be the solution of the eikonal equation given by Theorem 3.2, and e be a $\mathcal{S}^0$-function in $B_\infty(0,R')$ for some $0 < R' < R$, with $e(0) = 0$. Then for each m the equation*

$$
\begin{cases}
\langle \nabla \phi_0^{(m)}(x), \nabla \phi^{(m)}(x,y) \rangle = e^{(m)}(x) \\
\phi^{(m)}(0) = 0
\end{cases}
$$

has a unique solution ϕ^m, and $\phi = (\phi^{(m)})$ is an $\mathcal{S}^0$-function in $B_\infty(0,R')$.

These two propositions can be proved the same way using the properties of the flow of each of the vector fields involved. Actually what we need is an estimate like (3.4), which holds also for the flow of $\langle (y - x), \nabla_y \rangle$. Summing up the results of this Section we can state the

Theorem 3.5 *Suppose V satisfies assumptions (A1) and (A2) for some $R > 0$. Then there exists an $0 < R' < R$, and for each m a unique sequence of real numbers $(F_j^{(m)})$, a unique sequence of C^∞ functions $(f_j^{(m)})$ on $(B_\infty^{(m)}(0, R'))^2$ and a unique sequence of C^∞ functions $(\phi_j^{(m)})$ on $B_\infty^{(m)}(0, R')$ such that*

1. $\phi_0 = (\phi_0^{(m)})$ is the function given by Theorem 3.2,

2. $f_j(x, x) = 0$ for all $j \in \mathbb{N}$,

3. Equation (3.3) holds at the level of asymptotic expansions in h.

Moreover for each j, the function $\phi_j = (\phi_j^{(m)})$ is a $\mathcal{S}_\mathcal{R}^0$ function in $B_\infty(0, R')$, and the function $f_j = (f_j^{(m)})$ is a $\mathcal{S}_{\mathcal{R}\times\mathcal{R}}^0$-function in $B_\infty(0, R') \times B_\infty(0, R')$.

As an immediate consequence we have the

Corollary 3.6 *([He4, Theorem 10.1]) Under the assumptions (A1) and (A2), for any k there exits a constant $C_k > 0$ such that*

$$\forall m \in \mathbb{N}^*, |F_k^{(m)}| \leq C_k m$$

Proof: Recall that

$$F_k^{(m)} = L_k^{(m)}(0, 0) - D_k^{(m)}(0, 0)$$

where L_k is defined in Lemma 3.1:

$$L_k(x, y) = \sum_{n=1}^{k} \frac{(-1)^{n-1} 2^n}{n} \sum_{j_1+j_2+\ldots+j_n=k} \text{tr}\, (M_{j_1} M_{j_2} \ldots M_{j_n})$$

with $M_j = \nabla_{yy}^2 f_j^{(m)}(x, y)$, and D^k is given by

$$D_k^{(m)}(x, y) = \sum_{j=2}^{k} \langle \nabla_y f_j^{(m)}(x, y), \nabla_y f_{k+2-j}^{(m)}(x, y) \rangle.$$

With Lemma 2.9 we have first, for any $(x, y) \in B_\infty(0, R') \times B_\infty(0, R')$,

$$|L_k(x, y)| \leq \sum_{n=1}^{k} \frac{2^n}{n} \sum_{j_1+j_2+\ldots+j_n=k} |M_{j_1} M_{j_2} \ldots M_{j_n}|_{\mathcal{L}(\ell^\infty, \ell^1)}$$

$$\leq m \sum_{n=1}^{k} \frac{2^n}{n} \sum_{j_1+j_2+\ldots+j_n=k} |M_{j_1} M_{j_2} \ldots M_{j_n}|_{\mathcal{L}(\ell^\infty, \ell^\infty)}. \qquad (3.9)$$

