Journées Équations aux dérivées partielles

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Journées Équations aux dérivées partielles (1999), p. 1-16

<http://www.numdam.org/item?id=JEDP_1999____A14_0>

Journées *Équations aux dérivées partielles*
Saint-Jean-de-Monts, 31 mai–4 juin 1999
GDR 1151 (CNRS)

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Abstract

The aim of this work is threefold. First we set up a calculus for partial differential operators with nonsmooth coefficients which is based on the FBI (Fourier-Bros-Iagolnitzer) transform. Then, using this calculus, we prove a weaker version of the Strichartz estimates for second order hyperbolic equations with nonsmooth coefficients. Finally, we apply these new Strichartz estimates to second order nonlinear hyperbolic equations and improve the local theory, i.e. prove local well-posedness for initial data which is less regular than the classical threshold.

1. Introduction.

The first goal of these notes is to introduce a new approach for the analysis of partial differential operators with nonsmooth coefficients, which is based on the FBI transform. The idea is quite simple, namely to use the FBI transform to transform the "principal" part of a partial differential equation into a scalar or a ordinary differential equation in the "FBI" space. Thus one needs to produce approximate conjugates of pseudodifferential operators with respect to the FBI transform. This in turn requires appropriate error estimates, which are described in the next section.

In the second part we show how this method can be used to obtain Strichartz type estimates for second order hyperbolic equations with nonsmooth coefficients. In the "FBI" space the problem reduces to a subelliptic ode away from the characteristic cone and a transport equation along the Hamilton flow. Solving these ode's reduces the problem nicely to certain oscillatory integral estimates which, at least in spirit, are not far from the classical ones arising in the constant coefficient case.

Finally, we explain how these Strichartz estimates lead to improvements in the local theory for nonlinear hyperbolic equations. This is a simple argument based on the energy estimates.

The results presented here are contained in three articles of the author, [16], [18] and [17].

Research partially supported by NSF grant DMS-9622942 and by an Alfred P. Sloan fellowship

2. A calculus for operators with nonsmooth coefficients.

The FBI transform is, in a way, similar to the complex Fourier transform, in that for each function in R^n it provides a representation as a holomorphic function in R^{2n}. However, in the case of the FBI transform we can identify naturally R^{2n} with the phase space T^*R^n. For a pseudodifferential operator with smooth symbol acting on functions in R^n one can produce by conjugation a corresponding formal series acting on functions in R^{2n}, for which the first term is exactly the multiplication by the symbol. This series converges and has a nice representation in the Weyl calculus provided that the symbol of the operator is analytic. This is how the FBI transform has been used in the study of partial differential operators with analytic coefficients; see [10], [11], where this machinery is developed. Here we do the opposite: we look at operators with nonsmooth coefficients, approximate the conjugated operator by a partial sum of the formal series, and then we prove error estimates.

The calculus we develop is dependent on the frequency; thus, in order to use it for general pseudodifferential operators one needs to start with a Paley-Littlewood decomposition and then use the calculus for each dyadic piece separately. The parameter λ below represents the size of the frequency.

The FBI transform of a temperate distribution f is a holomorphic function in C^n defined as

$$(T_\lambda f)(z) = \lambda^{\frac{3n}{4}} 2^{-\frac{n}{2}} \pi^{-\frac{3n}{4}} \int e^{-\frac{\lambda}{2}(z-y)^2} f(y)\, dy \tag{1}$$

To understand better how the FBI transform works, consider the L^2 normalized function

$$f_{x_0,\xi_0}(y) = \lambda^{\frac{n}{4}} \pi^{-\frac{n}{4}} e^{-\frac{\lambda}{2}(y-x_0)^2} e^{i\lambda(y-x_0)}$$

which is localized in a $\lambda^{-\frac{1}{2}}$ neighborhood of x_0 and frequency localized in a $\lambda^{\frac{1}{2}}$ neighborhood of $\lambda\xi_0$.[1] Then

$$(T_\lambda f)(z) = \lambda^{-\frac{n}{4}} \pi^{\frac{n}{4}} e^{\frac{\lambda}{4}(z-x_0+i\xi_0)^2 - \frac{\lambda}{2}(z-x_0)^2} = \lambda^{-\frac{n}{4}} \pi^{\frac{n}{4}} e^{-\frac{\lambda}{4}|z-x_0+i\xi_0|^2} e^{\frac{\lambda}{2}|\Im z|^2} e^{-i\frac{\lambda}{2}(\Re z - x_0)(\Im z - \xi_0)}$$

Modulo the common factor $e^{\frac{\lambda}{2}|\Im z|^2}$ this is localized in a $\lambda^{-\frac{1}{2}}$ neighborhood of $x_0 - i\xi_0$. Hence, it is natural to introduce the notation

$$z = x - i\xi.$$

Like the Fourier transform, the FBI transform has good L^2 properties. Set

$$\Phi(z) = e^{-\lambda\xi^2}$$

Then the operator T_λ is an isometry from $L^2(R^n)$ onto the closed subspace of holomorphic functions in $L^2_\Phi(C^n)$. One inversion formula is provided by the adjoint operator:

$$f(y) = \lambda^{\frac{3n}{4}} 2^{-\frac{n}{2}} \pi^{-\frac{3n}{4}} \int \Phi(z) e^{-\frac{\lambda}{2}(\bar{z}-y)^2} (T_\lambda f)(z)\, dx d\xi$$

This is of course not the only possible inversion formula since the range of T_λ consists only of holomorphic functions.

