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Abstract

In this note we discuss some recent and ongoing joint work with Thalia Jeffres concerning the existence of Kähler-Einstein metrics on compact Kähler manifolds which have a prescribed incomplete singularity along a smooth divisor D. We shall begin with a general discussion of the problem, and give a rough outline of the 'classical' proof of existence in the smooth case, due to Yau and Aubin, where no singularities are prescribed. Following this is a discussion of the geometry of the conical or edge singularities and then some discussion of the new elements of the proof in this context.

1. Background material.

Let (X, g_0) be a compact Kähler manifold of complex dimension n. Thus if ω_0 is the corresponding Kähler form, then $d\omega_0 = 0$ and $[\omega_0]$ is a class in $H^{1,1}(X)$. A familiar question in geometric analysis is to find canonical metrics on a manifold. In this setting, this is commonly regarded as the search for Kähler-Einstein (KE) metrics. More specifically, a Kähler metric g on X is said to be KE if its Ricci tensor is a multiple of the metric. Using the complex structure J to turn the Ricci tensor into a $(1,1)$-form $\rho = \rho_g$, then we ask that

$$\rho_g = \lambda \omega_g$$

for some constant λ. By scaling, we can always assume that $\lambda = -1$, 0 or $+1$.

We narrow the search somewhat by requiring that the Kähler form ω_g lie in the same cohomology class as the Kähler form ω_0 for the original metric. Hence $\omega_g - \omega_0$ is exact, and using the $\partial\bar{\partial}$-Poincare lemma we can write

$$\omega_g - \omega_0 = i\partial\bar{\partial}u$$

for some scalar function u.

There is an immediate obstruction to the existence of such metrics. Let us define a class $c \in H^{1,1}(X)$ to be *positive* if it contains a representative $\eta \in c$ such

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that $\eta(Z, JZ) > 0$ for all nonzero vectors Z in TX. We write this as $c > 0$. The set of all positive classes forms an open cone in $H^{1,1}(X)$. The negative of this cone is the set of *negative* classes. Next, it is known that if $\tilde{g}$ is any Kähler metric on X, then the associated Ricci form $\tilde{\rho}$ represents the first Chern class of X, i.e.

$$[\tilde{\rho}] = c_1(X).$$

In the lowest dimensional case, when X is a Riemann surface, this is essentially just the Gauss-Bonnet theorem. Suppose now that g is a KE metric, so that $\rho = \lambda\omega$. If $\lambda = +1$, then $[\rho] = [\omega] = c_1(X)$, and so $c_1(X) > 0$. Similarly, if $\lambda = -1$, then $c_1(X) < 0$ and if $\lambda = 0$, then $c_1(X)$ must vanish.

It was conjectured by Calabi in the 1950's that this is the only obstruction. Through the work of Aubin and Yau in the 1970's, this is now known to be true in the cases $c_1(X) \leq 0$. The case $c_1(X) = 0$, done by Yau, is of particular interest because the metrics are Ricci flat. Compact Kähler manifolds with vanishing first Chern class are known as Calabi-Yau manifolds, and play an important role in mathematical physics and elsewhere. The case $\lambda = +1$ is much more difficult, and there are only partial results in this case, through the work of Tian. It is known that there are further obstructions in this case to the existence of KE metrics.

This conjecture was resolved by proving the existence of a solution of a complex Monge-Ampere equation. As explained above, for any Kähler metric g with ω_g in the same cohomology class as ω_0, we have $\omega_g = \omega_0 + i\partial\bar{\partial}u$. It is not hard to calculate that the metric g is KE if this scalar function u satisfies

$$\mathcal{M}(u) = \log\left(\frac{\det\left((g_0)_{i\bar{j}} + \sqrt{-1}u_{i\bar{j}}\right)}{\det\left((g_0)_{i\bar{j}}\right)}\right) - F + \lambda u = 0, \tag{1}$$

such that in addition

$$\left((g_0)_{i\bar{j}} + \sqrt{-1}u_{i\bar{j}}\right) > 0. \tag{2}$$

The function F in the definition of the operator $\mathcal{M}$ is determined solely by the initial metric g_0, and in a sense measures the defect by which g_0 fails to be a KE metric. Also, this equation has been written relative to any local holomorphic coordinate system, and the functions $u_{i\bar{j}}$ are the components of the complex Hessian in these coordinates, but it may easily by checked that the whole expression is independent of coordinate system.

