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Supersymmetry, Witten Complex and Asymptotics for Directional Lyapunov Exponents in $\mathbf{Z}^d$

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Abstract

By using a supersymmetric Gaussian representation, we transform the averaged Green's function for random walks in random potentials into a 2-point correlation function of a corresponding lattice field theory. We study the resulting lattice field theory using the Witten Laplacian formulation. We obtain the asymptotics for the directional Lyapunov exponents.

In this talk, I report on my quantitative study of random walks in random potentials in dimensions $d \geq 1$ in [Wa].

1. The analytical formulation and the main results.

In analytical language, which is the language we adopt, this corresponds to the study of the operator

$$H = \Delta + \gamma V, \text{ on } \ell^2(\mathbf{Z}^{\mathrm{d}}), \tag{1.1}$$

where

$$\begin{aligned} \Delta_{ij} &= 1 \quad |i-j|_{\ell_1} = 1 \\ &= 0 \quad \text{otherwise;} \end{aligned} \tag{1.2}$$

γ is a positive parameter, the potential function V is a diagonal matrix: $V = \mathrm{diag}(\mathrm{v_j})$, $\mathrm{j} \in \mathbf{Z}^{\mathrm{d}}$, where $\{v_j\}$ is a family of independently identically distributed (iid) real random variables with distribution dg. (dg is only assumed to be a measure.) From now on, we write $|\,|$ for $|\,|_{\ell_1}$. ℓ_2 norms will be denoted by $\|\,\|$. The probability measure is taken to be

$P = \prod_{j \in \mathbf{Z}^d} dg(v_j)$. We use the notation $\langle\,\rangle_g$ to denote the expectation value with respect to P.

As is well known, the spectrum $\sigma(\Delta)$ is $[-2d, 2d]$. (Note that the Laplacian defined here differs from the one usually used by the probabilists by the constant $2d$ times the identity. So the spectrum differs by $2d$.) Let $\mathrm{supp}\, dg$ be the support of dg, then we have the well established fact (See e.g., [CFKS, PF].) that

$$\sigma(H) = [-2d, 2d] + \mathrm{supp}\, dg \tag{1.3}$$

almost surely. Assume $E \notin \sigma(H)$, then we define $G(E) = (E - H)^{-1}$ and

$$G(E, \mu, \nu) = (\delta_\mu, (E - H)^{-1}\delta_\nu), \qquad \mu,\, \nu \in \mathbf{Z}^d, \tag{1.4}$$

to be the Green's function (or correlation function) of H at energy E. In this paper, we take $E \in \mathbf{R}\backslash\sigma(H)$ (assuming $\mathbf{R}\backslash\sigma(H) \neq \emptyset$) and study the asymptotics of $\langle G(E, \mu, \nu)\rangle_g$, as $|\mu - \nu| \to \infty$ for $\gamma << 1$.

Our result, which is to be stated precisely in the Theorem below, shows that $\langle G(E, \mu, \nu)\rangle_g$ can be expressed as the inverse of an effective *convolution matrix*. Our work in this paper can therefore be categorized as *quantitative* homogenization. But in order to achieve that, we need some technical conditions on dg.

We need to impose conditions on dg to ensure that the resolvent set: $\mathbf{R}\backslash\sigma(H) \neq \emptyset$. This is satisfied if suppdg is bounded either from below or above. We define the Laplace transform of dg, $\hat{g}(t)$ for $t \geq 0$ to be

$$\hat{g}(t) = \int e^{-tv} dg(v), \tag{1.5}$$

if suppdg is bounded from below; and

$$\hat{g}(t) = \int e^{tv} dg(v), \tag{1.6}$$

if suppdg is bounded from above. We also require that dg has bounded moments, in order that the derivatives of $\hat{g}$ have the required properties at infinity.

Remark. This is needed for the Witten Laplacian construction and subsequently for computing the asymptotics of the Lyapunov exponents. For the present paper as we do not work to all orders of γ, it is sufficient to assume that dg has finite n^{th} moment, for some fixed n. But as the construction could be extended to all orders, we assume that all moments of dg are finite.

Due to the presence of the parameter E, without loss of generality, we may then assume that

(H1) $\operatorname{supp} dg \subseteq (-\infty, 0]$ or $\operatorname{supp} dg \subseteq [0, +\infty)$

(H2) dg has finite moments, i.e.,

$$\left|\int v^n dg(v)\right| < \infty, \qquad \text{for all n} \geq 0.$$

Define

$$g(a) = \int_{-a}^{0^+} dg(v), \qquad (a \geq 0) \tag{1.7}$$

in the first case of (H1) and

$$g(a) = \int_{0^-}^{a} dg(v), \qquad (a \geq 0) \tag{1.8}$$

in the second case of (H1). We assume further

(H3) dg is such that g is of regular variation at 0 with exponent ρ $(0 \le \rho < \infty)$, i.e.,

$$\frac{g(Ca)}{g(a)} \to f(C) = C^\rho \tag{1.9}$$

as $a \to 0$ for all $C > 0$.