But since all the f_j's are standard, there exists for any j a constant $C_j > 0$, independent of (x, y) and m such that

$$|\nabla^2_{yy} f_j^{(m)}(x, y))|_{\mathcal{L}(\ell^\infty, \ell^\infty)} \le C_j.$$

Thus for some $\tilde{C}_k > 0$ independent of (x, y) and m, we have $|L_k(x, y)| \le \tilde{C}_k m$. Now Hölder's inequality gives for D_k:

$$|\langle \nabla_y f_j^{(m)}(0, 0), \nabla_y f_{k+2-j}^{(m)}(0, 0)\rangle| \le |\nabla_y f_j^{(m)}(0, 0)|_1 |\nabla_y f_{k+2-j}^{(m)}(0, 0)|_\infty$$

$$\le m |\nabla_y f_j^{(m)}(0, 0)|_\infty |\nabla_y f_{k+2-j}^{(m)}(0, 0)|_\infty$$

The standardness of the f_j's ensures the existence of constants $C_j > 0$, independent of m such that $|\nabla_y f_j^{(m)}(0, 0)|_\infty \le C_j$, and the corollary follows easily. $\qquad \square$

4. A formal asymptotic expansion

So far we have proved that the sequences $(F_k^{(m)}/m)_m$ are bounded as $m \to \infty$, and we want now to show that they converge. We recall that the potentials we have in mind can be written as

$$V(x) = \sum_{j=1}^m v(x_j) + w(x_j, x_{j+1})$$

and our study will now strongly rely on this particular form.

Let us first consider the case where $w = 0$ (no interaction). For each fixed n we write $V^{(n)} \tilde{\oplus} V^{(m)}$ the function on $\mathbb{R}^{n+m}$ defined by

$$V^{(n)} \tilde{\oplus} V^{(m)}(x_1, x_2, \ldots x_n, x_{n+1}, \ldots x_{m+n}) =$$
$$V^{(n)}(x_1, x_2, \ldots x_n) + V^{(m)}(x_{n+1}, \ldots x_{m+n}) \tag{4.1}$$

In that case we get immediately

$$V^{(n)} \tilde{\oplus} V^{(m)}(x) = V^{(m+n)}(x) \tag{4.2}$$

Then, and because we have uniqueness at each step of the procedure, it is straightforward that if we denote by $f_k^{(m,n)}$, $\phi_k^{(m,n)}$ and $F_k^{(m,n)}$ the objects which were defined in Section 3 but now for the standard potential $V_n = (V^{(n)} \tilde{\oplus} V^{(m)})_m$, we get

$$f_k^{(m,n)} = f_k^{(n)} \tilde{\oplus} f_k^{(m)}, \phi_k^{(m,n)} = \phi_k^{(n)} \tilde{\oplus} \phi_k^{(m)}$$

and $F_k^{(m,n)} = F_k^{(n)} + F_k^{(m)}$. Thus in that case $\dfrac{F_k^{(m)}}{m} = F_k^{(1)}$. When w is not identically 0, (4.2) does not hold anymore, but

$$\Phi^{(m,n)}(x) := V^{(m+n)}(x) - V^{(n)} \tilde{\oplus} V^{(m)}(x)$$
$$= w(x_n, x_{n+1}) + w(x_{m+n}, x_1) - w(x_n, x_1) - w(x_{m+n}, x_{n+1}) \tag{4.3}$$

and we see that the analysis will not be too different from the previous no-interaction case. We introduce a sequence of one-parameter family of standard potentials $(V_{n,\theta})_n$ where $V_{n,\theta}$ is defined for $\theta \in [0,1]$ by:

$$V_{n,\theta}^{(n+m)}(x) = \theta V^{(n+m)}(x) + (1-\theta)V^{(n)}\tilde{\oplus}V^{(m)}(x), \quad x \in \mathbb{R}^{n+m} \qquad (4.4)$$

and we want to obtain the result for the case with interaction ($\theta = 1$) by interpolation from the case with partial decoupling ($\theta = 0$). Notice that

$$V_{n,\theta}^{(m)} = V^{(n+m)} + \theta\Phi^{(m,n)}, \partial_\theta V_{n,\theta}^{(n+m)} = \Phi^{(m,n)}.$$

Our first task is to go back to the WKB constructions of Section 3 for this sequence of one-parameter family of $\mathcal{S}^0$-potential $(V_{n,\theta})$. Here follow our precise assumptions.