[1] Due to the uncertainty principle this is the best one can do when trying to localize in both space and frequency

Let $a(x,\xi)$ be a compactly supported symbol. Then

$$a_\lambda(x,\xi) = a(x, \frac{\xi}{\lambda})$$

is a symbol supported at frequency λ.

What we want is to determine an approximate conjugate $\tilde{A}_\lambda$ of $A_\lambda(y,D)$ with respect to F_λ,

$$T_\lambda A_\lambda(y,D) \approx \tilde{A}_\lambda T_\lambda$$

It is useful to see what happens for some simple symbols. For instance

$$T_\lambda(yf)(z) = (x + \frac{1}{-i\lambda}(\partial_\xi - \lambda\xi))T_\lambda f$$

The conjugate of $\frac{D}{\lambda}$ is of course $\frac{D_x}{\lambda}$, but we shall write it as

$$T_\lambda(\frac{D}{\lambda}f)(z) = (\xi + \frac{1}{\lambda}(\frac{1}{i}\partial_x - \lambda\xi))T_\lambda f$$

Based on this, one can use a Taylor series expansion of the symbol to produce the formal asymptotics

$$T_\lambda A_\lambda(x,D) \approx \sum_{\alpha,\beta}(\partial_\xi - \lambda\xi)^\alpha \frac{\partial_x^\alpha \partial_\xi^\beta a(x,\xi)}{\alpha!\beta!(-i\lambda)^{|\alpha|}\lambda^{|\beta|}}(\frac{1}{i}\partial_x - \lambda\xi)^\beta T_\lambda$$

Now we want to make these asymptotics rigorous for (nonsmooth) symbols a which are of class C^s with respect to x. Thus we define our candidate for the conjugate of A_λ with respect to T_λ to be the partial sum

$$\tilde{a}_\lambda^s = \sum_{|\alpha|+|\beta|<s} (\partial_\xi - \lambda\xi)^\alpha \frac{\partial_x^\alpha \partial_\xi^\beta a(x,\xi)}{\alpha!\beta!(-i\lambda)^{|\alpha|}\lambda^{|\beta|}}(\frac{1}{i}\partial_x - \lambda\xi)^\beta$$

Then we need to obtain good estimates for the remainder

$$R_{\lambda,a}^s = T_\lambda A_\lambda - \tilde{a}_\lambda^s T_\lambda$$

Our main result is

Theorem 1 *Assume that* $a \in C_x^s(C_0^\infty)$. *Then*

$$\|R_{\lambda,a}^s\|_{L^2 \to L^2_\phi} \leq c\lambda^{-\frac{s}{2}} \tag{2}$$

In other words, this theorem shows that the order s approximation is precise up to $s/2$ derivatives. Such an error estimate is sharp, as one can see from the following straightforward bounds on the terms in the partial sum:

Lemma 1 *If $u \in L^2_\phi$ is holomorphic then*

$$\|(\partial_\xi - \lambda\xi)^\alpha u\|_{L^2_\Phi} = c_\alpha \lambda^{\frac{|\alpha|}{2}} \|u\|_{L^2_\Phi}$$

$$\|(\partial_x - i\lambda\xi)^\alpha u\|_{L^2_\Phi} = c_\alpha \lambda^{\frac{|\alpha|}{2}} \|u\|_{L^2_\Phi}$$

If $s > 2$ then the conjugated operator has order 2 or higher, which makes the analysis more complicated. The two simpler cases are $0 \leq s \leq 1$, when

$$\tilde{a}^s_\lambda = a \qquad s \leq 1 \tag{3}$$

and $1 < s \leq 2$, when

$$\tilde{a}^s_\lambda = a + \frac{1}{-i\lambda} a_x(\partial_\xi - \lambda\xi) + \frac{1}{\lambda} a_\xi(\frac{1}{i}\partial_x - \lambda\xi)$$

Since we only consider this operator on holomorphic functions, the operators $(\partial_\xi - \lambda\xi)$ and $(\frac{1}{i}\partial_x - \lambda\xi)$ coincide. Then we can also rewrite it in a complex fashion as

$$\tilde{a}^s_\lambda = a + \frac{2}{\lambda}(\bar{\partial} a)(\partial - i\lambda\xi), \qquad 1 < s \leq 2 \tag{4}$$

Observe that there are two ways to match the factors in order to obtain real coefficients for the derivatives. Correspondingly we obtain two ode's along the gradient flow of a, generated by

$$a_x\partial_x + a_\xi\partial_\xi$$

respectively the Hamilton flow of a, generated by

$$a_x\partial_\xi - a_\xi\partial_x$$

3. Strichartz estimates for the wave equation with nonsmooth coefficients.

The Strichartz estimates are $L^p(L^q)$ estimates for solutions to the wave equation. These estimates have been very useful in the study of semilinear hyperbolic equations. One form of the estimates applies to solutions to the homogeneous wave equation,

$$\Box u = 0, \qquad u(0) = u_0, \quad u_t(0) = u_1$$

Then

$$\|u\|_{L^p(L^q)} \leq c\|u_0\|_{H^\rho} + \|u_1\|_{H^{\rho-1}} \tag{5}$$

provided that $2 \leq p \leq \infty$, $2 \leq q \leq \infty$ and

$$\frac{1}{p} + \frac{n}{q} = \frac{n}{2} - \rho, \qquad \frac{2}{p} + \frac{n-1}{q} \leq \frac{n-1}{2} \tag{6}$$

with the sole exception of the pair $(1, 2, \infty)$ in dimension $n = 3$.