Remark. Since dg is a probability measure, g is a positive monotone function on $[0, \infty)$. For such functions, condition (1.9) is much less restrictive than it seems at first sight. This is because once we assume the limit of $g(Ca)/g(a)$ exists and is finite as $a \to 0$ for a dense set of C's in $\mathbf{R}^+$, then the limit $f(C)$ is necessarily of the form C^ρ for some $0 \le \rho < \infty$. For more details on this, see p. 268 of [F].

(H3) enables us to use Tauberian type theorems to deduce the desired properties of the derivatives of $\hat{g}$ as $t \to \infty$. We note in particular that the Bernoulli distribution

$$\begin{aligned} dg(v) &= \frac{\delta(v) + \delta(v+1)}{2} \\ &= \frac{dh(v) + dh(v+1)}{2} \end{aligned} \tag{1.10}$$

where δ is the Dirac distribution at 0 and h is the Heaviside function, is of regular variation with exponent $\rho = 0$.

For all $\delta > 0$, define

$$\begin{aligned} I_\delta &= (2d + \delta, \infty) && \text{if } \operatorname{supp} dg \subseteq (-\infty, 0], \\ &= (-\infty, -2d - \delta) && \text{if } \operatorname{supp} dg \subseteq [0, \infty). \end{aligned} \tag{1.11}$$

Assuming (H1-3), our main result is

Theorem 1 *For all $\delta > 0$, there exists $\gamma_0 > 0$, such that for all $0 < \gamma < \gamma_0$, all $E \in I_\delta$, all μ, $\nu \in \mathbf{Z}^d$,*

$$\begin{aligned} \log\langle G(E, \mu, \nu)\rangle_g &= \log\langle G(E, \mu - \nu, 0)\rangle_g \\ &= \log \left(E - \Delta - \gamma\langle v\rangle_g - \gamma^2\langle (v - \langle v\rangle_g)^2\rangle_g[(E - \Delta - \gamma\langle v\rangle_g)^{-1}(0,0)]\right. \\ &\quad \left. -\gamma^3\langle (v - \langle v\rangle_g)^3\rangle_g[(E - \Delta - \gamma\langle v\rangle_g)^{-1}(0,0)]^2\right)^{-1}(\mu - \nu, 0) \\ &\quad +\mathbf{O}(\gamma^4)(|\mu - \nu| + 1) \\ &= \log(\widetilde{E} - \Delta)^{-1}(\mu - \nu, 0) + \mathbf{O}(\gamma^4)(|\mu - \nu| + 1), \end{aligned} \tag{1.12}$$

where

$$\begin{aligned} \widetilde{E} &\stackrel{\text{def}}{=} E - \gamma\langle v\rangle_g - \gamma^2\langle (v - \langle v\rangle_g)^2\rangle_g[(E - \Delta - \gamma\langle v\rangle_g)^{-1}(0,0)] \\ &\quad -\gamma^3\langle (v - \langle v\rangle_g)^3\rangle_g[(E - \Delta - \gamma\langle v\rangle_g)^{-1}(0,0)]^2 \\ &\quad \notin [-2d, 2d]. \end{aligned} \tag{1.13}$$

We stress here that the order of the quantifiers are important, in particular γ_0 is *uniform* in $|\mu - \nu|$.

Remark. We do not require any regularity conditions on dg other than (H1-3) for the Theorem to hold, in marked contrast to the random Schrödinger case, see e.g., [SW1, SW2]. Instead the conditions on dg, in particular (H3) are reminisent of the conditions that one imposes to obtain Lifshitz tails, see e.g., [PF]. The main reason for this contrast as we will see in (1.30, 1.31) is that when $E \in \mathbf{R}\backslash\sigma(H)$, it is the Laplace transform of dg that is relevant; while when $E \in \mathbf{C}\backslash\sigma(H)$, $\Re E \in \sigma(H)$, it is the Fourier transform of dg.

Note the interesting $\mathbf{O}(\gamma^2)$, $\mathbf{O}(\gamma^3)$ terms, which we will give a simple explanation in (1.15)-(1.17). The $\mathbf{O}(\gamma)$ term just corresponds to a shift in E. Note also that

$$|E - \gamma\langle v\rangle_g| = |E| + \gamma|\langle v\rangle_g| > |E| \tag{1.14}$$

for $E \in I_\delta$. So the rate of decay is better than the naive one; recall that the almost sure spectrum is contained in $[-2d, \infty)$ or $(-\infty, 2d]$. Hence in some sense, the "effective" spectrum is further away than the almost sure one. This is related to the so called Lifshitz tails for the density of states in random Schrödinger operators, see e.g., [PF].