For each $n \in \mathbb{N}$ we let $\mathcal{R}^n = (\mathcal{R}_{(n+m)}^n)_m$ where $\mathcal{R}_{(n+m)}^n$ is the set of all functions $\rho_n^{(n+m)} : \{1, \ldots, m+n\} \to \mathbb{R}^+$ such that for all $j \in \{1, \ldots, m+n\}$,

$$e^{-\alpha} \leq \frac{\rho_n^{(n+m)}(j)}{\rho_n^{(n+m)}(j+1)} \leq e^{\alpha}$$

and

$$e^{-\alpha} \leq \frac{\rho_n^{(n+m)}(m+n)}{\rho_n^{(n+m)}(n+1)} \leq e^{\alpha}, \qquad e^{-\alpha} \leq \frac{\rho_n^{(n+m)}(n)}{\rho_n^{(n+m)}(1)} \leq e^{\alpha}$$

for some $\alpha > 0$.

We denote by σ_n the weight in $\mathcal{R}^n$ defined by

$$\sigma_n^{(n+m)}(j) = \begin{cases} e^{\alpha \min(j, n+1-j)} & \text{when } 1 \leq j \leq n \\ 1 & \text{when } j > n \end{cases}$$

We shall suppose that the potential V satisfies the following assumpitons

(B1) For each fixed n, $V_{n,\theta} = (V_{n,\theta}^{(n+m)})$ belongs to $\mathcal{S}_{\mathcal{R}^n, \sigma_n}^0(B_\infty(0,R) \times [0,1])$. Moreover the constants in the estimates of standardness (see (2.4)) can be chosen independent of n.

(B2) For all $\theta \in [0,1]$ and any n, $V_{n,\theta}(0) = \nabla V_{n,\theta}(0) = 0$. Moreover there exist two constants $r_0 > r_1 > 0$ such that for all $\theta \in [0,1]$ and any n there is a diagonal matrix $D_{n,\theta}$ such that $D_{n,\theta} > r_0$ and for any $p \in [1, +\infty]$, any $\rho \in \mathcal{R}^n$:

$$\|\nabla^2 V_{n,\theta}(0) - D_{n,\theta}\|_{\mathcal{L}(\ell_\rho^p)} < r_1$$

Notice that for $n = 0$ these assumptions are precisely those of Section 3. First we can prove as in Theorem 3.2, the existence of a unique solution of the eikonal equation in this class of functions. Then, as in Section 3, we can obtain uniform estimates of standardness for all the functions we have defined there. The following theorem is really what will enable us to prove the forthcoming results in a quite elementary way.

Theorem 4.1 *Suppose the assumptions (B1) and (B2) are satisfied. Then the set of functions $f_{n,\theta,k}^{(m+n)}$ defined as above for $V = V_{n,\theta}$ is a subset of $\mathcal{S}_{\mathcal{R},\sigma_n}^0\left(B_\infty(0, R') \times [0,1]\right)$ for some $0 < R' \le R$, uniformly bounded with respect to n.*

Now we add an assumption about the isotropy of the interaction which is natural in the framework of statistical mechanics:

(B3) For all m, and for all $x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$, the function $V^{(m)}$ satisfies

$$V(x_1, x_2, \ldots, x_m) = V(x_m, x_1, x_2, \ldots, x_{m-1})$$

Under this assumption, the first eigenfunction of Kac's operator, which is simple, must have the same property. Thus we can search for a $u^{wkb}(x, h) = \exp(-\phi(x, h)/h)$ having the invariance property. Then the $f_{n,1,k}$ (in fact those written in Section 3) are also invariant by circular shift of the coordinates. This fact is an easy consequence of the uniqueness result in Theorem 3.5. At last, we have the