2. The proof in the smooth case.

The proof is by the method of continuity. One constructs a family of nonlinear operators

$$\mathcal{M}_\epsilon(u) = \log\left(\frac{\det\left((g_0)_{i\bar{j}} + \sqrt{-1}u_{i\bar{j}}\right)}{\det\left((g_0)_{i\bar{j}}\right)}\right) - \epsilon F + \lambda u = 0, \tag{3}$$

where the parameter $\epsilon \in [0, 1]$. When $\epsilon = 0$, there is the obvious solution $u = 0$, and so we seek to show that the set of ϵ in this interval for which there is a solution is both open and closed.

The openness argument is quite straightforward in this context, when $\lambda \leq 0$. Thus suppose that for some value ϵ' there is a solution u' of $\mathcal{M}_{\epsilon'}(u') = 0$ which also satisfies the positivity condition (2). Let g' be the associated metric. Then

$$D\mathcal{M}_{\epsilon'}\big|_{u'}(\phi) = \Delta_{g'}\phi + \lambda\phi. \tag{4}$$

If $\lambda < 0$ then this operator is obviously an isomorphism, say from $\mathcal{C}^{2,\alpha}(X)$ to $\mathcal{C}^{0,\alpha}(X)$, while if $\lambda = 0$ then it turns out that we may restrict to functions u with total integral zero, and so we get an isomorphism again. The fact that this operator need not be an isomorphism when $\lambda > 0$ is the first hint of trouble in this case. At any rate, a straightforward application of the implicit function theorem now shows that for all ϵ near to ϵ' there is also a solution to this problem.

The closedness argument is much more difficult. Here, if $\epsilon_j \to \epsilon'$, and if u_j are the corresponding solutions, then we would like to show that at least some subsequence of these functions converges in $\mathcal{C}^2(X)$, for then we could take a limit in the equation and hence obtain a solution at the limiting value ϵ'. This is accomplished by deriving a priori $\mathcal{C}^3$ bounds for solutions u of this equation, independent of the parameter ϵ. The $\mathcal{C}^0$ bound is trivial when $\lambda < 0$, but quite subtle when $\lambda = 0$. The bounds for the higher derivatives proceed by obtaining differential inequalities for various quantities, including in the end the norm squared of the third derivatives of u.

We shall not give any further details of this argument, but refer to [1], where the complete details, as well as much background material, are to be found.

3. Edge singularities.

We are now finally ready to describe our problem. Suppose that in addition to the Kähler manifold X we are also given a smooth irreducible divisor $D \subset X$. Thus D is a smooth complex hypersurface. We shall frequently use local coordinates $(\zeta, w_1, \dots, w_{n-1}) = (\zeta, w)$ near points of D, where D is locally defined by $\{\zeta = 0\}$, and w restricts to a complex coordinate chart along D. Fix a real number α in the interval $(0, 1)$. Then we may write the local model

$$g_\alpha = \frac{|d\zeta|^2}{|\zeta|^{2\alpha}} + |dw|^2 \tag{5}$$

for the type of singularity we are trying to prescribe. Note that for $0 < \alpha < 1$, g_α is incomplete, and is the product of a Euclidean factor with a cone of opening less than $\pi/2$ (the cone angle is easy to describe in terms of α). Note also that when $\alpha = 0$, then g_α is smooth across D, while if $\alpha \geq 1$ then g_α is complete near $\zeta = 0$.

Our problem, roughly speaking, is to find a KE metric g with $\lambda = -1$ which is asymptotic to g_α as $\zeta \to 0$. Existence of KE metrics with $\lambda \leq 0$ in the complete cases where $\alpha \geq 1$ was accomplished in a series of papers by Tian and Yau [14], [15], [16]. In addition, quite recently Joyce has constructed Ricci flat KE metrics on ALE spaces which are resolutions of noncompact orbifolds of the form $\mathbb{C}^n/\Gamma$ [6], [7]. Some previous results on the existence of KE metrics in this incomplete case, for orbifolds, were obtained by Kobayashi [8].