We now give a simple explanation for the terms $\mathbf{O}(\gamma)$, $\mathbf{O}(\gamma^2)$ and $\mathbf{O}(\gamma^3)$ in (1.12). Let $G_0 = (E - \Delta - \gamma\langle v\rangle_g)^{-1}$ and $\bar{V} = V - \gamma\langle v\rangle_g$. By the resolvent series, we have

$$\begin{aligned} G(E,\mu,\nu) &= G_0(\mu,\nu) + \gamma(G_0\bar{V}G_0)(\mu,\nu) + \gamma^2(G_0\bar{V}G_0\bar{V}G_0)(\mu,\nu) \\ &\quad +\gamma^3(G_0\bar{V}G_0\bar{V}G_0\bar{V}G_0)(\mu,\nu) + \gamma^4(G_0\bar{V}G_0\bar{V}G_0\bar{V}G_0\bar{V}G)(\mu,\nu), \end{aligned} \tag{1.15}$$

Let $G_0 = (E - \Delta - \gamma\langle v\rangle_g)^{-1}$ and $\bar{V} = V - \gamma\langle v\rangle_g$. Note that the last factor in order $\mathbf{O}(\gamma^4)$ is G itself. So

$$\begin{aligned} \langle G(E,\mu,\nu)\rangle_g &= G_0(\mu,\nu) + \gamma^2\langle(v - \langle v\rangle_g)^2\rangle_g G_0(0,0)(G_0G_0)(\mu,\nu) \\ &\quad +\gamma^3\langle(v - \langle v\rangle_g)^3\rangle_g (G_0(0,0))^2(G_0G_0)(\mu,\nu) + \mathbf{O}(\gamma^4). \end{aligned} \tag{1.16}$$

We hence obtain that for all μ, ν with $|\mu - \nu|$ fixed

$$\begin{aligned} \langle G(E,\mu,\nu)\rangle_g = \big(&E - \Delta - \gamma\langle v\rangle_g - \gamma^2\langle(v - \langle v\rangle_g)^2\rangle_g(E - \Delta - \gamma\langle v\rangle_g)^{-1}(0,0) \\ &-\gamma^3\langle(v - \langle v\rangle_g)^3\rangle_g((E - \Delta - \gamma\langle v\rangle_g)^{-1}(0,0))^2\big)^{-1}(\mu,\nu) + \mathbf{O}(\gamma^4). \end{aligned} \tag{1.17}$$

We see that indeed, the $\mathbf{O}(\gamma)$, $\mathbf{O}(\gamma^2)$, $\mathbf{O}(\gamma^3)$ terms are identical to the ones in (1.12). The trouble comes when we let $|\mu - \nu| \to \infty$, as the estimate for the remainder $\mathbf{O}(\gamma^4)$ is not in the appropriate weighted space due to the appearance of G there. We see therefore that the Theorem extends the result obtained from perturbation series, which is valid only for *fixed* $|\mu - \nu|$.

Using the usual method in field theory, in order to obtain the correct behaviour for $\langle G(E,\mu,\nu)\rangle_g$ for $|\mu - \nu| \to \infty$, one needs to expand the infinite series and then resum it. The advantage of our method is that one establishes first the existence of an effective convolution matrix. It is only when one wants to derive an expression for this convolution matrix that one expands. Therefore to obtain an accuracy of the given order, one only needs to expand a finite number of terms. It is *not* an infinite series.

For $E \notin \sigma(H)$, define

$$\beta_E(j) = \lim_{n\to\infty} -\frac{\log\langle G(E,0,nj)\rangle_g}{n}, \tag{1.18}$$

if the limit in the RHS exists. Using the FKG inequality [FKG, Li] as stated in Remark 3.2 on p. 241-242 of [Sz1] and subadditivity ergodic theorem, it can be shown, similar to [Sz1, Z] that for all $E \notin \sigma(H)$, the limit in (1.18) exists and moreover by patching of limits:

$$\lim_{j\to\infty} \frac{|\beta_E(j) + \log\langle G(E,0,j)\rangle|}{||j||} = 0. \tag{1.19}$$

The $\beta_E(j)$ are called the annealed directional Lyapunov exponents. They are in some sense, natural generalizations of the traditional Lyapunov exponent to higher dimensions. For a fixed E, β_E defines a norm on $\mathbf{R}^d$ [Sz1, Z]. One can similarly define α_E, the directional Lyapunov exponents in the almost sure case, which we will not address in this paper. α_E are called the quenched Lyapunov exponents. For more detailed statements, see Theorem 3.4, p. 244 of [Sz1] and Theorem A of [Z]. (We remark here that for $E \in I_\delta$, (1.19) can also be proved by using the constructions used in the proof of the Theorem, which we will describe in subsect. 3.)

Remark. There also appears to be some connection between the Lyapunov exponents defined here and the Lyapunov exponents of d-dimensional lattices of coupled, non-linear oscillators. See [EW1, EW2] and references therein for a discussion and some confirmation of this connection.

Define

$$D_{|\widetilde{E}|} = \{y \in R^{d^*} | \sum_{i=1}^{d} \cosh y_i < |\widetilde{E}|\}, \tag{1.20}$$

where $\widetilde{E}$ is as in (1.13). $D_{|\widetilde{E}|}$ is a convex set. We then obtain as a direct consequence of the Theorem

Corollary 2 *For all $\delta > 0$, there exists $\gamma_0 > 0$, such that for all $0 < \gamma < \gamma_0$, all $E \in I_\delta$,*

$$\beta_E(j) = \sup_{y\in \bar{D}_{|\tilde{E}|}} y \cdot j + \mathbf{O}(\gamma^4)||j||, \tag{1.21}$$

where $\tilde{E}$ is as in (1.13). (Note that $||y|| = \mathbf{O}(1)$.) Let

$$\hat{j} = \frac{j}{||j||} \in S^{d-1}. \tag{1.22}$$

$\beta_E(\hat{j})$ extends to an analytic function in E for E such that $\Re E \in I_\delta$.

