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Normal forms and inverse spectral theory

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Abstract

This talk will describe some results on the inverse spectral problem on a compact Riemannian manifold (possibly with boundary) which are based on V.Guillemin's strategy of normal forms. It consists of three steps: First, put the wave group into a normal form around each closed geodesic. Second, determine the normal form from the spectrum of the Laplacian. Third, determine the metric from the normal form. We will try to explain all three steps and to illustrate with simple examples such as surfaces of revolution.

1. Introduction.

Our purpose in this report is to survey some relatively new results on the classical inverse spectral problem: can one determine a compact Riemannian manifold (M, g) (with or without boundary) from the spectrum $\mathrm{Sp}(\Delta_g)$ of its Laplacian Δ_g? More precisely, is the correspondence $(M, g) \to Sp(\Delta_g)$ injective as (M, g) ranges over a class $\mathcal{M}$ of compact Riemannian manifolds? It is well-known that the answer is 'no' if $\mathcal{M}$ is the entire class of (M, g) (see [Go]), and there are few positive inverse results for any reasonably broad class $\mathcal{M}$. Recently, however, a new approach to the inverse spectral problem has developed which offers hope of obtaining positive results for at least some special classes of metrics or domains. To give it a name for this article, we will call it the 'strategy of normal forms'. The basic idea (due essentially to V. Guillemin, but with antecedents in the work of Colin de Verdière [CV.1], Melrose [M][M.M] and others) is that for generic (M, g), $Sp(\Delta_g)$ should determine the (quantum) Birkhoff normal form of Δ_g around each closed geodesic γ. The inverse spectral problem then reduces to the problem of determining the metric from the normal forms. This is still a difficult problem but for certain classes $\mathcal{M}$ one expects that it can be solved. In particular, we will describe a special class $\mathcal{M}$ of metrics, namely the analytic simple surfaces of revolution, where the strategy has succeeded to a reasonable degree [Z.3]. This class of surfaces is in some ways analogous to (but much simpler than) the class of real analytic plane domains, where the connection between Laplace spectrum and Birkhoff normal forms was first made in [CV.1][M.M]. We will not discuss domains here but it is hoped that the strategy

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of normal forms extends to them and we sometimes include them in our general remarks.

In view of the considerable number of counterexamples to the inverse spectral problem, and of the wide gap between the many negative and the few positive results, it may help orient the reader if we begin by discussing the restrictions we will place on the metrics (or domains). First, we will always assume that the length spectrum $\mathrm{Lsp}(M,g)$ is simple in the sense that the length functional on the loop space takes on distinct values on distinct components of its critical point set. E.g., in the typical case where all geodesics are non-degenerate, simplicity means that (up to time reversal) there is just one closed geodesic of each given length in $\mathrm{Lsp}(M,g)$. This is a generic condition which is necessary to separate out terms in the trace formula (§1). As far as the author knows, all known examples of non-isometric isospectral pairs have multiple length spectra, so this condition appears to eliminate the known obstacles to inverse spectral results. Secondly, we will often assume that the metrics (or domains) are real analytic. This is because we do not know how to combine information from distinct closed geodesics, so we must determine the metric from information on its germ at just one closed geodesic. Third, it is natural at this stage to restrict to classes $\mathcal{M}$ of metrics (or domains) where the unknown has just one functional degree of freedom (i.e. is a function of one variable). Our experience with surfaces of revolution (and symmetric plane domains) seems to indicate that the spectral invariants coming from just one closed geodesic just suffices to determine this amount of the unknown object.

We now divide up our discussion of results according to the relevant class of metrics.

1.1 Compact Riemannian manifolds with non-degenerate closed geodesics

A closed geodesic γ will be called 'non-degenerate' if 1 is not an eigenvalue of its linear Poincaré map P_γ, and 'strongly non-degenerate' if the eigenvalues of P_γ are independent (with π) over the rational numbers (§1). The main result for this class of Riemannian manifolds is the following:

Theorem 1 *([G.1-2]; see also [Z.1-2]) Suppose that (M,g) is a compact Riemannian manifold without boundary, and suppose that γ is a non-degenerate closed geodesic whose length L_γ is of multiplicity one in $Lsp(M,g)$. Then the quantum Birkhoff normal form of $\sqrt{\Delta}$at γ is a spectral invariant.*

The definition of 'quantum Birkhoff normal form' will be given in §2. Roughly speaking, it is the expression of $\sqrt{\Delta}$ as a function $F(D_s, \hat{I}_1, \dots, \hat{I}_n)$ of (micro-)locally defined transversal 'action operators' $\hat{I}_j$ and of the tangential derivative $D_s = \frac{1}{i}\frac{\partial}{\partial s}$ along γ. In the elliptic case, $\hat{I}_j = \frac{1}{2}(D_{y_j}^2 + y_j^2)$. Since the classical Birkhoff normal form of the symbol $H(x,\xi) = \sqrt{\sum_{ij} g^{ij}(x)\xi_i\xi_j}$ is the principal symbol of the quantum normal form, it follows that the classical BNF of the metric Hamiltonian H at γ is also a spectral invariant.

The question then arises to what degree a metric (or domain) is determined by the normal form of $\sqrt{\Delta}$at one (or all) closed geodesic(s)? In particular, to what degree does the classical Birkhoff normal form of the geodesic flow near γ determine the metric? The normal form coefficients were determined in [Z.1,2] and are very messy metric invariants even in dimension two (§2). Hence one looks for simple model examples where the normal forms have simple geometric or 'quantum' interpretations.