Before we can state the theorem more precisely, we must discuss the analogue of the compatibility condition $c_1(X) \leq 0$ in the smooth case. As a guide we consider again the case where X is a Riemann surface. A 'smooth, irreducible' divisor D in this case is a finite collection $\{p_1, \ldots, p_k\}$ of distinct points on X. Suppose that our problem is to construct a hyperbolic (constant curvature -1) metric on $X \setminus \{p_1, \ldots, p_k\}$ which has the specified asymptotic conic form $|d\zeta_i|^2/|\zeta_i|^{2\alpha_i}$, $i = 1, \ldots, k$, where ζ_i is a local complex coordinate near p_i. Suppose first that such a metric exists. Then by applying the Gauss-Bonnet theorem on a domain obtained by excising small balls around each of the p_i and letting the radius of each of these balls go to zero, we obtain

$$\chi(X) < \sum_{i=1}^{k} \alpha_i.$$

Actually, one gets an exact formula, where the term $\int_X K\, dA = -\text{Area}\,(X)$ appears on the right. This is a necessary condition for the existence of such a metric. It is also sufficient, as demonstrated through the work of Troyanov [17] and McOwen [10]. The equation to be solved in this lowest dimensional case is semilinear and it may be handled by barrier methods.

The form of this answer is not apparently immediately adaptable to the higher dimensional setting, but it motivates the compatibility condition we shall impose:

$$c_1(X) < \alpha\, c_1(L_D). \tag{6}$$

Here L_D is the line bundle associated to the divisor D, and the inequality is meant in the sense that $\alpha\, c_1(L_D) - c_1(X)$ is a positive class.

We may now state our result.

Theorem 1 *Suppose (X, g_0) is a compact Kähler manifold and $D \subset X$ a smooth irreducible divisor. Suppose also that for some $\frac{1}{2} \leq \alpha < 1$, $c_1(X) < \alpha c_1(L_D)$. Then there exists a Kähler-Einstein metric g with $\lambda = -1$ on $X \setminus D$ which is asymptotically equivalent to the singular metric g_α along D. Moreover, the metric g has bounded curvature, and indeed has a complete polyhomogeneous expansion upon approach to D.*

We make a few remarks. First of all, the restriction that $\alpha \geq 1/2$ is unfortunate, but is almost certainly caused by purely technical difficulties. We expect the result to be true for α in the full range $(0, 1)$. A similar restriction occurs in an analogous real three dimensional hyperbolic problem [5]. Secondly, we also fully expect that we will be able to prove the existence of Ricci flat Kähler-Einstein metrics, i.e. with $\lambda = 0$, however some points remain to be checked. Finally, the situation here is in many ways analogous to the problem of finding complete KE metrics with $\lambda = -1$ on the interior of strictly pseudoconvex domains. Existence of such metrics is due to Cheng and Yau [3], and the refined regularity statement in our theorem is parallel to the results of Fefferman [4] and Lee and Melrose [9]. In both cases (when the boundary of the pseudoconvex domain is real analytic, which is automatic for the divisor), the polyhomogeneous expansion is convergent, as follows from the work of Byde [2].

4. Sketch of the proof.

As in the smooth case, we proceed by the continuity method. However, there is an initial step, which is to show that by using the compatibility condition we may write down an initial guess for the solution metric $\tilde{g}$. This will be a Kähler metric which has the correct singularity, and thus is asymptotic to g_α along D.

To do this, we begin with the volume density V_0 for the background smooth Kähler metric g_0. Fix an Hermitian metric on the fibres of the line bundle L_D and choose a global section $s \in \Gamma(L_D)$ such that $D = s^{-1}(0)$. Finally, set

$$\beta = 1 - \alpha.$$

The constant β appears more frequently than α hereafter. Now write the new volume density

$$\tilde{V} = \frac{V_0}{||s||^{2\alpha}(1 - \epsilon||s||2\beta)^2},$$

where ϵ is any number sufficiently small to guarantee that the denominator does not vanish. Finally we may define the *Kähler cone metric* $\tilde{g}$ by its Kähler form

$$\tilde{\omega} = i\partial\bar{\partial}\tilde{V}.$$

While this is certainly a $(1,1)$ form, it is precisely the compatibility condition which ensures that we may choose our initial smooth metric g_0 and Hermitian metric on L_D so that it is also positive definite away from D. The fact that it is asymptotic to g_α follows from a straightforward calculation.

It is also interesting to note that already $\tilde{g}$ has the property that both $\tilde{\omega}$ and the associated Ricci form $\tilde{\rho}$ have a distributional (or rather, current) part supported on D, of the form $\alpha\delta_D$. Our final solution metric will also have this property, and so the Kähler-Einstein condition will be satisfied globally in the sense of currents, rather than just pointwise on $X \setminus D$.

