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Abstract

We study the theory of scattering for the Hartree equation with long range potentials. We prove the existence of modified wave operators with no size restriction on the data and we determine the asymptotic behaviour in time of solutions in the range of the wave operators.

In this lecture, we report on some recent work [5] [6] on the theory of scattering for the Hartree equation

$$i\partial_t u + \frac{1}{2}\Delta u = (V \star |u|^2)u \tag{1}$$

where u is a complex valued function defined in space time $I\!R^{n+1}$, Δ is the Laplacian in $I\!R^n$, V is a real valued even function defined in $I\!R^n$ and $\star$ denotes the convolution in space. The stationary version of (1) has been introduced in the late twenties as a simplified model for complex atoms. An improved version taking into account the Pauli principle, namely the Hartree-Fock equation, has been introduced in the early thirties and is still used currently in the theory of nuclear collisions. If V in (1) is replaced by a delta function, the equation becomes the cubic nonlinear Schrödinger equation so that (1) can also be regarded as a regularized version of the latter.

We restrict our attention here to potentials V of the form

$$V(x) = \lambda |x|^{-\gamma} \tag{2}$$

for some $\gamma > 0$, although the results presented below extend to time dependent V of the type

$$V(x) = \lambda\, t^{\mu-\gamma}\, |x|^{-\mu} \tag{3}$$

for suitable μ with $0 < \mu < n$.

The Cauchy problem for the equation (1) is known to be globally well posed in H^1 under assumptions on V which in the special case (2) reduce to $0 < \gamma < \mathrm{Min}(4, n)$,

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and in addition $\gamma < 2$ for negative λ [1] [4]. It is then a natural question to study the asymptotic behaviour in time of the global solutions, and that problem can be addressed through the theory of scattering. One of the basic problems in that theory is that of classifying the previous asymptotic behaviours by relating them to a set of model functions $\mathcal{V} = \{v = v(u_+)\}$ parametrized by some data u_+ and with suitably chosen and preferably simple asymptotic behaviour in time. For each v in $\mathcal{V}$ one looks for a solution u of (1) such that $u(t)$ behaves as $v(t)$ when $t \to +\infty$ in a suitable sense. If $v(t)$ can be parametrized by its value $v(0)$ at $t = 0$, one can then define the wave operator for positive time as the map $\Omega_+ : v(0) \to u(0)$. A similar question can be considered for $t \to -\infty$. We restrict our attention to positive time.

In the same way as in linear scattering theory, two different situations arise. If V decays sufficiently fast at infinity, the previous construction can be performed by taking for $\mathcal{V}$ the set of solutions of the free Schrödinger equation, namely

$$v(t) = U(t)\ u_+ \equiv \exp\left(i\ \frac{t}{2}\ \Delta\right) u_+ \quad . \tag{4}$$

This is the short range case, corresponding to $\gamma > 1$ in (2). Existence of the wave operators for (1) in that case is proved in [11].

If V decays too slowly at infinity, namely for $0 < \gamma \leq 1$, the previous (ordinary) wave operators are known not to exist. In fact, one can prove the following result [9].

Proposition 1. *Let $u \in \mathcal{C}(I\!R, L^2)$ be a solution of (1) with V given by (2) and $0 < \gamma \leq 1$, and let $u_+ \in L^2$ be such that*

$$\lim_{t\to\infty} \parallel u(t) - U(t)\ u_+ \parallel_2 = 0$$

Then $u_+ = 0$ and $u = 0$.

This is the long range case. In that case it is known in linear scattering theory that the asymptotic model functions v have to be modified by a suitable phase, thereby leading to the construction of modified wave operators. There is a vast literature on that subject for which we refer to [2] [10]. Here we want to perform the same construction for the equation (1), namely to construct modified wave operators for that equation.

The introduction of a phase in the model functions can be done in several ways. Let ϕ be a real function of space time. One can replace v in (4) by

$$v_1(t) = U(t) \exp\left[-i\phi(t, -i\nabla)\right] u_+ \quad . \tag{5}$$

This is the choice made in most of the literature on linear scattering.