1.2 Simple analytic surfaces of revolution

The simplest situation is that of real analytic surfaces of revolution of 'simple type'. Roughly, 'simple type' means that there is just one circle (geodesic) of points of critical distance to the axis of rotation, which we will call the equator, and that its associated Poincaré map is of twist type (§3). Such surfaces were studied in [CV.2] where it was shown that there exist global action operators $\hat{I}_1, \hat{I}_2$ and a (polyhomogeneous) function F such that $\sqrt{\Delta}= F(\hat{I}_1, \hat{I}_2)$.

The inverse result above applies to the normal form of $\sqrt{\Delta}$ at the equator, but it is simpler to take advantage of the complete integrability of the geodesic flow (and wave group) and prove an inverse normal form result in the integrable setting:

Theorem 2 *([Z.3]) Let (S^2, g) be an analytic simple surface of revolution with simple length spectrum. Then the normal form $F(I_1, I_2)$ is a spectral invariant.*

In this example, both the classical and quantum Birkhoff normal forms have simple expressions in terms of the metric. We have:

Theorem 3 *([Z.3]) Suppose that g_1, g_2 are two real analytic metrics on S^2 such that (S^2, g_i) are simple surfaces of revolution with simple length spectra. Then $Sp(\Delta_{g_1}) = Sp(\Delta_{g_2})$ implies $g_1 = g_2$.*

Here, equality means isometry of metrics. An obvious shortcoming of this result is that both metrics were assumed to belong to $\mathcal{R}$. One would like to know if it suffices to assume just that $g_1 \in \mathcal{R}$. In other words, is a metric $g \in \mathcal{R}$ spectrally determined? At the present time, the only metric on S^2 which is known to be spectrally determined is the round metric g_{can}, so we do not expect to fully answer the question for general $g \in \mathcal{R}$. However, in a work in progress with G.Forni, we prove some partial results in the direction that g_2 has completely integrable geodesic flow. At the present time, our result is:

Theorem 4 *Let $g_1 \in \mathcal{R}^*$ and suppose that g_2 is a real analytic metric on S^2 with $Sp(\Delta_{g_1}) = Sp(\Delta_{g_2})$. Then:*

(a) Up to time reversal, g_2 has precisely one elliptic non-degenerate closed geodesic;
(b) The Poincaré map for this orbit is completely integrable in the C^0 sense.

In other words, the Poincaré map (and geodesic flow near the orbit) preserve a continuous foliation by tori, at least near the 'equator'. Even if one could improve this result to show that the geodesic flow of g_2 was completely integrable, it would still not be enough to show that $g_2 \in \mathcal{R}^*$; indeed, at this time, the completely integrable metrics on S^2 have not been classified.

2. Preliminaries.

We begin with some definitions.

2.1 Jacobi fields and Poincaré map for non-degenerate closed geodesics

The Jacobi equation along a closed geodesic $\gamma(t)$ is the equation

$$\frac{D^2}{dt^2}J + R(T,J)T = 0$$

where $\frac{D}{dt}$ denotes covariant differentiation, $T = \gamma'(t)$ and $R(\cdot,\cdot)$ is the curvature tensor. A solution is called a Jacobi field, and a solution $J \perp T$ is called an orthogonal Jacobi field. We let $\mathcal{J}_\gamma^\perp \otimes \mathbb{C}$ denote the space of complex normal Jacobi fields along γ. It is a symplectic vector space of (complex) dimension $2n$ ($n = \dim M - 1$) with respect to the Wronskian

$$\omega(X,Y) = g(X, \frac{D}{ds}Y) - g(\frac{D}{ds}X, Y).$$

The linear Poincaré map P_γ is then the linear symplectic map on $\mathcal{J}_\gamma^\perp \otimes \mathbb{C}$ defined by $P_\gamma Y(t) = Y(t + L_\gamma)$.

A closed geodesic is called *non-degenerate* if $det(I - P_\gamma) \neq 0$. Since P_γ is symplectic, its eigenvalues come in 4-tuples (inverses and complex conjugates, which may coincide), i.e. the spectrum of P_γ) has the form: $Sp(P_\gamma) = \{e^{\pm\mu_j \pm i\alpha_j}, j = 1,...,n\}$. P_γ is called elliptic if all of its eigenvalues have modulus one, in which case $Sp(P_\gamma) = \{e^{\pm i\alpha_j}, j = 1,...,n\}$. It is (real) hyperbolic if the eigenvalues come in pairs $Sp(P_\gamma) = \{e^{\pm\mu_j}, j = 1,...,n\}$. For the sake of simplicity we will generally assume that P_γ is elliptic.

The Jacobi eigenvectors of P_γ will be denoted $\{Y_j, \overline{Y_j},\ j = 1,...,n\}$. In the elliptic case they may be normalized to satisfy:

$$P_\gamma Y_j = e^{i\alpha_j}Y_j \qquad P_\gamma \overline{Y}_j = e^{-i\alpha_j}\overline{Y}_j \quad \omega(Y_j, \overline{Y}_k) = \delta_{jk}.$$

We introduce a fixed parallel normal frame $e(s) := (e_1(s),...,e_n(s))$ along γ and write the eigenvectors in the form $Y_j(s) = \sum_{k=1}^n y_{jk}(s)e_k(s)$.