The above result is new for $d > 1$. For $d = 1$, there are similar results in the continuum by using probabilistic methods [Fr, Po, Sz2]. The case $d = 1$ is special, as the (random walk) process is additive. We use lattice field theory (see subsect. 3) to prove the Theorem; the Corollary is a straight forward consequence.

Proof of the Corollary. (1.21) follows directly from the Theorem and standard convex analysis as in [Sj1], so we do not repeat it here. (For a general reference on convex analysis, see e.g., [H].) Analyticity in E ($\Re E \in I_\delta$) for $\beta_E(\hat{j})$ is a direct consequence of the above Theorem, (1.21) and the Cauchy Theorem. □

Remark. We note from the Corollary that, to $\mathbf{O}(\gamma^4)$ the Lyapunov exponent is the support function of a convex set corresponding to the effective convolution matrix: $\widetilde{E}-\Delta$. Hopefully it will become clear after the proof of the Theorem that to *all orders* in γ, the Lyapunov exponent is the support function of a convex set corresponding to an effective convolution matrix by iterating the procedure used here. (cf. [Sj2] for a related situation.) Starting from $\mathbf{O}(\gamma^4)$, the off-diagonal elements of the effective matrix will be different from Δ, i.e., there will be corrections to the generator itself.

2. The probabilistic aspect.

Before explaining the proof of the Theorem, we first take a brief detour to elaborate on the probabilistic content of the problem. The proof of the Theorem and its Corollary are purely analytical, so they could be understood independently. However in order to put the results here in a more general context, it is the probabilitic point of view which is most natural and most useful. This is not surprising as the problem originates there.

Definition of random walks in random potentials.
Define P on $\mathbf{Z}^d \times \mathbf{Z}^d$ by

$$P(i,j) = \frac{\Delta_{ij}}{2d}, \tag{2.1}$$

for all $(i,j) \in \mathbf{Z}^d \times \mathbf{Z}^d$, where Δ_{ij} is as defined in (1.2). We have from (1.2) that

$$P(i,j) = P(0, i-j), \quad P(0,i) \geq 0, \quad \sum_{i \in \mathbf{Z}^d} P(0,i) = 1, \tag{2.2}$$

i.e., $P(i,j)$ defines the transition probability of a (simple) random walk (see e.g., [Spi]).

Using this point of view, the Green's function for Δ (i.e., the free Green's function) has the following path representation (see e.g., p. 169 of [La]):

$$G_0(E,\mu,\nu) = (E-\Delta)^{-1}(\mu,\nu) = \sum_{w:\,\mu\to\nu} E^{-|w|}, \qquad E > 2d, \tag{2.3}$$

where the sum is over all walks w from μ to ν and $|w|$ is the length (i.e., total number of steps) of the walk w. Similarly the Green's function for H defined earlier in (1.4) has the representation (see e.g., (1) of [Z]):

$$
\begin{aligned}
G(E,\mu,\nu) &= (E-H)^{-1}(\mu,\nu) \\
&= \sum_{n=0}^{\infty} \sum_{w:\,\mu\to\nu;\,|w|=n} \prod_{j=0}^{n} (E-\gamma V(w_j))^{-1}, \qquad E>2d,
\end{aligned}
\tag{2.4}
$$

where w_j is the position of the walk w after j steps. Note that due to the discrete time formulation, (2.3) and (2.4) are slightly different from the usual Feynmann-Kac formula. Clearly (2.4) reduces to (2.3) when $V=0$, and can be seen as defining a weighted (by V) path measure. (In fact, (2.3), (2.4) are, in analytical language, the resolvent (or Neumann) series about E or $E-\gamma V$ written in a slightly different way.) We say that H defines a random walk in the random potential γV.

Remark. The book *Brownian Motion, Obstacles and Random Media* by A. S. Sznitman [Sz1] provides an excellent in-depth reference on the subject. There is also a Bourbaki seminar [Kom] by Komorowski centered on the work of Sznitman, which can serve as a short introduction to the subject. The method that we use to prove the Theorem is substantially different–it is purely analytical. However much of our understanding of the subject comes from reading the book, which we frequently refer to in this section. Also when we make references to works of Sznitman covered in the book [Sz1], we do not in general trace back to the original reference.

For a given realization of the potential V, $G(E,\mu,\nu)$ is the probability of finding the random walk at site ν conditioned that it starts at site μ. (Recall that the Green's function is the integral over all time t of the heat kernel.) For more details, see e.g.,[La, MS, Spi]. It follows then $\langle G(E,\mu,\nu)\rangle_g$ is the *expected* (with respect to the random potential V) probability of finding the random walk at site ν conditioned that it starts at site μ. In the limit $|\mu-\nu|\to\infty$, $\langle G(E,\mu,\nu)\rangle_g$ is also essentially the normalization constant for the measure defined as the tensor product of the probability measure $P=\prod_{i\in\mathbf{Z}^d} dg(v_i)$ with the path measure for H in (2.4) (see (7), p. XI and (2.7), p. 323 of[Sz1]).

Comments on the Theorem.