In the sequel, call Strichartz pairs all the triplets (ρ, p, q) satisfying the above relations except for the forbidden endpoint $(1, 2, \infty)$ in dimension $n = 3$. If the equality holds in the second relation,

$$\frac{1}{p} + \frac{n}{q} = \frac{n}{2} - \rho, \quad \frac{2}{p} + \frac{n-1}{q} = \frac{n-1}{2}, \tag{7}$$

then we call (ρ, p, q) a sharp Strichartz pair. The estimates for any Strichartz pair follow by Sobolev embeddings from the estimates for sharp Strichartz pairs.

A special role is played in dimension $n \geq 4$ by the sharp Strichartz pair $(\frac{n+1}{2(n-1)}, 2, \frac{2(n-1)}{n-3})$ which we call the endpoint. Then all Strichartz estimates can be recovered from the endpoint estimate and the energy estimate (which corresponds to $(0, \infty, 2)$) by interpolation and Sobolev embeddings. The 3-dimensional correspondent is the forbidden endpoint $(1, 2, \infty)$.

The second form of the estimates applies to solutions to the inhomogeneous wave equation,

$$\Box u = f, \quad u(0) = 0, \quad u_t(0) = 0$$

Then

$$\|D^{1-\rho-\rho_1} u\|_{L^p(L^q)} \leq \|f\|_{L^{p_1'}(L^{q_1'})} \tag{8}$$

for all Strichartz pairs (ρ, p, q), (ρ_1, p_1, q_1).

Estimates of this type were first obtained in [3], [14]. Further references can be found in a more recent expository article [4]. The endpoint estimate was only recently proved in [6] $(n \geq 4)$.

Consider now a variable coefficient second order hyperbolic equation

$$P(x, D)u = 0, \qquad u(0) = u_0, \quad u_t(0) = u_1 \tag{9}$$

where

$$P(x, D) = -\partial_i g^{ij}(x) \partial_j$$

If the coefficients g^{ij} are smooth then the estimates hold locally, see [9] (except for the endpoint). For time independent $C^{1,1}$ coefficients, in dimension $n = 2, 3$, the estimates are proved in [12]. Furthermore, in [13] they are shown to fail for C^s coefficients, $s < 2$.

In what follows we assume that the matrices $(g^{ij}(x))$, $(g^{ij}(x))^{-1}$ are uniformly bounded and of signature $(1, n)$. Furthermore, we also assume that the surfaces $x_0 = const$ are space-like uniformly in x, i.e. that $g^{00} > c > 0$.

Our first result shows that the full Strichartz estimates hold provided that $D^2 g \in L^1(L^\infty)$.

Theorem 2 *Assume that $D^2 g \in L^1(L^\infty)$. Let (ρ, p, q) be a Strichartz pair. Then*

$$\||D|^{1-\rho} u\|_{L^p(0,T;L^q)} \leq \mu^{\frac{1}{p}} \|\nabla u\|_{L^\infty(L^2)} + \mu^{-\frac{1}{p'}} \|P(x, D)u\|_{L^1(L^2)}, \tag{10}$$

provided that $\mu \geq 1$ and

$$T\|D^2 g\|_{L^1(\infty)} \leq \mu^2$$

The estimates in Theorem 2 also lead to some weaker Strichartz estimates in the case when the coefficients have less regularity. Following the terminology in [19] define the microlocalizable scale of spaces $\mathcal{X}^s$ by

$$\|u\|_{X^s} = \sup_\lambda \lambda^s \|S_\lambda u\|_{L^1(L^\infty)}$$

where

$$1 = \sum_{\lambda=2^j} S_\lambda$$

is a standard Paley Littlewood decomposition. Then we consider operators with coefficients in the $\mathcal{X}^s$ spaces for $0 < s < 2$.

Theorem 3 *Assume that P is in divergence form and that $g \in \mathcal{X}^s$, $0 < s < 2$. Let (ρ, p, q) be a Strichartz pair and*

$$\sigma = \frac{2-s}{2+s}$$

Then

$$\||D|^{1-\rho-\frac{\sigma}{p}} u\|_{L^p(0,T;L^q)} \le \mu^{\frac{1}{p}} \|\nabla u\|_{L^\infty(L^2)} + \mu^{-\frac{1}{p'}} \||D|^{-\sigma} P u\|_{L^1(L^2)} \tag{11}$$

for all $\mu \ge 1$, $T > 0$ satisfying

$$T^s \|g\|^2_{\mathcal{X}^s} \le \mu^{2+s}$$

The estimates for solutions to the homogeneous equation follow easily from the above theorems combined with the energy estimates. Uniform energy estimates for a time T hold for instance if $1 \le s \le 2$ and

$$\|D_t g\|_{L^1(L^\infty)} \le 1$$

Now let us turn our attention to the estimates for the inhomogeneous problem. Our first result is a generalization of (10).