2.2 Wave trace invariants in the non-degenerate case on a manifold without boundary

The wave group of a compact Riemannian manifold is the unitary group $U(t) = e^{it\sqrt{\Delta}}$. As is well-known, it has a distribution trace $\mathrm{Tr}U(t)$ which is a Lagrangean distribution on $\mathbb{R}$ with singularities at lengths of closed geodesics [D.G]. In the case of a 'bumpy' riemannian manifold (M,g), i.e. one for which all closed geodesics are non-degenerate, the wave trace has the singularity expansion

$$\mathrm{Tr}\, U(t) = e_0(t) + \sum_{L \in \mathrm{Lsp}(M,g)} e_L(t) \tag{1}$$

with

$$e_0(t) = a_{0,-n}(t+iO)^{-n} + a_{0,-n+1}(t+iO)^{-n+1} + \cdots \tag{2}$$

$$\begin{aligned} e_L(t) &= a_{L,-1}(t-L+iO)^{-1} + a_{L,0}\log(t-(L+iO)) \\ &+ a_{L,1}(t-L+iO)\log(t-(L+iO)) + \cdots \\ a_{L,-k} &= \textstyle\sum_{\gamma:L_\gamma=L} a_{\gamma,-k}, \end{aligned} \tag{3}$$

where $\cdots$ refers to homogeneous terms of ever higher integral degrees ([DG]). The coefficients $a_{L,-k}$ and $a_{\gamma,-k}$ are known as wave invariants. The coefficients $a_{0,k}$ at $t=0$ are essentially the same as the heat coefficients, i.e. the coefficients of the small t expansion of $\mathrm{Tr}e^{-t\Delta}$ and are well-known to be given by integrals over M of curvature polynomials. The question arises of similarly characterizing the wave trace invariants associated to closed geodesics.

The principal wave invariant at $t = L_\gamma$ was determined in [D.G]. It is given by

$$a_{\gamma,-1} = \exp^{\frac{i\pi m_\gamma}{4}} L_\gamma^{\#}|\det(I-P_\gamma)|^{-1/2},$$

where $L_\gamma^{\#}$ is the primitive period (once around), where m_γ is the Maslov (= Morse) index, and (as above) where P_γ is the linear Poincaré map.

The following describes the wave trace invariants associated to a non-degenerate closed geodesic. Undefined terminology is reviewed below.

Theorem 5 *Let γ be a strongly non-degenerate closed geodesic. Then $a_{\gamma k} = \int_\gamma I_{\gamma;k}(s;g)ds$ where:*
(i) $I_{\gamma;k}(s;g)$ is a homogeneous Fermi–Jacobi–Floquet polynomial of weight $-k-1$ in the data $\{y_{ij}, \dot{y}_{ij}, D^m_{s,y}g\}$ with $m=(m_1,\dots,m_{n+1})$ satisfying $|m| \leq 2k+4$;
(ii) The degree of $I_{\gamma;k}$ in the Jacobi field components is at most $6k+6$;
(iii) At most $2k+1$ indefinite integrations over γ occur in $I_{\gamma;k}$;
(iv) The degree of $I_{\gamma;k}$ in the Floquet invariants β_j is at most $k+2$.

Let us define the term 'Fermi-Floquet-Jacobi polynomial'. First, we write the metric coefficients g_{ij} relative to Fermi normal coordinates (s,y) along γ. The vector fields $\frac{\partial}{\partial s}, \frac{\partial}{\partial y_j}$ and their real linear combinations will be referred to as Fermi normal vector fields along γ and contractions of tensor products of the $\nabla^m R$'s with these vector fields will be referred to as *Fermi curvature polynomials.* The m-th jet of g along γ is denoted by $j^m_\gamma g$, the curvature tensor by R and its covariant derivatives by $\nabla^m R$. Such polynomials will be called *invariant* if they are invariant under the action of $O(n)$ in the normal spaces. Invariant contractions against $\frac{\partial}{\partial s}$ and against the Jacobi eigenfields $Y_j, \overline{Y}_j$, with coefficients given by invariant polynomials in the components y_{jk}, are called *Fermi–Jacobi polynomials.* We will also use this term for functions on γ given by repeated indefinite integrals over γ of such FJ polynomials. Finally, FJ polynomials whose coefficients are given by polynomials in the Floquet invariants $\beta_j = (1-e^{i\alpha_j})^{-1}$ are *Fermi–Jacobi–Floquet* polynomials.

The role of the Floquet invariants should be particularly emphasized. The fact that the wave invariants are polynomials in the β_j's is a crucial ingredient in Guillemin's (and subsequent) proof(s) of the inverse result.

The 'weights' referred to above describe how the various objects scale under $g \to \epsilon^2 g$. Thus, the variables g_{ij}, $D^m_{s,y} g_{ij}$ (with $m := (m_1, \dots, m_{n+1})$), $L := L_\gamma, \alpha_j, y_{ij}, \dot{y}_{ij}$ have the following weights: $\mathrm{wgt}(D^m_{s,y} g_{ij}) = -|m|$, $\mathrm{wgt}(L) = 1$, $\mathrm{wgt}(\alpha_j) = 0$, $\mathrm{wgt}(y_{ij}) = \frac{1}{2}, \mathrm{wgt}(\dot{y}_{ij}) = -\frac{1}{2}$. A polynomial in this data is homogeneous of weight s if all its monomials have weight s under this scaling. Finally, τ denotes the scalar curvature, τ_ν its unit normal derivative, $\tau_{\nu\nu}$ the Hessian $\mathrm{Hess}(\tau)(\nu, \nu)$; Y denotes the unique normalized Jacobi eigenfield, $\dot{Y}$ its time-derivative and δ_{j0} the Kronecker symbol (1 if $j = 0$ and otherwise 0).