A different choice, better suited for our purpose, has been proposed in [12] [13]. It uses the following structure of the group $U(t)$. In fact $U(t)$ can be represented as

$$U(t) = M(t)\ D(t)\ F\ M(t) \tag{6}$$

where $M(t)$ is the operator of multiplication by the function

$$M(t) = \exp\left[ix^2/2t\right] \quad , \tag{7}$$

$D(t)$ is the dilation operator defined by

$$(D(t)f)(x) = (it)^{-n/2}\, f(x/t) \tag{8}$$

and F is the Fourier transform. The fact that $M(t)$ tends strongly to $\mathbb{1}$ (for instance in L^2) when $t \to \infty$ suggests to replace v_1 by

$$\begin{aligned} v_2(t) &= M(t)\ D(t)\ F \exp[-i\phi(t,-i\nabla)]u_+ \\ &= M(t)\ D(t) \exp[-i\phi(t,x)]w_+ \\ &= \exp[-i\phi(t,x/t)]M(t)\ D(t)\ w_+ \end{aligned} \tag{9}$$

where $w_+ = Fu_+$.

In order to construct solutions u of (1) that are asymptotic to v_2, it is useful to represent those solutions as

$$u(t) = M(t)\ D(t) \exp[-i\ \varphi(t,x)]w(t) \tag{10}$$

and try to ensure that $w(t) \to w_+$ and that $\varphi(t)$ behaves as $\phi(t)$ as $t \to \infty$. The parametrization (10) has been introduced in [7] [8] and used there to prove the existence of global solutions with small initial data for the equation (1) in long range situations.

The evolution equation for (w, φ) is obtained by substituting (10) into (1). Using the fact that

$$D(t)^{-1} \left(V \star |u|^2\right) D(t) = t^{-\gamma}\ g_0(w,w) \tag{11}$$

where

$$g_0(w_1,w_2) = \lambda \ \mathrm{Re}\ |x|^{-\gamma} \star (w_1 \bar{w}_2) \tag{12}$$

we obtain

$$\left\{ i\partial_t + (2t^2)^{-1}\ \Delta - t^{-\gamma}\ g_0(w,w) \right\} \exp(-i\varphi) w = 0 \tag{13}$$

or equivalently

$$\begin{aligned} &\left\{ i\partial_t + \left(2t^2\right)^{-1} \Delta - i \left(2t^2\right)^{-1} (2\nabla\varphi\cdot\nabla + (\Delta\varphi)) \right\} w \\ &+ \left\{ \partial_t\varphi - \left(2t^2\right)^{-1}\ |\nabla\varphi|^2 - t^{-\gamma}\ g_0(w,w) \right\} w = 0 \quad . \end{aligned} \tag{14}$$

However, the parametrization (10) is redundant and we have only one evolution equation for two unknown functions. In the same way as in gauge theory, we arbitrarily add a second equation by taking the last bracket in (14) to be zero, thereby obtaining the following auxiliary system

$$\left\{ \begin{aligned} \partial_t w &= i\,(2t^2)^{-1}\ \Delta w + (2t^2)^{-1}\,(2\nabla\varphi\cdot\nabla w + (\Delta\varphi)w) \\ \partial_t \varphi &= (2t^2)^{-1}\ |\nabla\varphi|^2 + t^{-\gamma}\ g_0(w,w) \quad . \end{aligned} \right. \tag{15}$$

The construction of modified wave operators for the equation (1) will be performed by first constructing wave operators for the auxiliary system (15) and then recovering u from (10).

The first step consists in showing that the auxiliary system yields a well defined dynamics for large time. In fact, the Cauchy problem for (15) is well posed for large time in suitable spaces, with $w(t) = O(1)$ and $\varphi(t) = O(t^{1-\gamma})$ for large t (see Proposition 2 below). That asymptotic behaviour for large time is obviously compatible with (15).