Theorem 4 *Assume that the coefficients satisfy $D^2 g \in L^1(L^\infty)$. Let (ρ, p, q) be a Strichartz pair. Then*

$$\||D|^{1-\rho} u\|_{L^p(L^q)} \le \mu^{\frac{1}{p}} \|\nabla u\|_{L^\infty(L^2)} + \mu^{-\frac{1}{p'}} \|f_1\|_{L^1(L^2)} + \||D|^{\rho} f_2\|_{L^{p'}(L^{q'})} \tag{12}$$

whenever

$$P(x, D) u = f_1 + f_2$$

and

$$T \|D^2 g\|_{L^1(L^\infty)} \le \mu^2, \qquad \mu \ge 1$$

The analogue of this result in the case when the coefficients are in $\mathcal{X}^s$, with $0 \le s \le 2$, is

Theorem 5 *Assume that P is in divergence form and that $g \in \mathcal{X}^s$, $0 \leq s < 2$. Let (ρ, p, q) be a Strichartz pair. Then*

$$|||D|^{1-\rho-\frac{\sigma}{p}} u\|_{L^p(L^q)} \leq \mu^{\frac{1}{p}} \|\nabla u\|_{L^\infty(L^2)} + \mu^{-\frac{1}{p'}} |||D|^{-\sigma} f_1\|_{L^1(L^2)} + |||D|^{\rho-\frac{\sigma}{p}} f_2\|_{L^{p'}(L^{q'})} \quad (13)$$

whenever

$$P(x, D)u = f_1 + f_2$$

and

$$T^s \|g\|^2_{\mathcal{X}^s} \leq \mu^{2+s}, \qquad \mu \geq 1$$

Applied to solutions for the initial value problem (9) this shows that the full Strichartz estimates hold for operators with C^2 coefficients.

Corollary 6 *Assume that $D^2 g \in L^1(0, T; L^\infty)$. Let $(\rho, p, q), (\rho_1, p_1, q_1)$ be Strichartz pairs. Then the following estimate holds*

$$|||D|^{1-\rho_1} u\|_{L^{p_1}(0,T;L^{q_1})} \leq |||D|^\rho P(x, D)u\|_{L^{p'}(0,T;L^{q'})} + \|u_0\|_{H^1} + \|u_1\|_{L^2} \quad (14)$$

The corresponding result for $1 \leq s < 2$ is

Corollary 7 *Assume that the operator P has $\mathcal{X}^s$ coefficients in $[0, T]$ with $1 \leq s \leq 2$. Let $(\rho, p, q), (\rho_1, p_1, q_1)$ be Strichartz pairs. Then*

$$|||D|^{1-\rho_1-\frac{\sigma}{p_1}} u\|_{L^{p_1}(0,T;L^{q_1})} \leq |||D|^{\rho+\frac{\sigma}{p}} Pu\|_{L^{p'}(0,T;L^{q'})} + \|u_0\|_{H^1} + \|u_1\|_{L^2} \quad (15)$$

4. Proof of the Strichartz estimates.

Here we sketch the proof of Theorems 2, 3. The rest of the results are proved in a similar fashion.

Localization and truncation The first part of the proof of Theorem 2 involves several localization type arguments. First we reduce the estimate to the case when $\mu = 1$, $T = 1$, $\|D^2 g\|_{L^1(L^\infty)} \leq 1$ and u is supported in a cube of size 1. Then we use a Paley-Littlewood decomposition to reduce the problem to the corresponding dyadic estimates at fixed frequency λ,

$$\lambda^{-\rho} \|S_\lambda u\|_{L^p(L^q)} \leq \|S_\lambda u\|_{L^\infty(L^2)} + \lambda^{-1} \|P S_\lambda u\|_{L^1(L^2)} \quad (16)$$

where S_λ is the multiplier which selects the frequencies of size aproximatively equal to λ.

Next we observe that (16) remains unchanged if we truncate the coefficients of P at frequency $\sqrt{\lambda}$. Thus, without any restriction in generality we can assume that

$$\|\partial_x^\alpha g\|_{L^1(L^\infty)} \leq c_\alpha \lambda^{\frac{|\alpha|-2}{2}}$$

which also gives

$$|\partial_x^\alpha g| \leq c_\alpha \lambda^{\frac{|\alpha|-1}{2}}$$

The same steps apply for the proof of Theorem 3, with the only difference that the coefficients are truncated at frequency $\mu^{\frac{1}{2}} \lambda^{\frac{2}{2+s}}$. But then the corresponding dyadic estimate follows directly from Theorem 2.

Using the FBI transform The second stage of the proof of Theorem 2 is to obtain good L^2 estimates for the FBI transform of u. Conjugating the operator P with respect to the FBI transform T_λ we get two ode's in the "FBI" space. One of these ode's is along the gradient flow of $p(x,\xi)$ and provides an elliptic estimate away from the characteristic cone. The other one is along the Hamilton flow of p and corresponds to propagation of singularities. Exploiting the L^2 information coming from the two ode's we can reduce the dyadic inequalities to certain oscillatory integral estimates.