For instance, in dimension 2 (where there is only one Floquet invariant β) the residual wave invariant $a_{\gamma 0}$ is given by:

$$a_{\gamma 0} = \frac{a_{\gamma,-1}}{L^\#}[B_{\gamma 0;4}(2\beta^2 - \beta - \frac{3}{4}) + B_{\gamma 0;0}] \tag{4}$$

where:
(a) $a_{\gamma,-1}$ is the principal wave invariant;
(b) $L^\#$ is the primitive length of γ; σ is its Morse index; P_γ is its Poincaré map;
(c) $B_{\gamma 0;j}$ has the form:

$$B_{\gamma 0;j} = \frac{1}{L^\#}\int_o^{L^\#} [a\ |\dot{Y}|^4 + b_1\ \tau |\dot{Y}\cdot Y|^2 + b_2\ \tau \mathrm{Re}\,(\bar{Y}\dot{Y})^2 + c\,\tau^2 |Y|^4 + d\ \tau_{\nu\nu}|Y|^4 + e\ \delta_{j0}\tau]ds$$

$$+ \frac{1}{L^\#}\sum_{0 \le m,n \le 3; m+n=3} C_{1;mn}\frac{\sin((n-m)\alpha)}{|(1-e^{i(m-n)\alpha})|^2}\left|\int_o^{L^\#} \tau_\nu(s)\bar{Y}^m \cdot Y^n(s)ds\right|^2$$

$$+ \frac{1}{L^\#}\sum_{0 \le m,n \le 3; m+n=3} C_{2;mn}\,\mathrm{Im}\left\{\int_o^{L^\#} \tau_\nu(s)\bar{Y}^m \cdot Y^n(s)\left[\int_o^s \tau_\nu(t)\bar{Y}^n \cdot Y^m(t)dt\right]ds\right\}$$

for various universal (computable) coefficients.

The wave invariants (and normal form coefficients) are obviously very messy. The beauty of the normal forms strategy is that it organizes these coefficients into a potentially meaningful invariant, namely the classical (and quantum) Birkhoff normal form.

2.3 Wave trace expansion on a simple surface of revolution

The wave trace expansion on a simple surface of revolution is quite different since the geodesic flow is completely integrable and closed geodesics (other than the equator) fill out tori in the cosphere bundle.

To explain this, let us be more precise about the definition of 'simple surface of revolution.' First, we will assume that there is an effective action of S^1 by isometries of (S^2, g). The two fixed points will be denoted N, S and (r, θ) will denote geodesic polar coordinates centered at N, with $\theta = 0$ some fixed meridian γ_M from N to S. The metric may then be written in the form $g = dr^2 + a(r)^2 d\theta^2$ where $a : [0, L] \to \mathbb{R}^+$ is defined by $a(r) = \frac{1}{2\pi}|S_r(N)|$, with $|S_r(N)|$ the length of the distance circle of radius r centered at N. We define the class $\mathcal{R}$ of *simple analytic* surfaces of revolution as follows:

Definition 6 *$\mathcal{R}$ is the moduli space of metrics of revolution (S^2, g) with the properties:*
(i) g (equivalently a) is real analytic;
(ii) a has precisely one non-degenerate critical point $r_o \in (0, L)$, with $a''(r_o) < 0$, corresponding to an 'equatorial geodesic' γ_E;
(iii) the (non-linear) Poincaré map $\mathcal{P}_{\gamma_E}$ for γ_E is of twist type.
We denote by $\mathcal{R}^* \subset \mathcal{R}$ the subset of metrics with 'simple length spectra' in the sense above.

From a geometric (or dynamical) point of view, the principal virtue of metrics in $\mathcal{R}$ is described by the following:

Proposition 7 *Suppose that $g \in \mathcal{R}$. Then the Hamiltonian $|\xi|_g := \sqrt{\sum g^{ij}\xi_i\xi_j}$ on T^*S^2 is completely integrable and possesses global real analytic action-angle variables.*

By an action variable we mean a homogeneous function on $T^*S - 0$ whose Hamiltonian flow is 2π-periodic. There is an obvious action variable given by the Clairaut integral $p_\theta(v) := \langle v, \frac{\partial}{\partial\theta}\rangle$. Since the Poisson bracket $\{p_\theta, |\xi|_g\} = 0$, the geodesics are constrained to lie on the level sets of p_θ. With the assumption on a, the level sets are compact and the only critical level is that of the equatorial geodesics $\gamma_E^{\pm} \subset S_g^*S^2$ (traversed with either orientation). The other level sets consist of two-dimensional tori.

The second action variable is less familiar but is obtained in a standard way by integrating the action form over a homology basis of the invariant tori. The formula for it is given by (cf. [CV.2, §6])

$$I_2(I_1, E) = \frac{1}{\pi}\int_{r_-}^{r_+}\sqrt{E^2 - \frac{I_1^2}{a(r)^2}}dr + |I_1|$$

where $r_\pm$ are the extremal values of r on the annulus $\pi(F_{I_1,E})$ (with $\pi : S_g^*S^2 \to S^2$ the standard projection and $F_{I_1,E}$ the torus given by $|\xi| = E, p_\theta = I_1$).

The pair $\mathcal{I} := (I_1, I_2)$ generate a global Hamiltonian torus $(S^1 \times S^1)$-action commuting with the geodesic flow. The singular set of $\mathcal{I}$ equals $\mathcal{Z} := \{I_2 = \pm p_\theta\}$, corresponding to the equatorial geodesics. The map

$$\mathcal{I} : T^*S^2 - \mathcal{Z} \to \Gamma := \{(x, y) \in \mathbb{R} \times \mathbb{R}^+ : |x| < y\}$$

is a trivial torus fibration. Henceforth we write T_I for the torus $\mathcal{I}^{-1}(I)$ with $I \in \Gamma$. Using the product structure we may equip each T_I with angle variables ϕ_j which are symplectically dual to the action variables.