Now that we have defined this initial metric, we may write down the same Monge-Ampere operator $\mathcal{M}(u)$ and its deformations $\mathcal{M}_\epsilon(u)$, where $\mathcal{M}_1 = \mathcal{M}$. This is now a singular, fully nonlinear elliptic operator, because of the appearance of the coefficients of the metric $\tilde{g}$ in the determinant terms.

We may regularize at least the appearance of this expression by choosing a new singular coordinate system as follows. This simplifies many of the calculations. Recalling the smooth complex coordinate ζ, we set

$$z = \zeta^\beta.$$

Then

$$dz = \beta\zeta^{-\alpha}d\zeta, \qquad \text{and so} \qquad \beta^{-2}|dz|^2 = \frac{|d\zeta|^2}{|\zeta|^{2\alpha}}. \tag{7}$$

So we see that using this new coordinate, along with the other coordinates $(w_1, \ldots, w_{n-1}$ as before, we obtain a singular coordinate system in which the model cone metric g_α takes a particularly simple form. We write this more completely, and add a final trivial change of variables. Let $z = t\,e^{i\phi}$; then $0 \leq \phi \leq 2\pi\beta$, and so we define

$\theta = \phi/\beta$ to make θ an element of the 'standard' circle S^1. We also multiply g_α by the constant factor β^2 and scale the w coordinates by β. Performing all of these changes, we now have

$$g_\alpha = dt^2 + \beta^2 t^2 d\theta^2 + |dw|^2. \tag{8}$$

Now we proceed with the proof. At this point, using the new singular coordinate system, the Monge-Ampere equation no longer appears as singular. Of course, we have just shifted the difficulties elsewhere, but we will see the advantage of this formulation particularly when we study the linearization. At any rate, we again use the continuity method.

It turns out that the closedness argument does not require vast changes from the smooth case, so although this step is still not easy, at least the work has been done for us! There are two main obstacles in making this part of the argument work. Recall that we are ultimately interested in estimating third derivatives of the solutions u of $\mathcal{M}_\epsilon(u) = 0$. Cutting to the end of the argument, we define $S(u) = |\nabla_g^3 u|^2$. Then in both Yau and Aubin's work, a (rather intricate) differential inequality is derived for this quantity. In truth, this inequality depends on already having obtained certain types of a priori estimates for lower derivatives. In the smooth case one considers the point on X where $S(u)$ attains its maximum and uses the differential inequality (the coefficients of which depend only on previously estimated quantities) to conclude an a priori upper bound for $S(u)$ itself. In our case, two different aspects of this argument could fail.

- $S(u)$ might not be bounded on $X \setminus D$

- Even if it is, the maximum of $S(u)$ might occur on D.

Although the same differential inequality as in the smooth case remains valid on $X \setminus D$, it yields no information if the maximum occurs somewhere on D.

The first of these difficulties is resolved by carefully choosing the function space in which u lies. We shall describe this later when we come to the linear analysis. Having made this choice, we may now assume at least that $S(u)$ is bounded on $X \setminus D$. If its maximum does occur on D, then we modify the function $S(u)$ by adding to it some explicit quantity H which has the effect that the maximum of $S(u) + H$ occurs away from D. The point here is that again by the choice of function space, we know the asymptotic behaviour of $S(u)$ near D, and so it is easy to find a suitable function H which moves the maximum away from the divisor. Without going into further details, this finally allows us to conclude the a priori C^3 estimates for solutions u.

The openness part of the continuity argument requires substantial new work. As before, we would like to be able to apply the implicit function theorem to $\mathcal{M}_\epsilon$, and this requires an understanding of the mapping properties of the linearization of this operator. As in the smooth case,

$$D\mathcal{M}_\epsilon|_u(\phi) = (\Delta_{g_\epsilon} - 1)\phi,$$

where g_ϵ is the metric associated with $\tilde{\omega} + i\partial\bar{\partial}u$ where $\mathcal{M}_\epsilon(u) = 0$. For reasons of homogeneity, we shall consider instead the equivalent operator

$$L = t^2(\Delta_{g_\epsilon} - 1). \tag{9}$$

This operator L is an elliptic, differential edge operator, of the type studied extensively in [11]. (A parallel study of these sorts of operators has also been undertaken by Schulze et al. [13], and various special cases have been studied by many other researchers.) We record that near $t = 0$ it has the form

$$L = (t\partial_t)^2 + \beta^{-2}\partial_\theta^2 + t^2\Delta_w + ...,$$

where the omitted terms on the end are higher order in the sense that they are products of at most two of the basic factors $t\partial_t$, ∂_θ and $t\partial_w$ with at least one extra factor of t in front. We briefly state some of the main results for operators of this sort.