Set

$$w = \Phi^{\frac{1}{2}} T_\lambda S_\lambda u$$

Then we try to get good L^2 estimates for w. The function $S_\lambda u$ can then be recovered from

$$S_\lambda u = T_\lambda^* \Phi^{\frac{1}{2}} w$$

Observe first that w is concentrated in the region

$$U = \{|x| \leq 2, \ \frac{1}{4} \leq |\xi| \leq 4\}$$

Outside this region we have exponential decay,

$$\|w\|_{L^2(U^c)} \leq e^{-c\lambda} \|S_\lambda u\|_{L^2}.$$

Hence it suffices to get good estimates for w in the region U. Set

$$(\lambda p + 2(\bar{\partial} p)(\partial - i\lambda\xi))w = g$$

where

$$g = \Phi^{\frac{1}{2}}(R_\lambda S_\lambda u + T_\lambda f)$$

Since $\Phi^{-\frac{1}{2}} w$ is holomorphic, we obtain in effect two pieces of information, namely

$$[(p_x \partial_\xi - p_\xi \partial_x) - i\lambda(p - p_\xi \cdot \xi)]\, w = g \tag{17}$$

respectively

$$[(p_x \partial_x + p_\xi \partial_\xi) + \lambda(p - i p_x \cdot \xi)]\, w = g \tag{18}$$

The first equation is an ode along the Hamilton flow of p, while the second equation is an ode along the gradient curves of p. Our strategy is now to use the (17) to obtain good estimates for w on the characteristic cone K, and then to use (18) to obtain good decay rates away from the cone.

We use (18) to decompose w into two parts,

$$w = w_1 + w_2$$

where w_1 solves the inhomogeneous equation

$$[(p_x \partial_x + p_\xi \partial_\xi) + \lambda(p - i p_x \cdot \xi)]\, w_1 = g, \qquad w_{1|K} = 0 \tag{19}$$

and w_2 solves the homogeneous equation

$$[(p_x \partial_x + p_\xi \partial_\xi) + \lambda(p - i p_x \cdot \xi)]\, w_2 = 0, \qquad w_{2|K} = w \tag{20}$$

Correspondingly split $S_\lambda u$ into $u_1 + u_2$ with

$$u_i = T_\lambda^* \Phi^{\frac{1}{2}} w_i$$

The estimate for w_1 If we multiply (19) by $\frac{w_1}{p}$ and integrate by parts then we obtain

$$2\Re < \frac{w_1}{p}, g >= \lambda\|w_1\|^2 + \|\frac{|\nabla p| w_1}{p}\|^2$$

which yields

$$\lambda\|w_1\|^2 \leq c\|g\|^2 \tag{21}$$

On the other hand if we just square (19) and integrate by parts then we obtain

$$\|g\|^2 = \lambda^2\|pw_1\|^2 + \|((p_x\partial_x + p_\xi\partial_\xi) - i\lambda p_x \cdot \xi)w_1\|^2 - \lambda\||\nabla p| w_1\|^2 \tag{22}$$

Summing up the last two inequalities we get

$$\lambda^2\|pw_1\|^2 + \lambda\|w_1\|^2 \leq \|g\|^2$$

If one uses appropriately the remainder estimate, on the other hand, we get a bound for $\lambda^{-\frac{1}{4}}\|g\|^2$. Hence it remains to prove that

$$\|T_\lambda^* \Phi^{\frac{1}{2}} w_1\|_{L^p(L^q)} \leq \lambda^{\rho+\frac{1}{4}}(\lambda^{\frac{1}{2}}\|pw_1\|^2 + \|w_1\|^2)$$

which is equivalent to

$$\|T_\lambda^* \Phi^{\frac{1}{2}} \frac{\lambda^{\frac{1}{4}} a(x,\xi)}{1+\lambda^{\frac{1}{2}}|p|}\|_{L^2 \to L^p(L^q)} \leq \lambda^{\rho+\frac{1}{2}}$$

and further, by the "TT^*" argument, to

$$\|T_\lambda^* \Phi \frac{\lambda^{\frac{1}{2}} a^2(x,\xi)}{(1+\lambda^{\frac{1}{2}}|p|)^2} T_\lambda\|_{L^{p'}(L^{q'}) \to L^p(L^q)} \leq \lambda^{2\rho+1}$$

The weight inside is integrable across the level sets of p therefore we can foliate with respect to the level sets of p and reduce this to

$$\|T_\lambda^* \Phi a^2(x,\xi) \delta_{p=0} T_\lambda\|_{L^{p'}(L^{q'}) \to L^p(L^q)} \leq \lambda^{2\rho+1}$$

which follows by standard interpolation arguments and oscillatory integral estimates. One can compare the kernel arising here,

$$K(y,\tilde{y}) = \lambda^{\frac{3(n+1)}{2}} \int_K a(x,\xi) e^{i\lambda\xi(y-\tilde{y})} e^{-\frac{\lambda}{2}(x-y)^2} e^{-\frac{\lambda}{2}(x-\tilde{y})^2} dx d\xi \tag{23}$$

with the corresponding kernel arising in the constant coefficient case,

$$K_1(y,\tilde{y}) = \lambda^{n+1} \int_K a(\xi) e^{i\lambda\xi(y-\tilde{y})} d\xi$$

and observe that the Gaussians have at most a regularizing effect above the $\lambda^{-\frac{1}{2}}$ scale.