The metric Hamiltonian may be written in the form $H(I_1, I_2)$. The vector $\omega_I = \nabla_I H$ is called the frequency vector. The equations of motion for a geodesic on T_I are given by $\dot{I}_j = 0, \phi_j = \omega_j(I)$.

We now come to the definition of length spectrum and simple length spectrum for a completely integrable geodesic flow.

Definition 8 *(a) A torus T_I is a* periodic torus *if all the geodesics on it are closed.*
(b) The period L *of the periodic torus is then the common period of its closed geodesics.*
(c) The length spectrum $\mathcal{L}$ *of the completely integrable system is the set of these lengths.*
(d) The completely integrable system has a simple length spectrum *if there exist a unique periodic torus (up to time reversal) of each length $L \in \mathcal{L}$.*

Consider now the wave trace formula for a metric on a general Riemannian manifold of dimension n with completely integrable geodesic flow. The dimension of each periodic torus $\mathcal{T}$ of period L equals $dim\mathcal{T} = n$ and the wave trace has the form:

$$TrU(t) = e_o(t) + \sum_{\mathcal{T}} e_{\mathcal{T}}(t)$$

where the sum runs over the periodic tori in S^*M and where

$$e_{\mathcal{T}}(t) = a_{\mathcal{T};-\frac{n+1}{2}}(t - L + i0)^{-\frac{n+1}{2}} + a_{\mathcal{T};-\frac{n+1}{2}+1}(t - L + i0)^{-\frac{n+1}{2}+1} + \ldots.$$

More precisely, it takes this form if n is even; if n is odd, the positive powers of $(t - L + i0)$ should be multiplied by $log(t - L + i0)$. Thus, if $dimM = 2$, the wave trace expansion at a torus $\mathcal{T}$ has the form

$$e_{\mathcal{T}}(t) = a_{\mathcal{T},-\frac{3}{2}}(t - L + i0)^{-\frac{3}{2}} + a_{\mathcal{T},-\frac{1}{2}}(t - L + i0)^{-\frac{1}{2}} + \ldots$$

while if $dimM = 3$, the wave trace expansion at $\mathcal{T}$ has the form

$$e_{\mathcal{T}}(t) = a_{\mathcal{T},-2}(t - L + i0)^{-2} + a_{\mathcal{T},-1}(t - L + i0)^{-1} + a_{\mathcal{T},0}log(t - L + i0) + \ldots.$$

3. Wave invariants as non-commutative residues.

In connecting the wave invariants to the normal form, it will prove useful to interpret the wave invariants as non-commutative residues of the wave group and its time derivatives. We digress to recall the definitions and basic results.

The non-commutative residue of a Fourier Integral operator is an extension of the well-known non-commutative residue of a pseudodifferential operator A on a compact manifold M, defined by

$$\text{res}(A) = 2\,\text{Res}_{s=0}\,\zeta(s, A)$$

where

$$\zeta(s, A) = \text{ Tr } A\Delta^{-s/2} \ (\text{Re } s >> 0)\,.$$

Let A be a Fourier Integral operator in $I^m(M \times M, \Lambda)$ for some homogeneous canonical relation $\Lambda \subset T^*(M \times M)\backslash 0$ and $m \in \mathbb{Z}$ and let diag$(X \times X)$ denote the diagonal in $X \times X$. We have:

Theorem 9 *Let $A \in I^m(M \times M, \Lambda)$, where Λ is a homogeneous canonical relation intersecting* diag*$(T^*M \times T^*M)$ cleanly. Then $\zeta(s, A, P) := \operatorname{Tr} AP^{-s}$ is analytic for* Re s *large for any $P \in \Psi^1$ positive elliptic. It admits a meromorphic extension to $\mathbb{C}$ with simple poles only among the points $s = m + 1 + \frac{e_o - 1}{2} - j$, with $e_o = dim \Lambda \cap$* diag*$(S^*M \times S^*M)$, and with $j = 0, 1, 2, \ldots$ The residues are given by local invariants of the germ of the complete symbol of a and Λ near the intersection with the diagonal.*

In the case of the wave group we then have:

Theorem 10 *Let Δ be the Laplacian for a metric g on M, and let $U_t = \exp it\sqrt{\Delta}$ be the wave group. Let the eigenvalues of $\sqrt{\Delta}$ be denoted $0 = \lambda_o < \lambda_1 < \ldots$. Then the zeta function*

$$\zeta(s,t) := \sum_{j=0}^{\infty} e^{it\lambda_j} \lambda_j^{-s}$$

has a meromorphic continuation to $\mathbb{C}$ with poles only among $s = 1 + 1/2(\dim \operatorname{SFix}(G^t) - 1) - j$.

The non-commutative residue of the Fourier integral operator A is then defined by:

Definition 11

$$res(A) := Res_{s=0}\, \zeta(s, A)$$

The residue res(A) has the properties:

- it is independent of the choice of Δ;
- if either A or B is associated to a local canonical graph, then res$(AB) =$ res(BA)
- there is a local formula for res(A).

Corollary 12 *In the case of a non-degenerate closed geodesic we have:*

$$a_{L,k} = \; res(D_t^k U(t)|_{t=L})$$

$(D_t = \frac{1}{i}\frac{d}{dt})$.