The next step in the construction of the wave operators for (15) consists in choosing an asymptotic comparison dynamics. This is obtained by dropping all the terms that are integrable at infinity in time in the RHS of (15). For orientation we first consider the simple case where $1/2 < \gamma < 1$. In that case we obtain the asymptotic system

$$\begin{cases} \partial_t w_0 = 0 \\ \\ \partial_t \varphi_0 = t^{-\gamma} \ g_0(w_0, w_0) \end{cases} \tag{16}$$

which is immediately solved by

$$\begin{cases} w_0(t) = w_+ \\ \\ \varphi_0(t) = (1-\gamma)^{-1} \ (t^{1-\gamma} - 1) \ g_0(w_+, w_+) \end{cases} \tag{17}$$

with initial condition $\varphi_0(1) = 0$. One sees easily that (17) provides an adequate description of the asymptotic behaviour of solutions of (15). In fact, it follows from the first equation of (15) that $\partial_t w = O(t^{-1-\gamma})$ so that $w(t)$ has a limit w_+ as $t \to \infty$ and that

$$w(t) - w_+ = O\left(t^{-\gamma}\right) \quad . \tag{18}$$

Taking the difference of the second equations of (15) and (16), one obtains

$$\partial_t(\varphi - \varphi_0) = \left(2t^2\right)^{-1} \ |\nabla\varphi|^2 + t^{-\gamma} \ g_0(w - w_+, w + w_+) \tag{19}$$

the RHS of which is $O(t^{-2\gamma})$ and therefore integrable at infinity for $\gamma > 1/2$. Therefore $\varphi - \varphi_0$ has a limit ψ_+ as $t \to \infty$ and

$$\varphi(t) - \varphi_0(t) - \psi_+ = O\left(t^{1-2\gamma}\right) \tag{20}$$

(see Proposition 3 below).

The main step in the construction of the wave operator for the system (15) now consists in obtaining solutions of that system satisfying (18) (20) for given (w_+, ψ_+). The main difficulty consists in the fact that we want to solve the Cauchy problem for the system (15) with infinite initial time, with initial data for φ which blow up at that time. That difficulty shows up already when solving the Cauchy problem for the system (15) with finite initial time t_0 with the appropriate initial condition at t_0 (see Proposition 2). In that situation, the solution turns out to be defined only in an interval $[T, \infty)$ where $T = O(t_0^{1-\gamma})$ when $t_0 \to \infty$. In order to circumvent that difficulty, we proceed as follows. For given (w_+, ψ_+) we choose t_0 large and we construct the solution (w_{t_0}, φ_{t_0}) of (15) with initial condition $(w_+, \varphi_0(t_0) + \psi_+)$ at t_0. Estimating the difference $(w_{t_0} - w_+, \varphi_{t_0} - \varphi_0 - \psi_+)$, we show that the solution can be extended to a fixed interval $[T, \infty)$ independent of t_0. We can then take the limit of (w_{t_0}, φ_{t_0}) as $t_0 \to \infty$, thereby obtaining the required solution (w, φ) of (15) (see Proposition 4).

The extension of the previous construction to the general case $0 < \gamma \leq 1$ proceeds as follows. Let $p \geq 0$ be an integer. We expand w and φ as

$$\begin{cases} w = \sum\limits_{0 \leq m \leq p} w_m + q_{p+1} = W_p + q_{p+1} \\ \varphi = \sum\limits_{0 \leq m \leq p} \varphi_m + \psi_{p+1} = \phi_p + \psi_{p+1} \end{cases} \tag{21}$$

with the understanding that as $t \to \infty$

$$\begin{cases} w_m(t) = O\left(t^{-m\gamma}\right) \quad , \quad q_{p+1}(t) = o\left(t^{-p\gamma}\right) \\ \varphi_m(t) = O\left(t^{1-(m+1)\gamma}\right) \quad , \quad \psi_{p+1}(t) = o\left(t^{1-(p+1)\gamma}\right) \quad . \end{cases} \tag{22}$$