We believe quantitative results like (1.12, 1.13) were not know before for random walks in (time-independent) random potentials in $d>1$. If one considers random walks in time-*dependent* random potentials (cf. [?, IS]) , which is a type of the so called *directed* random walks, then Sinai in [Sin] obtained a result which is similar to the above Theorem. (See also (2.22) on p. 327 of [Sz1].) The term directed refers to the fact that the walk is parametrized by time and that the graph of the walk in $\mathbf{Z}\times\mathbf{Z}^{d-1}$ moves at a constant rate in the time direction.

The main and also the crucial difference between the directed case and our (non-directed) case is that in the directed case, one can always, in some sense, reduce to a transfer matrix type of situation, as there is a prefered direction; while in our case, the *non*-directed case, one cannot avoid treating walks that return (loops or self-intersections), even though they are unlikely due to the condition $E\notin\sigma(H)$. (See (2.4).)

In spite of the difference stressed above, for high dimensions and potentials which have small variations ($\gamma << 1$), the result for the correlation function in the non-directed case (1.12) is similar to that obtained in the directed case in [Sin]. This partially proves a commonly held belief (see e.g., p.327-328 of [Sz1]).

In the mathematical physics literature, there are results which are similar to ours, e.g., [CC, P-L, Sch]. We mention in particular the result obtained by Chayes and Chayes in [CC] for self-avoiding walks (walks with no self-intersections), as the setting is sufficiently close to ours. (For a precise definition and general references, see [La, MS].) In [CC], the correlation function is defined to be:

$$G(\beta, \mu, \nu) = \sum_{\omega:\, \mu \to \nu} e^{-\beta|\omega|}, \qquad \beta \in [\beta_c, \infty), \tag{2.5}$$

where the sum is over all *self-avoiding* walks ω from μ to ν, $|\omega|$ is the length (i.e., total steps) of ω, β_c is the critical parameter beyond which the sum in (2.5) converges.

Clearly G defined in (2.5) is translationally invariant. So we may take $\nu = 0$. In [CC], for $\nu \to \infty$ in a conic neighborhood of a given direction, which apparently without loss of generality may be chosen to be one of the axis in $\mathbf{Z}^d$, it is shown that for all $\beta < \beta_c$, the correlation function for walks that are conditioned so that their projections onto the axis of the cone contains only points of self-intersections (for more precise definition, see [CC]) decays *strictly* faster than the correlation function for the unconditioned walk. (Questions of uniformity with respect to different directions do not seem to be addressed explicitly in the paper.) From this they deduce the equality of the upper and lower bound of the rate of decay for the correlation function, although no explicit expression is given for this rate.

The decay rate of the walk is customarily called *the gap*; the decay rate of the conditioned walk is called *the upper gap.* The key result of [CC] can therefore be rephrased as stating that the upper gap is strictly greater than the gap "in a cone".

We will come back to this question of gap and upper gap later in subsect. 3, when we describe our method, which reduces the problem to the study of the spectral theory of a self-adjoint operator, the so called Witten Laplacian. As we will see, our construction is *uniform* in all directions. Gap and upper gap appear naturally in this construction. We only preview now that the fact that we have an asymptotics for $\log\langle G(E, \mu, \nu)\rangle_g$ uniformly in $|\mu - \nu|$ as in (1.12), and not just an estimate is a direct consequence of our precise control not only over the gap, but also over the upper gap, or more precisely the spectrum beyond the gap.

Comments on the Corollary.

As mentioned earlier, the results in the Corollary are not known before for $d > 1$. For $d = 1$, in the continuum, there are some results due to Friedlin, Povel and Sznitman [Fr, Po, Sz2], which are summarized on p. 233,325 of [Sz1]. The case $d = 1$ is special as the Brownian motion or the random walk process is additive. The above Theorem and its Corollary confirm a few conjectures raised in the past in the annealed (averaged) case. For a sample of such conjectures in the quenched (almost sure) case, see p. 219, 233 of [Sz1].

In [Sz2], Sznitman proved that in $d = 1$ and for Brownian motions in Poissonian potentials, i.e., for

$$H = -\frac{d^2}{dx^2} + V,$$

where $V \geq 0$ is a random potential with Poisson distributions, the quenched Lyapunov exponent $\alpha_E(1)$ is analytic in E for E such that $\Re E > 0$. He used an explicit formula obtained in [Fr] to derive the result. The analyticity of $\beta_E(\hat{j})$ proved in the above Corollary extends analyticity results to the annealed case in $\mathbf{Z}^d$ for all $d \geq 1$, when $\gamma << 1$ and for E such that $\Re E \in I_\delta$.

However, the analyticity of $\beta_E(\hat{j})$ for E such that $\Re E \in I_\delta$, as it stands (contrary to the almost sure case) does not yet contain much information. This is because for E complex, $G(E, 0, j)$ is in general complex. A priori, the signs of its real and imaginary part depend on V. However, since there is good reason [Spe] and also p. 326 of [Sz1] to believe that $\alpha_E = \beta_E$ almost surely for $d > 2$, the Corollary would then imply that $\alpha_E(\hat{j})$ is analytic for $d > 2$, almost surely, which certainly carries interesting informations.