In the case of a completely integrable system, we recall, the dimension of each periodic torus $\mathcal{T}$ of period L equals $e_o = dim\mathcal{T} = n$. Hence we have:

Corollary 13

$$\sum_{\pm} a_{\pm\mathcal{T}, -(\frac{n+1}{2})+k} = \; res(\sqrt{\Delta}^{-\frac{n+1}{2}+k} U(t)|_{t=L})$$

Thus, in the case of simple surface of revolution the first wave trace invariants at a periodic torus $\mathcal{T}$ have the form:

$$\sum_{\pm} a_{\pm\mathcal{T},-\frac{3}{2}} = res\sqrt{\Delta}^{-\frac{3}{2}} e^{iL\sqrt{\Delta}}, \quad \sum_{\pm} a_{\pm\mathcal{T},-\frac{1}{2}} = res\sqrt{\Delta}^{-\frac{1}{2}} e^{iL\sqrt{\Delta}}, \ldots$$

The wave invariants for a closed geodesic γ (or periodic torus T_L) are exactly the same as for its time reversal, hence the same residue formulae also give the individual wave invariants.

4. Normal forms.

In this section we review the normal forms of $\sqrt{\Delta}$ around closed geodesics in the three rather different cases. For detailed discussions see [G.1-2], [Z.1-4].

4.1 Quantum Birkhoff normal forms: non-degnerate case

Roughly speaking, to put $\sqrt{\Delta}$ into microlocal normal form around γ is to conjugate it into a maximal abelian subalgebra $\mathcal{A}$ of pseudodifferential operators on the normal bundle $N_\gamma \sim S^1 \times \mathbb{R}^n$ of γ. The appropriate algebra $\mathcal{A}$ is determined by the spectrum of P_γ. In the tangential direction, one takes D_s where $D_x = \frac{1}{i}\partial/\partial x$. In the transverse directions $\mathbb{R}^n$ to γ, one chooses a generating set of action operators $\hat{I}_j$, that is, quadratic Hamiltonians in the transverse variables y_j, D_{y_j}. Here, $n = \dim M - 1$, and the coordinates (s, y_j) are the Fermi normal coordinates around γ.

In the elliptic directions, $\hat{I}_j$ will be a Harmonic oscillator $\alpha_j(D_{y_j}^2 + y_j^2)$. In the real hyperbolic directions, $\hat{I}_j = \mu_j(y_j D_{y_j} + D_{y_j} y_j)$. In the complex hyperbolic directions, $\hat{I}_j$ involves two y-variables y_j, y_{j+1} and has the form $\mu_j(y_j D_{y_j} + y_{j+1} D_{y_{j+1}}) + \alpha_j(y_{j+1} D_{y_j} - y_j D_{y_{j+1}})$.

The normal form theorem in the non-degenerate case is the following:

Theorem 14 *Assume that γ is a non-degenerate closed geodesic and that its Floquet exponents $\{\alpha_j, \mu_j\}$, together with π are independent over Q. Then there exists a microlocally elliptic Fourier Integral operator W from a conic neighborhood V of $\mathbb{R}^+\gamma$ in $T^*N_\gamma - 0$ to a conic neighborhood of $T_+^*S^1$ in $T^*(S^1 \times \mathbb{R}^n)$ such that:*

$$W\sqrt{\Delta_\psi}W^{-1} :\equiv \mathcal{D} \equiv D_s + \frac{1}{L}H_\alpha + \frac{\tilde{p}_1(\hat{I}_1,\ldots,\hat{I}_n,L)}{D_s} + \frac{\tilde{p}_2(\hat{I}_1,\ldots,\hat{I}_n,L)}{D_s^2} + \ldots + \frac{\tilde{p}_{k+1}(\hat{I}_1,\ldots,\hat{I}_n,L)}{D_s^{k+1}} + \ldots$$

where the numerators $\tilde{p}_j(\hat{I}_1, ..., \hat{I}_n, L)$ are polynomials of degree $j+1$ in the variables $\hat{I}_1, ..., \hat{I}_n$, where W^{-1} denotes a microlocal inverse to W in V. The k-th remainder term lies in the space $\bigoplus_{j=0}^{k+2} O_{2(k+2-j)}\Psi^{1-j}$.

Here, $O_m\Psi^k$ is the space of transverse pseudodifferential operators of order k (in the y-variables) which vanish to order m at $y = 0$. Thus the error term is bigraded by pseudodifferential order and by order of vanishing. The remainder is small if in some combination it has a low pseudodifferential order or a high vanishing order at γ. This is a useful remainder estimate since a given wave invariant a_γ only involves a finite part of the jet of the metric and only a finite part of the complete symbol of $\sqrt{\Delta}$.

4.2 Normal form for $\sqrt{\Delta}$ on a simple surface of revolution

As mentioned in the introduction, $\sqrt{\Delta}$on a simple surface of revolution has a global Birhoff normal form. The reason for this is that the wave group is integrable on the quantum level and in fact commutes with a Fourier integral torus action. Let us first define this notion:

Definition 15 *The wave group $e^{it\sqrt{\Delta}}$ of a compact, Riemannian n-manifold (M,g) is* quantum torus integrable *if there exists a unitary Fourier-Integral representation*

$$\hat{\tau}: T^n \to U(L^2(M)), \qquad \hat{\tau}_{(t_1,\ldots,t_n)} = e^{i(t_1\hat{I}_1+\ldots t_n\hat{I}_n)}$$

of the n-torus and a symbol $\hat{H} \in S^1(\mathbb{R}^n - 0)$ such that $\sqrt{\Delta} = \hat{H}(I_1,\ldots,I_n)$.