The first main issue is mapping properties of L. It turns out to be natural to consider these mapping properties on a scale of weighted Hölder spaces (though we could equally well have used weighted Sobolev spaces). We define

$$\mathcal{C}_\delta^{k,\gamma} = t^\delta \mathcal{C}_0^{k,\gamma},$$

where $\mathcal{C}_0^{k,\gamma}$ is the Hölder space based on differentiations with respect to the vector fields $t\partial_t$, ∂_θ and $t\partial_w$. Alternately, the norms of these spaces with subscript 0 are invariant under the (local) dilations $(t, \theta, w) \to (\lambda t, \theta, \lambda w)$, $\lambda > 0$. It is elementary to check that

$$L : \mathcal{C}_\delta^{k+2,\gamma} \longrightarrow \mathcal{C}_\delta^{k,\gamma}$$

for any $0 < \gamma < 1$ and $\delta \in \mathbb{R}$.

Theorem 2 *This mapping is semi-Fredholm provided $\delta \notin \{j/\beta : j \in \mathbb{Z}\}$.*

Unfortunately the statement that L is only semi-Fredholm is optimal, because when $\delta < 0$, L has an infinite dimensional nullspace, while when $\delta > 0$, the range of L has infinite codimension. The cokernel, respectively nullspace, in these two cases is at most finite dimensional. When δ happens to equal one of these omitted values, then the mapping does not even have closed range. The numbers j/β are called the *indicial roots* of the problem, in analogy with the one-dimensional regular singular theory. Because the term in L of order zero is negative, we may easily conclude

Corollary 1 *L is injective when $\delta > 0$ and surjective when $\delta < 0$, $\delta \neq j/\beta$.*

Now we see rather clearly the dilemma with which we are faced. The function spaces most suited for the linear analysis, namely say $\mathcal{C}_\delta^{2,\gamma}$ with $\delta < 0$ where L is surjective, are very poorly suited for the nonlinear theory, since the Monge-Ampere operators $\mathcal{M}_\epsilon$ surely do not preserve any of these spaces. The spaces these nonlinear operators do preserve, when $\delta \geq 0$, are not well suited for the linear theory and the application of the implicit function theorem.

To proceed further, we quote an asymptotics result, again from [11].

Proposition 1 *If $\delta > 0$ and $f \in \mathcal{C}_\delta^{0,\gamma}$, then any solution $u \in \mathcal{C}_\eta^{2,\gamma}$ of $Lu = f$ for $-1/\beta < \eta < 0$ has a partial asymptotic expansion of the form*

$$u = \sum_{i<\delta} \tilde{u}_i t^i \log t + \sum_{i+j\beta<\delta} u_{ij} t^{i+j\beta} + v$$

where the $\tilde{u}_i$ and u_{ij} are functions of (θ, w) and $v = \mathcal{O}(t^\delta)$.

We omit discussion of the precise regularity of the coefficient functions and of v because this is a rather delicate issue.

As already noted, there are infinitely many solutions of the equation $Lu = f$ with $u \in \mathcal{C}^{2,\gamma}_{\eta}$. Most of these are still unsuitable for the linear analysis because they are unbounded on account of the log term $\tilde{u}_0 \log t$ in the expansion. Somewhat remarkable there is precisely one solution which is bounded

Proposition 2 *Given f as above, there is exactly one solution u of $Lu = f$ which has an expansion as above with the coefficients $\tilde{u}_i$ all equal to zero.*

The assignment

$$f \longrightarrow u \equiv Gf$$

is actually rather well-behaved because the operator G is a pseudodifferential edge operator in the calculus defined in [11]. Thus we can obtain rather specific information about the solution u in terms of f. More specifically, the Schwartz kernel of G, which is a distribution on $(X \setminus D)^2$, turns out to be the pushforward of a rather simple distribution on a geometric resolution (blow-up) of a compactification of this space. It has polyhomogeneous expansions at the various boundaries of this compactification as well as along the diagonal.