Substituting (21) into (15) and identifying the various powers of $t^{-\gamma}$ yields the system of equations

$$\begin{cases} \partial_t \, w_{m+1} = \left(2t^2\right)^{-1} \sum\limits_{0 \leq j \leq m} \left(2\nabla\varphi_j \cdot \nabla + (\Delta\varphi_j)\right) w_{m-j} \\ \partial_t \, \varphi_{m+1} = \left(2t^2\right)^{-1} \sum\limits_{0 \leq j \leq m} \nabla\varphi_j \cdot \nabla\varphi_{m-j} + t^{-\gamma} \sum\limits_{0 \leq j \leq m+1} g_0\left(w_j, w_{m+1-j}\right) \end{cases} \tag{23}$$

for $0 \leq m+1 \leq p$.

The system (23) is triangular and can be solved by successive integrations over time, with initial conditions

$$\begin{cases} w_0(\infty) = w_+ \quad , \quad w_m(\infty) = 0 \quad \text{for } m > 0 \\ \varphi_m(1) = 0 \quad \text{for } 0 \leq m \leq p \quad , \end{cases} \tag{24}$$

thereby reproducing the asymptotic behaviour (23) at least for $0 \leq m \leq p$ and $(p+1)\gamma < 1$. One can then show that for $(p+2)\gamma > 1$, one can construct solutions (w, φ) of (15) satisfying the asymptotic behaviour

$$\begin{cases} w(t) - W_p(t) = O\left(t^{-(p+1)\gamma}\right) \\ \varphi(t) - \phi_p(t) - \psi_+ = O\left(t^{1-(p+2)\gamma}\right) \end{cases} \tag{25}$$

(see Proposition 4).

The previous constructions enable us to define the preliminary wave operator for the system (15) as the map

$$\Omega_0 : (w_+, \psi_+) \to (w, \varphi) \tag{26}$$

where (w, φ) is the solution of (15) with asymptotic behaviour (25). Combining (26) with (10) yields a map

$$(w_+, \psi_+) \longrightarrow (w, \varphi) \longrightarrow u = M \; D \exp(-i\varphi) w$$

which is however unsatisfactory as a wave operator for φ for two reasons. Firstly it depends on too many variables, namely on (w_+, ψ_+), instead of only $w_+ = Fu_+$. Secondly it has no chance of being injective since the same u can be obtained from different pairs (w, φ). We now briefly discuss that problem. Two solutions (w, φ) and (w', φ') of the system (15) are said to be gauge equivalent if they give rise to the same u, namely if $\exp(-i\varphi)w = \exp(-i\varphi')w'$. Similarly, two pairs of asymptotic variables (w_+, ψ_+) and (w'_+, ψ'_+) are said to be gauge equivalent if $\exp(-i\psi_+)w_+ = \exp(-i\psi'_+)w'_+$. One can then show that gauge equivalent solutions of the system (15) have gauge equivalent asymptotic variables, and conversely that gauge equivalent pairs of asymptotic variables have gauge equivalent images under Ω_0. One then defines the wave operator for u as the map

$$\Omega : u_+ \longrightarrow (Fu_+, 0) \longrightarrow (w, \varphi) \longrightarrow u = M\ D\exp(-i\varphi)w \tag{27}$$

and that map is injective.

We now substantiate the previous heuristic discussion with mathematical statements. We first define the function spaces where to solve the system (15). We look for (w, φ) such that

$$\left(w, t^{\gamma-1}\varphi\right) \in (\mathcal{C} \cap L^\infty)\left(I, H^k \oplus Y^\ell\right) \tag{28}$$

for some interval $I = [T, \infty)$, where k and ℓ are positive integers, H^k is the usual Sobolev space, and

$$Y^\ell = L^\infty \cap \dot{H}^1_{r_0} \cap \dot{H}^{[n/2]+1} \cap \dot{H}^{\ell+2}$$