The generators $\hat{I}_j$ are first order commuting elliptic action operators with the property that $e^{2\pi i\hat{I}_j} = C_j Id$ for some constant C_j of modulus one. Hence their joint spectrum consists of lattice points in the image of the classical moment map $(I_1,\ldots,I_n) : T^*M - 0 \to \mathbb{R}^n - 0$. More correctly, the lattice can be shifted by a Maslov index.

Since $\hat{H}$ is a first order elliptic symbol on $\mathbb{R}^n - 0$ it has an asymptotic expansion in homogeneous functions of the form:

$$\hat{H} \sim H_1 + H_o + H_{-1} + \ldots, \qquad H_j(rI) = r^j H_j(I).$$

Thus we have:

Proposition 16 *([CV.2]) Suppose $g \in \mathcal{R}$. Then $\sqrt{\Delta_g}$ is quantum torus integrable and hence there exists a polyhomogeneous function $\hat{H}$ such that $\sqrt{\Delta_g} = \hat{H}(\hat{I}_1,\hat{I}_2)$. Moreover, $H_o = 0$.*

The spectrum of $\sqrt{\Delta}$ is therefore given as the values $\{\hat{H}(m,\ell+\frac{1}{2}) : |m| \leq \ell, \ell > 0\}$ of $\hat{H}$ at lattice points in the convex polyhedral cone Γ (shifted by $(0,\frac{1}{2})$.

To break the homogeneity of the normal form coefficients, we fix a point $I^o \in \{H = 1\}$, let ω^o denote the common frequency vector of the ray of torii $\mathbb{R}^+ T_{I^o}$ and put

$$I \cdot \omega^o := \sum_{k=1}^{2} \omega_k^o I_k.$$

The equation of the tangent line to $\{H = 1\}$ at I^o in the action cone Γ_o is then given by $\omega^o \cdot I = 1$. The conic neighborhood will be parametrized in the following way: we fix a basis (i.e. a non-zero vector) v of the line $I \cdot \omega^o = 0$, and define the map

$$(\rho,\xi) \to \rho(I^o + \xi v), \qquad \xi \in (-\epsilon,\epsilon).$$

For sufficiently small ϵ, this map sweeps out a conic neighborhood W_o of I^o with inverse given by

$$\rho = \omega^o \cdot I, \quad \xi v_j := \frac{I_j}{I \cdot \omega^o} - I_j^o.)$$

Since $H(I_1, I_2) = (\omega^o \cdot I)\ \ H(\frac{I_1}{\omega^o \cdot I}, \frac{I_2}{\omega^o \cdot I})$ and since $(\frac{I_1}{\omega^o \cdot I}, \frac{I_2}{\omega^o \cdot I}) \in \{\omega^o \cdot I = 1\}$ we may write

$$H(I_1, I_2) = \rho h_{I^o}(\xi)$$

where h_{I^o} is the function on $W_o \cap \{\omega^o \cdot I = 1\}$ defined by

$$h_{I^o}(\xi) := H(I^o + \xi v).$$

The C^∞ Taylor expansion of $h_{I^o}(\xi)$ around $\xi = 0$ is then a symplectic invariant of H.

Similary, we write the higher homogeneous terms as

$$H_j(I_1, I_2) = (\omega^o \cdot I)^j\ \ H_j(\frac{I_1}{\omega^o \cdot I}, \frac{I_2}{\omega^o \cdot I}) := (\omega^o \cdot I)^j\ \ h_j(\xi).$$

The Taylor expansion of $h_j(\xi)$ around $\xi = 0$

$$h_j(\xi) = \sum_{\alpha \geq 0} h_j^\alpha(0)\xi^\alpha$$

then defines the quantum Birkhoff normal form coefficients.

5. Normal form from spectrum.

5.1 Inverse results for non-degenerate manifolds

Let us now sketch the proof of Theorem 1. The following is somewhat schematic; we refer to [Z.1,2] for the detailed proof.

Using the residue description, we have

$$a_{\gamma k} = \text{res } D_t^k e^{it\sqrt{\Delta}} := \text{Res}_{z=0} \text{Tr } D_t^k e^{it\sqrt{\Delta}} \sqrt{\Delta}^{-z}. \tag{1}$$

Since res is invariant under conjugation by (microlocal) unitary operators, and depends on only a finite jet of the Laplacian near γ, it may be calculated by conjugating to the normal form, and only depends on a finite part of the normal form. Applying D_t^k and formally exponentiating the terms of order ≤ -1 in D_s we get

$$\text{res } D_t^k e^{it\mathcal{D}}|_{t=L} = \text{res } e^{i2\pi L D_s} e^{iH_\alpha} \mathcal{D}^k (I + iL\frac{\tilde{p}_1(I_1, ..., I_n, L)}{D_s} + ...). \tag{2}$$