In the case of Brownian motions in Poissonian potentials, Sznitman [Sz1] derived upper and lower bounds for α_E and β_E. In the case of random walks in random potentials (same setting as in this paper), Zerner [Z] derived upper and lower bounds for α_E. Our aim is actually to *compute* the *asymptotics* of β_E as $\gamma \searrow 0$. Our method is constructive.

Due to the lack of rotational symmetry in $\mathbf{Z}^d$, the directional dependence of the Lyapunov exponents are more complicated than in $\mathbf{R}^d$. (In $\mathbf{R}^d$, it is known to be proportional to the Euclidean norm, although the constant of proportionality is not known, see p. 219 of [Sz1].) In particular, the Corollary shows that the unit ball in the norm defined by β_E is *not* rotationally invariant. In [Z], when V is a *constant*, i.e., H is a convolution matrix, Zerner obtained a closed expression for the Lyapunov exponents. Moreover for $d = 2$, using this expression, Zerner numerically computed the Lyapunov exponents in the case where the potential V is a *constant.* Combining the Corollary with this numerical result, we see explicitly that for $d = 2$, the unit ball in the norm given by β_E approaches instead the shape of a diamond.

We mention another application of the Corollary to random walks in random potentials with a constant drift $h \in \mathbf{R}^d$. More precisely, we define the first order difference operator ∇ componentwise as

$$(\nabla_{e_\alpha} u)(n) = u(n + e_\alpha) - u(n),$$

where $e_\alpha \in \mathbf{Z}^d$, $e_\alpha(\beta) = \delta_{\alpha\beta}$, α, $\beta = 1, \cdots, d$, and we replace the generator Δ by $\Delta + \sum_{\alpha=1}^d h_\alpha \nabla_{e_\alpha}$. We define the dual norm β_E^* as

$$\beta_E^*(\ell) = \sup_{x \neq 0} \{ \frac{\ell \cdot x}{\beta_E(x)} \}, \qquad \ell \in \mathbf{R}^d, \quad E \notin \sigma(H).$$

The unit ball in the norm β_E^* is the critical unit ball [Sz1]: if h is such that $\beta_E^*(h) > 1$, then the motion is ballistic; if h is such that $\beta^*(h) < 1$, then the motion is sub-ballistic. The Corollary then allows us to compute the critical drifts h_c, which are directional dependent, for $E \in I_\delta$.

3. The constructions toward the proof of the Theorem.

We now describe briefly the construction of the proof of the Theorem. As usual we use finite dimensional approximations and take Λ to be a finite set in $\mathbf{Z}^d$ and define H_Λ to be the restricted operator with appropriate boundary conditions. For concreteness, assume supp$dg \subseteq (-\infty, 0]$ and $E > 2d$. (The other case works in exactly the same way.) Our starting point as in [SW1] is the following Gaussian integral representation of $G_\Lambda(E, \mu, \nu)$:

$$G_\Lambda(E, \mu, \nu) = \int x_\mu \cdot x_\nu [\det(E - H_\Lambda)]^{-1} e^{-\sum_{i,j}(E-H_\Lambda)_{ij} x_i \cdot x_j} \prod_{i \in \Lambda} \frac{d^2 x_i}{\pi}, \tag{3.1}$$

where $x_i \in \mathbf{R}^2$ and $\cdot$ is the usual scalar product in $\mathbf{R}^2$. Using Grassmann variables (see sect. II), the determinant can be further expressed as a Gaussian integral (of Grassmann variables). (This is first used in this context in [BCKP, KS].) This is the so called supersymmetry. Using this representation, we can explicitly take the expectation value of $G_\Lambda(E, \mu, \nu)$ with respect to dg. Let

$$\hat{g}(t) = \int e^{tv} dg(v) \tag{3.2}$$

be the Laplace transform of dg. Since $\hat{g}(t) > 0$, we define

$$k(t) = \log \hat{g}(t).$$

We obtain (For more details, see [Wa].)

$$\langle G_\Lambda(E, \mu, \nu) \rangle_g = \int x_\mu \cdot x_\nu e^{-2\phi(x)} \prod_{i \in \Lambda} \frac{d^2 x_i}{\pi}, \tag{3.3}$$

where

$$\begin{aligned} \phi(x) \;=\; & \tfrac{1}{2}[\textstyle\sum_{i,j}(E - \Delta_\Lambda)_{ij} x_i \cdot x_j - \sum_{j \in \Lambda} k(\gamma x_j \cdot x_j) \\ & - \log\det(E - \Delta_\Lambda - \gamma \operatorname{diag} k'(\gamma x_j \cdot x_j))]. \end{aligned} \tag{3.4}$$

We see that the above supersymmetric-Gaussian transform has mapped the averaged correlation function of a random walk in random potentials to a correlation function of statistical mechanics. The measure $\prod_{i \in \Lambda} dg(v_i)$ on $\mathbf{R}^\Lambda$ has been transformed to $e^{-2\phi}$ on $(\mathbf{R}^2)^\Lambda \overset{\text{def}}{=} \mathbf{R}^{2\Lambda}$. From now on we denote by $\langle\,\rangle$ without subscript, the expectation with respect to the measure $e^{-2\phi}$.