The gradient flow The next step in our analysis is to estimate w_2 using (20). To achieve that we need to compute the regularity of the gradient flow of p. Suppose we start with initial data (x, ξ) on the cone $K = \{p = 0\}$. Denote by q the natural parameter along the flow, chosen so that $q = 0$ on K. Set (x_q, ξ_q) the image of (x, ξ) along the flow. Then (x_q, ξ_q) solve the equations

$$\begin{cases} \partial_q x_q = p_x(x_q, \xi_q) \\ \partial_q \xi_q = p_\xi(x_q, \xi_q) \end{cases}$$

Since the first derivatives of p are bounded but the second derivatives are only bounded by $\sqrt{\lambda}$, the gradient flow remains smooth on the scale of $\lambda^{-\frac{1}{2}}$ and becomes exponentially "bad" afterwards. More precisely,

Theorem 8 *Assume that P has C^1 coefficients whose Fourier transforms are supported in $B(0, \sqrt{\lambda})$. Then*

$$\begin{array}{rcll} |\partial_x^\alpha \partial_\xi^\beta x_q| & \leq & c_{\alpha,\beta} \lambda^{\frac{|\alpha|-1}{2}} e^{c\sqrt{\lambda} q} & \qquad |\alpha| + |\beta| > 0 \\ |\partial_x^\alpha \partial_\xi^\beta (\xi_q - \xi)| & \leq & c_{\alpha,\beta} \lambda^{\frac{|\alpha|-1}{2}} e^{c\sqrt{\lambda} q} & \qquad |\alpha| + |\beta| \geq 1 \\ |\partial_x^\alpha \partial_\xi^\beta p(x_q, \xi_q)| & \leq & c_{\alpha,\beta} (1 + \lambda^{\frac{|\alpha|-1}{2}}) e^{c\sqrt{\lambda} q} & \end{array}$$

However, the fundamental solution for (20) exhibits Gaussian decay on the same scale which overrides the exponential growth corresponding to the flow.

The oscillatory integral for u_2 We have

$$u_2 = T_\lambda \Phi^{\frac{1}{2}} w_2$$

If we use the ode (20) to express w_2 in terms of the trace of w on the cone and then carry out the integration along the gradient flow of p then we obtain the following representation for w_2:

Theorem 9 *Assume that P has C^1 coefficients frequency localized in $\|\xi\| \leq \sqrt{\lambda}$. Then we have*

$$u_2 = \lambda^{-\frac{1}{2}} V_\lambda w_{|K}$$

where V_λ is an integral operator,

$$V_\lambda w = \lambda^{\frac{3(n+1)}{4}} \int_K e^{i\lambda\xi(x-y)} G(x, y, \xi) w \, dx d\xi$$

with a kernel G satisfying

$$|\partial_x^\alpha d_\xi^\beta G(x, y, \xi)| \leq c_{\alpha,\beta} \lambda^{\frac{|\alpha|}{2}} e^{-c\lambda(x-y)^2} \tag{24}$$

Then, using appropriately the error estimates, we need to prove an estimate of the form

$$\|V_\lambda w\|_{L^p(L^q)} \leq \lambda^{\rho+\frac{1}{4}} \|H_p w\|_{L^1_{x_0}(L^2_{x'\xi'})(K)} \tag{25}$$

for all w supported in $K \cap U$.

Given a pair (x, ξ) we denote by (x_t, ξ_t) its image along the Hamilton flow. This map is homogeneous of order 1 with respect to ξ. Then, (25) is equivalent to

Theorem 10 *Let $a(x, \xi)$ be a smooth compactly supported function, which is* 0 *near* $\xi = 0$ *and* 1 *in* $1/4 \leq |\xi| \leq 4$. *Then*

$$\|V_\lambda a(x, \xi) L\|_{L^2(K \cap \{x_0 = 0\}) \to L^p(L^q)} \leq \lambda^{\rho+\frac{1}{4}} \tag{26}$$

where L is the forward transport operator along the Hamilton flow given by

$$(Lw)(x, \xi) = \begin{cases} 0 & x_0 < 0 \\ w(x_t, \xi_t) & x_{0t} = 0, \quad t < 0 \end{cases}$$

This is further equivalent to the corresponding bound for the operator

$$Z = (V_\lambda a L)(V_\lambda a L)^* = V_\lambda a L L^* \delta_{p(x,\xi)=0} a V_\lambda^* \ , \tag{27}$$

namely

$$\|Z\|_{L^{p'}(L^{q'}) \to L^p(L^q)} \leq \lambda^{2\rho+\frac{1}{2}} \tag{28}$$

The operator LL^* is an integral operator along bicharacteristics, with kernel

$$l(t, s) = 1_{\{t \geq 0, \ s \geq 0\}}$$

Using a standard complex interpolation argument this can be obtained from a trivial $L^2 \to L^2$ estimate and an $L^1(L^{q_1'}) \to L^1(L^{q_1})$ estimate. The $L^1(L^{q_1'}) \to L^1(L^{q_1})$ estimate, in turn, reduces to a kernel bound for an operator which involves a shift along the Hamilton flow,