Theorem 4.2 *Suppose the assumptions (B1) to (B3) are satisfied. Then for any $k \ge 1$, the sequence $\left(\frac{F_k^{(m)}}{m}\right)$ converges. Moreover, denoting by $-\Lambda_k$ its limit, there exists a constant $C_k > 0$ such that*

$$\left| \frac{F_k^{(m)}}{m} + \Lambda_k \right| \le C_k e^{-\alpha m/2}$$

Sketch of proof: We recall that (see Corollary 3.6) $F_k^{(n+m)} = L_k^{(n+m)} - D_k^{(n+m)}$ where

$$\begin{cases} D_k^{(n+m)} = \sum_{j=2}^{k} \langle \nabla_y f_{n,1,j}^{(n+m)}(0,0), \nabla_y f_{n,1,k+2-j}^{(n+m)}(0,0) \rangle \\[2mm] L_k^{(n+m)} = \sum_{s=1}^{k} \frac{2^s}{s} \sum_{j_1+j_2+\cdots+j_s=k} |M_{n,\theta,j_1}^{(n+m)} M_{n,\theta,j_2}^{(n+m)} \cdots M_{n,\theta,j_s}^{(n+m)}|_{\mathcal{L}(\ell^\infty, \ell^1)} \end{cases}$$

and $M_{n,\theta,j}^{(n+m)} = \nabla_{yy}^2 f_{n,\theta,j}^{(n+m)}(0,0)$.

In order to illustrate how the standardness properties enters the game, we describe how to treat the D_k's. Under the assumption (B3) we have first

$$\langle \nabla_y f_{n,1,j}^{(n+m)}(0,0), \nabla_y f_{n,1,k+2-j}^{(n+m)}(0,0) \rangle = m \partial_{y_i} f_{n,1,j}^{(n+m)}(0,0) \partial_{y_i} f_{n,1,k+2-j}^{(n+m)}(0,0)$$

for any $i \in \{1, \ldots, m\}$. Since each $f_{n,\theta,j}$ is an $\mathcal{S}_{\mathcal{R},\sigma_n}$-function uniformly with respect to n, we also have,

$$|\partial_\theta \nabla_y f_{n,\theta,j}^{(n+m)}(0,0)|_{\infty,\sigma_n^{(n+m)}} \le A_j$$

for some constant A_j independent of m and n. Integrating with respect to $\theta \in [0,1]$ we get

$$|\nabla_y f_{n,1,j}^{(n+m)}(0,0) - \nabla_y f_{n,j}^{(n+m)}(0,0)|_{\infty,\sigma_n^{(n+m)}} \le A_j$$

so that for any $i \in \{0, \ldots, m+n\}$ we have

$$\left| \sigma_n^{(n+m)}(i) \partial_{y_i} \left(f_j^{(m+n)} - f_j^{(n)} \tilde{\oplus} f_j^{(m)} \right)(0,0) \right| \le A_j.$$

In particular for $i \leq n$ we obtain

$$|\partial_{y_i} f_j^{(m+n)}(0,0) - \partial_{y_i} f_j^{(n)}(0,0)| \leq \frac{A_j}{\sigma_n^{(n+m)}(i)}. \qquad (4.5)$$

Then we get

$$|\frac{D_k^{(m+n)}}{m+n} - \frac{D_k^{(n)}}{n}| \leq$$

$$\sum_{j=2}^{k} \frac{A_j}{\sigma_n^{(n+m)}(i)} |\partial_{y_i} f_{k+2-j}^{(n)}(0,0)| + \frac{A_{k+2-j}}{\sigma_n^{(n+m)}(i)} |\partial_{y_i} f_j^{(n)}(0,0)| + \frac{A_j A_{k+2-j}}{\sigma_n^{(n+m)}(i)^2}.$$