where $r_0 = \infty$ for n even, $r_0 = 2n$ for n odd, and $\dot{H}^\ell_r$ are the homogeneous Sobolev spaces. We shall use the notion $|\cdot|_k$ and $|\cdot|_\ell$ for the norms in H^k and Y^ℓ, ambiguity being eliminated by the context. Pairs of integers (k, ℓ) will be said to be admissible if

$$\begin{cases} k \leq \ell \quad , \quad \ell > n/2 \\ \ell + 2 + \gamma \leq \mathrm{Min}(n/2 + 2k, n + k) \\ k > n/2 \quad \text{if} \quad \ell + 2 + \gamma = n + k \\ k > 2 \quad \text{if} \quad \gamma = 1 \text{ and } n \text{ is even} \quad . \end{cases}$$

Admissible pairs exist only for $n \geq 3$, and our results hold only in that case. If (k, ℓ) is admissible, so is also $(k + j, \ell + j)$ for any positive integer j. For $n = 3$, $0 < \gamma \leq 1$, the pair (2,2) is admissible.

For simplicity, we state the results by giving the estimates in the form appropriate to the case where γ^{-1} is not an integer. If γ^{-1} is an integer, additional logarithms occur in the estimates. In all the subsequent results, we assume that $n \geq 3$ and $0 < \gamma \leq 1$.

We first state the results on the Cauchy problem for the system (15) with finite initial time.

Proposition 2. *Let (k,ℓ) be an admissible pair and let $(w_0,\widetilde{\varphi}_0) \in H^k \oplus Y^\ell$. Then there exists $T_0 = T_0(w_0,\widetilde{\varphi}_0)$ such that for all $t_0 \geq T_0$, the system (15) with initial data $w(t_0) = w_0$, $\varphi(t_0) = t_0^{1-\gamma}\widetilde{\varphi}_0$ has a unique solution (w,φ) such that*

$$\left(w, t^{\gamma-1}\varphi\right) \in (\mathcal{C} \cap L^\infty)\left([T,\infty), H^k \oplus Y^\ell\right) \tag{29}$$

for some $T \leq t_0$. One can take $T = C\ T_0^\gamma\ t_0^{1-\gamma}$ and the solution satisfies the estimate

$$|\varphi(t)|_\ell \leq C\,(\mathrm{Max}(t,t_0))^{1-\gamma} \quad . \tag{30}$$

Note that neither T nor the estimate (30) are uniform in t_0. The main ingredient in the proof of Proposition 2, as well as in those of the subsequent ones, consists of energy estimates. In the subsequent results, that method of proof will produce various losses of derivatives.

The next result is the existence of asymptotic variables (w_+,ψ_+) for the solutions of the system (15) obtained through Proposition 2.

Proposition 3. *Let (k,ℓ) be an admissible pair. Let p be an integer such that $(p+2)\gamma > 1$. Let (w,φ) be a solution of the system (15) such that*

$$\left(w, t^{\gamma-1}\varphi\right) \in (\mathcal{C} \cap L^\infty)\left([T,\infty), H^{k+\mathrm{Max}(p+1,2)} \oplus Y^{\ell+p}\right) \tag{31}$$

for some $T > 1$. Then

(1) The following limit exists

$$\lim_{t\to\infty} w(t) = w_+ \tag{32}$$

strongly in H^{k+p} and weakly in H^{k+p+1}.

(2) Let W_p and ϕ_p be defined by (21) in terms of the solutions of the system (23) with initial condition (24). Then the following limit exists

$$\lim_{t\to\infty} \varphi(t) - \phi_p(t) = \psi_+ \tag{33}$$

strongly in $Y^{\ell-1}$. Furthermore the following estimates hold

$$\begin{cases} |w(t) - W_p(t)|_{k-1} \leq C\ t^{-(p+1)\gamma} \\ \\ |\varphi(t) - \phi_p(t) - \psi_+|_{\ell-1} \leq C\ t^{1-(p+2)\gamma} \quad . \end{cases} \tag{34}$$

The main technical result is the existence of solutions (w,φ) of the system (15) with prescribed asymptotic variables (w_+,ψ_+).