Theorem 11 *Denote by F^t the translation by t along the Hamilton flow, and by H^t the kernel of the operator*

$$Z^t = V_\lambda a F^t \delta_{p(x,\xi)=0} a V_\lambda^*$$

Then the kernels H^t satisfy the following estimate:

$$|H^t(y, \tilde{y})| \leq c \lambda^{n+1} e^{-c\lambda(\tilde{y}_0 - y_0 - t)^2} (1 + \lambda |y - \tilde{y}|)^{\frac{n-1}{2}} \tag{29}$$

The kernel H^t has the form

$$H^t(y, \tilde{y}) = \lambda^{\frac{3(n+1)}{2}} \int_K G(x, y, \xi) G(x, \tilde{y}, \xi) e^{i\lambda\xi(x-y)} e^{-i\lambda\xi_t(x_t - \tilde{y})} dx d\xi$$

To estimate it we need to compute the regularity of the Hamilton flow.

The regularity of the Hamilton flow Here we obtain precise bounds on the derivatives of the flow map F^t with respect to ξ. Observe first that if the coefficients g^{ij} satisfy

$$D^2 g \in L^1(L^\infty)$$

and have Fourier transform supported in $|\xi| \leq \sqrt{\lambda}$ then the following relations hold:

$$\|\partial_x^\alpha g\|_{L^1(L^\infty)} \leq c_\alpha \lambda^{\frac{\alpha-2}{2}}, |\alpha| \geq 2 \tag{30}$$

This implies that

Lemma 2 *Assume that $D^2 g \in L^1(L^\infty)$ with Fourier transform supported in $B(0,\sqrt{\lambda})$. Then the following bounds hold*

$$\begin{array}{ll} |\partial_\xi^\alpha x_t| \leq t(1+t\sqrt{\lambda})^{|\alpha|-1}, & |\alpha| \geq 1 \\ |\partial_\xi^\alpha \xi_t| \leq (1+t\sqrt{\lambda})^{|\alpha|-1} & |\alpha| \geq 1 \end{array} \tag{31}$$

We can use the above Lemma to produce an expansion of x_t, ξ_t in terms of powers of t:

Lemma 3 *Assume that the coefficients of P satisfy $D^2 g \in L^1(L^\infty)$ with Fourier transform supported in $B(0,\sqrt{\lambda})$. Then the following estimates hold:*

$$x_t = x + tp_\xi + t^2 g(t,x,\xi) \tag{32}$$

$$\xi_t = \xi + th(t,x,\xi) \tag{33}$$

where g, h satisfy the following bounds:

$$|\partial_\xi^\alpha h(t,x,\xi)|, |\partial_\xi^\alpha g(t,x,\xi)| \leq (1+t\sqrt{\lambda})^{\alpha-1}, \tag{34}$$

Another straightforward consequence of Lemma 2 is the following bound for the exponents in the kernel G:

Lemma 4 *Let G be as in (24). Then*

$$|\partial_\xi^\alpha G(x_t,\xi_t,y)| \leq c_\alpha e^{-c\lambda(x_t-y)^2}(1+t\sqrt{\lambda})^\alpha,$$

In a similar manner we obtain the related result for the phase function in our kernel:

Lemma 5 *For ξ in a compact set and away from 0 we have*

$$|\partial_\xi \left(\lambda(x_t-\tilde{y})\xi_t\right) - \left(\lambda(x-\tilde{y}) + t^2\lambda g(t,x,\xi)\right)| \leq (1+\lambda(x_t-\tilde{y})^2)(1+t\sqrt{\lambda}) \tag{35}$$

and

$$|\partial_\xi^\alpha \lambda((x_t-\tilde{y})\xi_t)| \leq c_\alpha(1+\lambda(x_t-\tilde{y})^2)(1+t\sqrt{\lambda})^\alpha, |\alpha| \geq 2 \tag{36}$$

This is essential since it allows us to replace the nonlinear phase function with a linear one modulo a good factor.

Oscillatory integral estimates Given the above results, for $t \leq \lambda^{-\frac{1}{2}}$ the Hamilton flow stays smooth and we can represent the kernel H^t in the form

$$H^t(y,\tilde{y}) = \lambda^{\frac{3(n+1)}{2}} \int_K e^{-c\lambda(x-y)^2} e^{-c\lambda(x-\tilde{y})^2} e^{i\lambda\xi(y-\tilde{y})} f(y,\tilde{y},x,\xi)dxd\xi$$

where f is bounded, compactly supported away from $\xi = 0$ and has bounded derivatives in ξ. This behaves in the same way as the kernel in (23).

For $t > \lambda^{-\frac{1}{2}}$, on the other hand, we get the representation

$$H^t(y,\tilde{y}) = \lambda^{\frac{3(n+1)}{2}} \int_K e^{-c\lambda(x-y)^2} e^{-c\lambda(y_0-\tilde{y}_0-t)^2} e^{-c\lambda t^2(\xi-\tilde{\xi})^2} e^{i\lambda\xi(y-\tilde{y}-t^2 g(t,x,\tilde{\xi}))} f(y,\tilde{y},x,\xi)dxd\xi$$

where $\tilde{\xi}$ depends only on $x, y, \tilde{y}$ and f satisfies

$$|\partial_\xi^\alpha f| \leq c_\alpha(1+t\sqrt{\lambda})^\alpha \tag{37}$$

This is worse than before, but is compensated by the fact that the integrand is localized in ξ on the same scale.