Using again the uniform standardness of the $f_{n,\theta,j}^{(m)}$, we obtain that for some constant $C > 0$ independent of m and n, since $\sigma_n^{(n+m)}(i) \geq 1$,

$$|\frac{D_k^{(m+n)}}{m+n} - \frac{D_k^{(n)}}{n}| \leq \frac{C}{\sigma_n^{(n+m)}(i)}$$

With our choice for σ_n, choosing $i = [n/2]$, we obtain

$$|\frac{D_k^{(m+n)}}{m+n} - \frac{D_k^{(n)}}{n}| \leq C e^{-\alpha n/2}. \qquad (4.6)$$

This shows that $(\frac{D_k^{(n)}}{n})_n$ is a Cauchy sequence and gives the expected control on the speed of convergence. The other term L_k can be handled with similar arguments (see the proof of Corollary 3.6). $\qquad \square$

5. Estimates for the thermodynamic limit

We now want to obtain estimates on the thermodynamic limit

$$\Lambda(h) = -\lim_{m \to +\infty} \frac{\ln(\mu_1(h,m))}{m}$$

of the true first eigenvalue of Kac's operator. We suppose now that these quantities are well-defined (see [He2] for example), and that assumptions (B1) to (B3) hold. We follow a classical procedure: with the formal WKB solution that we have built, we can define approximate solutions considering finite sums: for each $N \in \mathbb{N}$ we let u^N be the function defined on $\Omega = B_\infty(0, R')$ by

$$u^N(x) = e^{-\phi^N(x,h)/h}, \phi^N(x,h) = \sum_{j=0}^{N} \phi_j(x)h^j,$$

and the $\phi'_j s$ are the functions we define on $B_\infty(0, R')$ in Section 3. We define also

$$\mu^N = e^{-F^N(h)/h}, F^N(h) = \sum_{j=0}^{N} F_j h^j$$

With the min-max principle it is immediately clear that since $\mu^{(m)}$ is the highest eigenvalue of $K^{(m)}$ as an operator on $L^2(\mathbb{R}^m)$, we have

$$\frac{\mu^{(m)}}{m} \geq \frac{1}{m}\frac{(K_\Omega^{(m)}u^N, u^N)}{(u^N, u^N)}$$

where we let K_Ω be the same operator K but acting on $L^2(\Omega)$. Computing the right-hand side and letting $m \to \infty$ we get

Proposition 5.1 *There is an $h_0 > 0$ such that, for any $N \in \mathbb{N}$ there exists a constant $C_N > 0$ such that,*

$$\forall h \in]0, h_0], \quad \Lambda(h) \geq \sum_{j=0}^{N-1} \Lambda_j h^j - C_N h^N.$$

Proof: Considering the linear combination of equations (T_0) to (T_{N+1}) which is obtained multiplying (T_j) by h^{j-2}, we get

$$-\tfrac{1}{h}\phi^N(y, h) + \tfrac{1}{h}\phi^N(x, h) - \tfrac{V(x)}{2} - \tfrac{V(y)}{2} - \tfrac{1}{4h^2}|x - y|^2 =$$
$$-|\nabla_y f^{N+1}(x, y, h)|^2 - F^{N-1}(h) + \log|\det \nabla_{yy}^2 f^{N-1}(x, y, h)| + m\ln 2$$

where we let also $f^N(x, y, h) = \sum_{j=0}^N f_j(x, y)h^j$. Thus

$$K^{(m)}u^N(x, h) = e^{-F^{N-1}(h)}u^N(x, h) \times$$
$$\times \frac{1}{(\pi h^2)^{m/2}}\int_\Omega e^{-|\nabla_y f^{N+1}(x,y,h)|^2/h^2}|\det \nabla_{yy}^2 f^{N-1}(x, y, h)|dy.$$