We comment on what is really going on. As we have already explained, the weight $\delta = 0$ is critical for this problem, and this value divides the ranges for which the analytic and geometric problems are well-posed. In this sense, this problem may be regarded as one with a 'critical exponent'. Our solution is to use this right inverse G described above. It is not bounded from $\mathcal{C}^{0,\gamma}_0$ to $\mathcal{C}^{2,\gamma}_0$, but it is bounded if we restrict the domain to the subspace $\mathcal{C}^{0,\gamma}_{\delta}$.

We now see the various constraints placed on the function space B of which u is an element. B should be defined as the image under G of some subspace $B' \subset \mathcal{C}^{0,\gamma}_{\delta}$ for some $\delta > 0$, so that automatically $LB = B'$. Also, $\mathcal{M}_\epsilon$ should carry B to B'. Finally, we also require that if $u \in B$, then the third order quantity S for the metric g associated to u must be bounded, and we also wish that the sectional curvatures of g be bounded. Accomplishing all of these objectives is not so easy, unfortunately, and although there are at least a few viable choices which work, it is really a matter of taste which of these is preferable. For example, if we use the most obvious choice, $B' = \mathcal{C}^{0,\gamma}_{\delta}$, then as we have already mentioned, the regularity of the coefficients in the expansion for $u = Gf$, $f \in B'$, is somewhat complicated, and it requires some delicate analysis to ensure that the nonlinear operator $\mathcal{M}_\epsilon$ acts properly. At the other extreme, we could choose B' to be the space $\mathcal{A}^\delta$ of conormal functions of weight δ. Elements of this space have complete regularity in the interior, as well as full tangential regularity as $t \to 0$. The space B is then a space of partially polyhomogeneous functions, where the coefficients u_{ij} are all smooth. The price here is that these are Frechet spaces, and we must check all the hypotheses of the Nash-Moser implicit function theorem, which again is rather cumbersome. We omit further discussion of these details.

We do wish to briefly discuss a bit more about the partial expansions and coefficients $u_{ij}(\theta, w)$. There are two main issues. The first is exactly how many terms of the expansion must be included, and the second is the dependence of these coefficients on θ. We leave aside the question of their regularity. Recalling that we want third derivatives (with respect to the variables (z, w)) to be bounded as $t \to 0$, we see that the integral powers $t^i u_{i0}$ do not cause any problems, but that any terms of the form $t^{i+j/\beta} u_{ij}$ will be problematic if $i + j/\beta < 3$, $j \neq 0$. Thus we must take particular care of the terms $u_{0j} t^{j/\beta}$ for $j/\beta < 3$. If $\beta < 1/3$ (so $\alpha > 2/3$), then there are no such terms at all, while if $1/3 < \beta < 1/2$, so $\alpha > 1/2$, then there is one such bad term. To control it we need to know something about its coefficient u_{01}. In fact, an argument involving substituting u into the nonlinear equation and collecting terms of the resulting function, which again has a partial expansion, yields first that u_{00} is a function of w alone, and then that $u_{01} = a(w) \cos \theta + b(w) \sin \theta$. It is then straightforward to check that the third derivatives of u are bounded, as desired. The cases $\beta = 1/3$, $1/2$ are handled by similar, but more involved calculations. It is almost surely the case that this same sort of argument carried further will allow us to extend the validity of the theorem to all values of β, and hence α, in $(0, 1)$.

We remark also that an additional regularity argument is needed. On the one hand, the openness argument produces solutions which have at least a partial expansion near D, while the solutions obtained in the limiting process in the closedness argument do not seem to necessarily have this property. In fact, it is possible to show that all solutions which are weakly asymptotic to g_α actually have a complete polyhomogeneous expansion. This uses the techniques of [9] and [12].

This concludes our discussion of the proof.

5. Further directions.

There are many areas where these investigations should be continued. The most obvious is to understand what happens in the more general case when D is an effective divisor with normal crossings. Given the delicacy of the arguments required even in the smooth case, this may be difficult since the requisite linear theory is not yet at an advanced enough state. It would also be interesting to understand the limit of these metrics as $\alpha \to 1$. Most likely, the limit should be one of the complete metrics obtained by Tian and Yau [14], and this fact might shed further light on the precise asymptotics of these metrics. Finally, recalling that one of the first major applications of the existence of KE metrics in the smooth case was to obtaining the so-called Miyaoke-Yau inequalities amongst the Chern classes of X, we would expect that our theorem should yield some ineresting inequalities of the same type involving the pair (X, D).

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