5. Quasilinear hyperbolic equations.

Consider a quasilinear second order hyperbolic equation in $R^n \times R$,

$$\partial_i g^{ij}(u)\partial_j u = N(u,\partial u) \tag{38}$$

with Cauchy data

$$u(0) = u_0, \qquad u_t(0) = u_1 \tag{39}$$

Then the classical theory (see [5], and also [19] and references therein) says that this problem is locally well-posed in $H^s \times H^{s-1}$ for $s > \frac{n}{2}+1$. This condition insures that the coefficients of the principal part are C^1 and that ∇u is bounded.

The question is whether the problem remains locally well-posed for initial data which is less regular than that. This can only be possible if we restrict the class of nonlinearities N. Thus, we assume that the nonlinearity is at most quadratic in ∇u,

$$N(u,u) = G(u)Q(\nabla u, \nabla u) \tag{40}$$

and that the functions G, g^{ij} are smooth, bounded and have bounded derivatives up to a sufficiently high order. Also we assume that the coefficients g^{ij} are uniformly hyperbolic in time. Then combining the new Strichartz estimates with the energy estimates it is fairly easy to prove that

Theorem 12 *The quasilinear problem (38)-(39) is locally well-posed in $H^s \times H^{s-1}$ for*

$$s \geq \frac{n}{2} + \frac{5}{6} \quad , \quad n = 2$$
$$s > \frac{n}{2} + \frac{2}{3} \quad , \quad n \geq 3$$

The first result in this direction, for

$$s \geq \frac{n}{2} + \frac{7}{8} \quad , \quad n = 2$$
$$s > \frac{n}{2} + \frac{3}{4} \quad , \quad n \geq 3$$

was independently proved in [16] and [2]. The above theorem is proved in [17]. We have recently learned that Bahouri and Chemin were also able to improve their first result in a second article [1]. Their new exponents (e.g. $\frac{n+1}{2} + \frac{4-\sqrt{13}}{2}$ for $n \geq 3$) are only slightly larger than the ones in the theorem.

The idea of the proof is quite simple. From the energy estimates we know that there is no blow up for as long as $\nabla g(u)$ remains in $L^1(L^\infty)$. Then in order to close the argument we use the Strichartz estimates to show that this also implies that $\nabla u \in L^2(L^\infty)$ $(n \geq 3)$ respectively $\nabla u \in L^2(L^\infty)$ $(n = 2)$. In the constant coefficient case this would require $s \geq \frac{n}{2} + \frac{1}{2}$ $(n \geq 3)$ respectively $s \geq \frac{n}{2} + \frac{3}{4}$ $(n = 2)$. However, in our case we loose $\frac{\sigma}{p}$ derivatives in the Strichartz estimates, i.e. $\frac{1}{6}$ $(n \geq 3)$ respectively $\frac{1}{12}$ $(n = 2)$. Thus we need

$$s > \frac{n}{2} + \frac{1}{2} + \frac{1}{6} \qquad n \geq 3$$

respectively

$$s > \frac{n}{2} + \frac{3}{4} + \frac{1}{12} \qquad n \geq 3$$

Is is perhaps interesting to compare the results for the quasilinear equation with those for the corresponding semilinear equation

$$\Box u = |\nabla u|^2 \tag{41}$$

	The semilinear problem (41)		The quasilinear problem (38)	
n	using only Strichartz estimates	best result	using only Strichartz estimates	best result
2	$\frac{n}{2} + \frac{3}{4}$	$\frac{n}{2} + \frac{3}{4}$	$\frac{n}{2} + \frac{5}{6}$	?
3	$\frac{n}{2} + \frac{1}{2}$	$\frac{n}{2} + \frac{1}{2}$[2]	$\frac{n}{2} + \frac{2}{3}$	?
4	$\frac{n}{2} + \frac{1}{2}$	$\frac{n}{2} + \frac{1}{4}$[3]	$\frac{n}{2} + \frac{2}{3}$	?
5+	$\frac{n}{2} + \frac{1}{2}$	$\frac{n}{2}$	$\frac{n}{2} + \frac{2}{3}$	?

One should note, though, that at this point the counterexamples for the quasilinear equation are no better than those for the semilinear equation, see [7], [8].

The same method can be used for second order hyperbolic equations of the form

$$g^{ij}(u, \nabla u)\partial_i \partial_j u = N(u, \nabla u) \tag{42}$$

[2]This can be easily proved in the framework of the $X^{s,\theta}$ spaces

[3]see [15]

Differentiating once we obtain equations which are essentially of the form (38), therefore

Theorem 13 *The quasilinear problem (42)-(39) is locally well-posed in $H^s \times H^{s-1}$ for*

$$\begin{aligned} s &\geq \frac{n+2}{2} + \frac{5}{6} \quad , \quad n = 2 \\ s &> \frac{n+2}{2} + \frac{2}{3} \quad , \quad n \geq 3 \end{aligned}$$

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