We put $z(x, y, h) = \nabla_y f^{N+1}(x, y, h)$ in the integral above and we get

$$K^{(m)}u^N(x, h) = \frac{e^{-F^{N-1}(h)}u^N(x, h)}{(\pi h^2)^{m/2}}\int_{\tilde\Omega(x,h)} e^{-z^2/h^2}\frac{|\det \nabla_{yy}^2 f^{N-1}(x, y(z), h)|}{|\det \nabla_{yy}^2 f^{N+1}(x, y(z), h)|}dz$$

where $\tilde\Omega(x, h)$ is the image of Ω under the map $y \mapsto z(x, y, h)$. Notice that the map z is a C^1-diffeomorphism in $B_\infty(0, R)$ for any h small enough since $\nabla_{yy}^2 f^{N-1}(x, y, h) = \tfrac{1}{2}I + O(h)$ or, more precisely,

$$|\nabla_{yy}^2 f^{N-1}(x, y, h) - \frac{1}{2}I|_{\mathcal{L}(\ell_\infty, \ell_\infty)} \leq Ch \tag{5.1}$$

for some $C > 0$ independent of m. We also have for some $C > 0$ independent of m

$$\left| \ln\left(\frac{\det \nabla_{yy}^2 f^{N-1}(x, y(z), h)}{\det \nabla_{yy}^2 f^{N+1}(x, y(z), h)} \right) \right| \leq mCh^N \tag{5.2}$$

Indeed, writing

$$L(t) := \nabla_{yy}^2 f^{N-1}(x, y, h) + t(\nabla_{yy}^2 \tilde{f}^N(x, y, h)h^N + \nabla_{yy}^2 \tilde{f}^{N+1}(x, y, h)h^{N+1})$$

we have

$$\ln\left(\frac{\det \nabla_{yy}^2 f^{N-1}(x,y,h)}{\det \nabla_{yy}^2 f^{N+1}(x,y,h)}\right) = \int_0^1 \partial_t \ln\det(L(t))dt = \int_0^1 \operatorname{tr}((L(t)^{-1}L'(t))dt.$$

With Lemma 2.9, since

$$|L'(t)|_{\mathcal{L}(\ell_\infty,\ell_1)} \leq m|L'(t)|_{\mathcal{L}(\ell_\infty,\ell_\infty)} \leq Cmh^N$$

for some $C > 0$ independent of m, and using also (5.1), we get (5.2).

We have proved that

$$(K^{(m)}u_N, u_N) \geq e^{-F_{N-1}(h)-mCh^N} \int_\Omega u_N^2(x,h) \int_{\tilde{\Omega}(x,h)} e^{-z^2/h^2}\frac{dzdx}{(\pi h^2)^{m/2}}$$

We will obtain a lower bound for the inner integral if we integrate on a ball $B_\infty(0,r)$ contained in all the $\Omega(x,h)$ for $x \in \Omega$ and h small enough. That such a ball exists follows easily from the uniform estimates on z and z^{-1} we mentioned above (see (5.1)). With that choice of domain we get, for some $\varepsilon > 0$,

$$(K^{(m)}u_N, u_N) \geq e^{-F_{N-1}(h)-mCh^N}(1 - e^{-\varepsilon/h^2})^m \int_\Omega u_N^2(x,h)dx$$

so that

$$\log\frac{\mu^{(m)}}{m} \geq \frac{1}{m}\left(-F_{N-1}(h) - mCh^N + \log(1 - e^{-\varepsilon/h^2})^m\right)$$

and the proposition follows letting $m \to +\infty$. $\qquad\square$

It would be quite natural that a similar upper bound held for the thermodynamic limit, at least when the potential V is globally strictly convex. At this time however we were not able to get such a result. To begin with, we will give an upper bound for the restriction of K to $L^2(\Omega)$, where Ω is the neighborhood of the origin where we have been able to construct our WKB solution. What is missing then is some uniform estimate on the decay of the true eigenfunction outside of this neighborhood.