Remark. In the physics literature (see e.g., [Ef]), supersymmetry has always played an important role in the study of disordered systems. In [BCKP, KS], supersymmetry was incorporated in a mathematically rigorous manner for the first time. One of the main differences between the present paper (also the earlier paper [SW1]) with [BCKP, KS], is that the Grassmann variables are further integrated over (see (3.3), (3.4)), so that "conventional" analysis becomes feasible.

By a now well known integration by parts initiated in [HS1, HS2, Sj1], which for completeness we rederive in sect. II, we have further (formally)

$$\langle G_\Lambda(E,\mu,\nu)\rangle_g = \sum_{i=1}^{2}\int (e^{-\phi}dx_\mu^{(i)}, (\Delta_\phi^{(1)})^{-1}(e^{-\phi}dx_\nu^{(i)})), \tag{3.5}$$

where

$$\begin{aligned} \Delta_\phi^{(1)} &= \Delta_\phi^{(0)}\otimes I + 2\phi'' && \text{on } \mathbf{C}_0^\infty(\mathbf{R}^{2\Lambda};\wedge\mathbf{R}^{2\Lambda}) \\ \Delta_\phi^{(0)} &= \sum z_j^* z_j, && \text{on } \mathbf{C}_0^\infty(\mathbf{R}^{2\Lambda}), \end{aligned} \tag{3.6}$$

where $z_j = \frac{\partial}{\partial x_j} + \frac{\partial\phi}{\partial x_j}$, $z_j^* = -\frac{\partial}{\partial x_j} + \frac{\partial\phi}{\partial x_j}$, z_j^* is the formal adjoint of z_j, we have identified 1-forms with functions with values in $\mathbf{R}^{2\Lambda}$ and $(\,,\,)$ denotes the inner product on $L^2(\mathbf{R}^{2\Lambda};\wedge\mathbf{R}^{2\Lambda})$. (For conditions on ϕ, domaines of $\Delta_\phi^{(0)}$, $\Delta_\phi^{(1)}$ etc., see [Wa]. It suffices to mention here that under conditions (H1-3) on g, $\Delta_\phi^{(0)}$, $\Delta_\phi^{(1)}$ have self-adjoint extensions.) $\Delta_\phi^{(0)}$, $\Delta_\phi^{(1)}$ are respectively the Witten Laplacians on 0, 1-forms. Note that when ϕ is quadratic, $\Delta_\phi^{(0)}$ is just the usual harmonic oscillator on $L^2(\mathbf{R}^{2\Lambda})$. More generally

$$\begin{aligned} &\Delta_\phi = d_\phi^* d_\phi + d_\phi d_\phi^*, \\ &d_\phi = e^{-\phi} d e^{\phi} = \sum_j z_j dx_j^\wedge, \\ &d_\phi^* = e^{\phi} d^* e^{-\phi} = \sum_j z_j^* dx_j^\rfloor, \end{aligned} \tag{3.7}$$

where d is the usual exterior differential, d^* its formal adjoint (and consequently d_ϕ^* is the formal adjoint of d_ϕ).

The spectra of $\Delta_\phi^{(0)}$, $\Delta_\phi^{(1)}$ play a crucial role in our construction. For now it suffices to mention that $\Delta_\phi^{(0)} \geq 0$ and that $\Delta_\phi^{(0)}$ has 0 as an eigenvalue of multiplicity 1 with $e^{-\phi}$ the unique eigenfunction. If ϕ is strictly convex, i.e., $\phi'' > c > 0$ *uniformly* in Λ as an operator, which is the case for $E \in I_\delta$, $(\delta > 0)$, we obtain that $\Delta_\phi^{(1)} \geq c > 0$ *uniformly* in Λ. This is the so called spectral gap, which is responsable for the exponential decay in the Theorem. Hence to obtain asymptotics for $\langle G_\Lambda\rangle$, we need to compute this spectral gap as an asymptotic series in γ. This requires precise control over the spectrum *beyond* the gap. The main difficulty here is that we need estimates which are uniform in Λ, so that we can pass to the limit $\Lambda \nearrow \mathbf{Z}^d$. This is achieved in sects. IV, V by using appropriate weighted spaces. We also note that for $E \notin \sigma(H)$,

$$\lim_{\Lambda\nearrow\mathbf{Z}^d} \langle G_\Lambda(E,\mu,\nu)\rangle_g = \langle G(E,\mu,\nu)\rangle_g$$

exists a priori by resolvent series and spectral theory, so we only need to ensure that we have uniform estimates in this paper.

We use a Grushin problem to reduce the study of $\Delta_\phi^{(1)}$ near the lower part of its spectrum to that of an effective operator. From the representation in (3.5), we are only interested in $(\Delta_\phi^{(1)}-z)^{-1}$ for $z=0$. We show that to order $\mathbf{O}(\gamma^4)$ in appropriate weighted spaces, this effective operator is just $\langle 2\phi''\rangle = \int e^{-2\phi}2\phi''$.