Proposition 4. *Let (k,ℓ) be an admissible pair. Let p be an integer such that $(p+2)\gamma > 1$. Let $w_+ \in H^{k+\mathrm{Max}(p+1,2)}$ and define W_p and ϕ_p by (21) (23) (24) as in Proposition 3. Let $\psi_+ \in Y^{\ell+1}$. Then there exists $T = T(w_+,\psi_+)$ and there exists a unique solution (w,φ) of the system (15) such that*

$$\left(w, t^{\gamma-1}\varphi\right) \in (\mathcal{C} \cap L^\infty)\left([T,\infty), H^k \oplus Y^\ell\right) \qquad (35) \equiv (29)$$

and such that the estimates (34) hold.

With Proposition 4 available, one can define the auxiliary wave operator Ω_0 by (26) and the wave operator Ω by (27). In order to describe the asymptotic behaviour of solutions u of (1) in the range of Ω, we need additional notation. We define the operator

$$J(t) \equiv x + it\nabla = M(t)\ D(t)\ i\nabla\ D(t)^*\ M(t)^* \tag{36}$$

so that in view of (10)

$$J(t)^m\ u(t) = M(t)\ D(t)(i\nabla)^m \exp\left(-i\varphi(t)\right) w(t) \quad .$$

One sees easily that if (w, φ) satisfy (35), then $\exp(-i\varphi)w \in \mathcal{C}([T, \infty), H^k)$, so that u obtained through (10) from such a (w, φ) belongs to the space $\mathcal{X}^k([T, \infty))$ defined by

$$\mathcal{X}^k(I) = \left\{u; < J(t) >^k u \in \mathcal{C}(I, L^2)\right\} \tag{37}$$

where $< \cdot > = < 1 + |\cdot|^2 >^{1/2}$. In particular it follows from Proposition 4 that the wave operator Ω maps $FH^{k+\mathrm{Max}(p+1,2)}$ into $\mathcal{X}^k([T, \infty))$ for suitable k and T.

We finally collect the properties of solutions u of (1) in the range of Ω that follow from the previous results.

Proposition 5. *Let k be the first member of an admissible pair. Let p be an integer such that $(p + 2)\gamma > 1$. Let $u_+ \in FH^{k+\mathrm{Max}(p+1,2)}$. Then*

(1) There exists $T = T(u_+)$ and there exists a unique solution u of the equation (1) defined in $[T, \infty)$ which can be represented through (10) in terms of a solution (w, φ) of the system (15) obtained in Proposition 4 with $w_+ = Fu_+$, $\psi_+ = 0$. The solution u belongs to $\mathcal{X}^k([T, \infty))$.

(2) The map $\Omega : u_+ \to u$ is injective.

(3) u satisfies the estimate

$$\| < J(t) >^k \left\{\exp\left(i\phi_p(t, x/t)\right) u(t) - M(t)\ D(t)\ F\ u_+\right\} \|_2 \leq C\ t^{1-(p+2)\gamma} \quad . \tag{38}$$

(4) Let r satisfy $0 \leq \delta(r) \equiv n/2 - n/r \leq \mathrm{Min}(k, n/2)$, $\delta(r) < n/2$ if $k = n/2$. Then u satisfies the estimate

$$\| \ u(t) - \exp\left(-i\phi_p(t, x/t)\right) M(t)\ D(t)\ F\ u_+ \ \|_r \leq C\ t^{1-(p+2)\gamma-\delta(r)} \tag{39}$$

where $\| \cdot \|_r$ denotes the norm in $L^r(I\!R^n)$.

Remark. The asymptotic estimates (34) (38) (39) hold in the form stated here for simplicity only under the condition $(p+1)\gamma < 1$. They have to be suitably modified if that condition does not hold (see [6]).

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