Let us recall that the highest eigenvalue $\mu_1^{(m)}(h)$ of the positive, compact operator $K_V^{(m)}$ is precisely its L^2 norm (this is part of the Krein-Rutman theorem). We will use the following version of Schur's lemma, that can be proved along the same lines.

Proposition 5.2 *Let Ω be an open bounded set in $\mathbb{R}^{(m)}$, and u a nowhere vanishing smooth function on Ω. Then*

$$\|K\|_{\mathcal{L}(L^2(\Omega))} \leq \sup_{x\in\Omega}\int_\Omega \frac{u(y)}{u(x)}|k(x,y)|dy$$

Now let $\Lambda_\Omega(h)$ be the thermodynamic limit of the first eigenvalue of the operator K acting on $L^2(\Omega)$. With the above lemma, and mimicking the proof of Proposition 5.1 we obtain the following result.

Proposition 5.3 *There is an $h_0 > 0$ such that, for any $N \in \mathbb{N}$ there exists a constant $C_N > 0$ such that,*

$$\forall h \in]0, h_0], \quad \Lambda_\Omega(h) \leq \sum_{j=0}^{N-1} \Lambda_j h^j + C_N h^N$$

References

[El] R. Ellis: *Entropy, Large Deviations, and Statistical Mechanics*, Grundlehren der math. Wiss. 271, Springer.

[He1]B. Helffer: *Around a stationnary phase theorem in large dimension*, Journal of Functional Analysis, Vol. 119, n.1, 1994, p. 217–252.

[He2]B. Helffer: *Spectral Properties of the Kac Operator in Large Dimension*, Proceedings on Mathematical Quantum Theory II: Schrödinger operators (August 1993), J. Feldman, R. Froese and L.M. Rosen (ed.), CRM Proc. Lect. Notes. 8, 179-211 (1995).

[He3]B. Helffer: *Recent results and open problems on Schrödinger operators, Laplace integrals and transfer operators in large dimension*.Michael Demuth (ed.) et al., Schrödinger operators, Markov semigroups, wavelet analysis, operator algebras. Berlin: Akademie Verlag. Math. Top. 11, 11-162 (1996).

[He4]B. Helffer: *Semiclassical analysis for the transfer operator: WKB constructions in large dimension*, Comm. in Mathematical Physics **187** (1997), p. 81–113.

[He-Sj]B. Helffer, J. Sjöstrand: Semiclassical expansions of the thermodynamic limit for a Schrödinger equation. I: The one well case. D. Robert (ed.), Methodes semi-classiques, Volume 2. Colloque international (Nantes, juin 1991), Asterisque. 210, 135-181 (1992).

[Ka]M. Kac: *Mathematical mechanisms of phase transitions*, Brandeis lectures (1966). New-York: Gordon and Breach, 1966.

[Mø]J. Møller: *The low-temperature limit of transfer operator in fixed dimension*. Prépublication de l'Université Paris Sud 2000-35.

[Ru]D. Ruelle: *Probability estimates for continuous spin systems*, Comm. in Mathematical Physics **50** (1976), p. 189–194.

[Sj1]J. Sjöstrand: *Potential wells in high dimensions I*, Ann. de l'Institut Henri Poincaré, Phys. Théor., Vol. 58 n.1 (1993), p.1–41.

[Sj2]J. Sjöstrand: *Potential wells in high dimensions II*, Ann. de l'Institut Henri Poincaré, Phys. Théor., Vol. 58 n.1 (1993), p.43–55.

[Sj3]J. Sjöstrand: *Evolution equations in a large number of variables*, Math. Nachr. **166** (1994), p. 17–53.

Département de Mathématiques
Université Paris Sud
(UMR CNRS 8628)
Thierry.Ramond@math.u-psud.fr
www.math.u-psud.fr/~osc