Sketchily, this is accomplished as follows. Let $\bar\Lambda = \Lambda \times \{1,2\}$ which we identify with $\{1,2,\cdots,|\Lambda|,\cdots,2|\Lambda|\}$. We first use the set of orthonormal 1-forms $A_1 = \{e^{-\phi}dx_j, j\in\bar\Lambda\}$ to pose the first Grushin problem for $\Delta_\phi^{(1)}$. (Recall that $e^{-\phi}$ is the unique eigenfunction of $\Delta_\phi^{(0)}$ with eigenvalue 0.) Consequently, we obtain that the effective operator is $\langle 2\phi''\rangle$ modulo $\mathbf{O}(\gamma^2)$, valid for a certain spectral interval I_1 at 0. (See [Wa].) This step is similar to that of [Sj1] (see also [BJS]) in the statistical mechanics context, where the effective operator is constructed to order $\mathbf{O}(h^{3/2})$, where h is the semi-classical parameter. The order $\mathbf{O}(h^{3/2})$ there approximately corresponds to the order $\mathbf{O}(\gamma^2)$ here.

However as it is clear from (1.12,1.13) and as we explained earlier, the terms which reveal the structure of the problem start at $\mathbf{O}(\gamma^2)$. So we need to carry the Grushin constructions a bit further. To $\mathbf{O}(\gamma^4)$, this is accomplished by enlarging the set of orthonormal forms to

$$A_3 = \{e^{-\phi}dx_j, j\in\bar\Lambda\}\cup\{z_k^*e^{-\phi}dx_j,\ z_k^*z_\ell^*e^{-\phi}dx_j,\ j,k,\ell\in\bar\Lambda,\ k\le\ell\}^\dagger,$$

where $\{\quad\}^\dagger$ denotes the set of orthonormal 1-forms obtained from $\{\quad\}$ by orthonormalization. These two sets are of the same dimension here. (The restriction $k\le\ell$ is due to the commutativity of the z_j^* $(j\in\bar\Lambda)$, defined in (3.6).)

More precisely we show that the well-posedness of the first Grushin problem for $\Delta_\phi^{(1)}$ using $\{e^{-\phi}dx_j, j\in\bar\Lambda\}$, implies the well-posedness of a Grushin problem for $\Delta_\phi^{(0)}$ using

$$\{e^{-\phi}\}\cup\{z_k^*e^{-\phi},\ k\in\bar\Lambda\}^\dagger.$$

So we obtain an effective operator for $\Delta_\phi^{(0)}$ valid in the same interval I_1. (See [Wa] .) Here we used in a crucial way that d_ϕ is a complex and therefore the spectrum of $\Delta_\phi^{(1)}$ and $\Delta_\phi^{(0)}$ are related.

Using the first equation of (3.6) and $\phi''>c>0$, we obtain an effective operator for $\Delta_\phi^{(1)}$ valid in a *larger* spectral interval $I_2\supset I_1$ by using

$$A_2 = \{e^{-\phi}dx_j, j\in\bar\Lambda\}\cup\{z_k^*e^{-\phi}dx_j,\ j,k\in\bar\Lambda\}^\dagger.$$

(See Proposition 4.7.)

Iterating this once more (See [Wa].), we obtain an effective operator for $\Delta_\phi^{(1)}$ modulo $\mathbf{O}(\gamma^4)$ when restricted to the subspace spanned by A_1, in an even larger interval $I_3\supset I_2\supset I_1$ by using A_3. (Although we do not pursue it in this paper, it is clear to us that this procedure can in fact be iterated to all orders.)

We stress that since we are interested in the asymptotics as $|\mu-\nu|\to\infty$, it is crucial that I_3 is large enough, as this translates into estimates which are in appropriate weighted spaces. The Theorem is then obtained by computing explicitly $\langle 2\phi''\rangle^{-1}$. The Corollary follows by using the Theorem and standard results on the inverse of convolution matrices.

We note also that for γ small, ϕ is close to a quadratic form, and $e^{-2\phi}$ is close to a Gaussian. So it should not be surprising that the set of functions

$$\{e^{-\phi},\ z_k^*e^{-\phi},\ z_k^*z_\ell^*e^{-\phi},\cdots k,\ell\in\bar\Lambda,\ k\le\ell,\cdots\}$$

play a special role here. This is because when $\gamma = 0$, this set of functions after proper orthonormalization is just a set of Hermite functions. What we did can therefore be seen as a perturbation theory around an "infinite" dimensional Gaussian or harmonic oscillator. In other words, what we are using is a Witten complex formulation of Euclidean lattice field theory, whose foundation was laid down by Sjöstrand in his seminal paper [Sj1].

After we finished the paper, we realized that the connections between random walks and lattice field theory are in fact well known, see e.g., [FFS], in particular [BF1, BF2], where they use the term oscillation modes, which roughly corresponds to the eigenfunctions of $\Delta_\phi^{(1)}$ in our vocabulary. The novelty here is that we have an *explicit* transformation of random walks in random potentials to a specific operator: $\Delta_\phi^{(1)}$, where ϕ is determined by the random walk and the random potential as in (3.4). $\Delta_\phi^{(1)}$ can in turn be interpreted as giving rise to a field theory. It is due to this equality in (3.5) that we are able to *compute* various quantities.

Finally, we like to add that we also hope to address the almost sure (quenched) behaviour of G by similar constructions in the future. As mentioned earlier, it is conjectured that $\alpha_E = \beta_E$ a.s., for $d > 2$. If it is true, then the Lyapunov exponents obtained here are also the Lyapunov exponents in the quenched case for $d > 2$.

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