Since $e^{i2\pi D_s} = I$ on $\mathbb{R}/2\pi L\mathbb{Z}$ the Fourier Integral factor in (2) is just e^{iH_α}. This is an operator in the metapletic representation and its trace is the character of this representation. Recall that for elements of the metaplectic group $Mp(n, \mathbb{R})$ not having 1 as an eigenvalue, the character is given by

$$Ch(x) = \frac{i^\sigma}{\sqrt{|det(I - x)|}}$$

where σ is a certain Maslov index. For non-degenerate x with p pairs of eigenvalues $e^{\pm i\alpha_j}$ of modulus one, q pairs of positive real eigenvalues $e^{\pm\lambda_j}$ and c quadruplets of eigenvalues $e^{\pm(\mu_j \pm i\nu_j)}$, $Ch(x)$ is therefore given (up to a Maslov factor) by

$$(5.6) \qquad T(\alpha,\lambda,(\mu,\nu)) = \Pi_{j=1}^{p} \frac{e^{\frac{1}{2}i\alpha_j}}{1-e^{i\alpha_j}} \cdot \Pi_{j=1}^{q} \frac{e^{\frac{1}{2}\lambda_j}}{1-e^{\lambda_j}} \cdot \Pi_{j=1}^{c} \frac{e^{\frac{1}{2}(\mu_j+i\nu_j)}}{1-e^{\mu_j+i\nu_j}} \frac{e^{\frac{1}{2}(\mu_j-i\nu_j)}}{1-e^{\mu_j-i\nu_j}}.$$

Here we have selected one eigenvalue ρ from each symplectic pair ρ, ρ^{-1}. The ambiguity is fixed by the Maslov factor i^{σ}, which can (and will) be ignored below for the sake of brevity. For simplicity let us assume that γ is elliptic.

The next observation is that the residue is the trace of the term of order D_s^{-1}. This is formally obvious and the detailed justification is given in [Z.1, 2]. Therefore the wave invariant has the form

$$a_{\gamma k} = Tr\mathcal{F}_{k,-1}(\hat{I})e^{i\alpha\hat{I}}$$

for a certain polynomial $\mathcal{F}_{k,-1}$ which can easily be determined from the $p_j(\hat{I})$'s. We then observe that

$$Tr\mathcal{F}_{k,-1}(\hat{I})e^{i\alpha\hat{I}} = \mathcal{F}_{k,-1}(D_{\alpha_1},...,D_{\alpha_n})Tre^{i\alpha\hat{I}}.$$

Hence we get the result:

$$a_{\gamma k} = \mathcal{F}_{k,-1}(D) \cdot Ch(x)|_{x=P_\gamma}.$$

The coefficients of the polynomials $\mathcal{F}_{k,-1}$ are evidently polynomials in the quantum Birkhoff normal form invariants. For the inverse result, one observes (with Guillemin) that the differentiation process produces polynomials in the β_j's. Moreover, it is important to observe that the expressions for the wave invariants at iterates γ^k of γ involve the same operator $\mathcal{F}_{k,-1}$ applied to $Ch(k\alpha)$. Determining the Birkhoff normal form coefficients from the wave invariants is then equivalent to determining the coefficients of these polynomials from their special values at $\beta_j = (1-e^{ik\alpha_j})^{-1}$ corresponding to γ and its iterates. Under the irrationality condition, the points $(e^{i\alpha_1},\dots,e^{i\alpha_n})$ are dense on the torus and hence the special values determine the polynomial.

5.2 Inverse results for simple surfaces of revolution

We now prove that the wave trace invariants of a metric $g \in \mathcal{R}^*$ determine its quantum normal form $\hat{H}$. Below ψ_L is a smooth cutoff to a neighborhood of the periodic torus $\mathcal{T}_L$ of period L in the 'action space'. Also, by the vector of winding numbers one means the homology class of a closed geodesic on $\mathcal{T}_L$ relative to a fixed basis (it may be fixed independently of the torus here).

Proposition 17 *Let $g \in \mathcal{R}^*$ and let $L \in \mathcal{L}$. Then we have:*

$$\sum_{\pm} a_{\pm T_L,-\frac{1}{2}+k} = Res_{s=0} \int_{\Gamma} \psi_L(I+\mu)e^{i\langle M_L,I\rangle}e^{-iL\hat{H}(I+\mu)}(\hat{H}(I+\mu))^{-\frac{1}{2}+k}(\omega_L\cdot(I+\mu))^{-s}dI$$

where as above M_L is the vector of winding numbers of $\mathcal{T}_L$ and ω_L is the frequency vector of T_L.

Proof: Since $\sqrt{\Delta} = \hat{H}(\hat{I}_1, \hat{I}_2)$ and since $\hat{\psi}_L$ is a function of the action operators, we have that

$$\sum_{\pm} a_{\pm T_L, -\frac{1}{2}+k} = Res_{s=0} \sum_{N \in \mathbb{Z}^2} \psi_L(N+\mu) e^{iL\hat{H}(N+\mu)} (\hat{H}(N+\mu))^{-\frac{1}{2}+k} (\omega_L \cdot (N+\mu))^{-s}.$$

We then apply the Poisson summation formula for Re $s >> 0$ to replace the sum over $N \in \mathbb{Z}^2$ by

$$Res_{s=0} \sum_{M \in \mathbb{Z}^2} J_{L,M,k}(s)$$

where

$$J_{L,M,k}(s) := \int_{\Gamma} \psi_L(I+\mu) e^{-i\langle M, I\rangle} e^{iL\hat{H}(I+\mu)} (\hat{H}(I+\mu))^{-\frac{1}{2}+k} (\omega_L \cdot (I+\mu))^{-s} dI.$$

By simplicity of the length spectrum, only the term with $M = M_L$ has a pole at $s = 0$. Indeed, only in this term does the phase $-\langle M, I\rangle + LH(I)$ have a critical point, since $M = L\nabla_I H(I_M)$ implies that the torus with actions I_M is periodic of period L. □

Now let use calculate the residue using the Taylor coefficients of the normal form at the point $I^o = (0,1)$, corresponding to the torus of meridians between the poles. Some computation based on the residue formula above shows that

$$a_{T_L, -\frac{3}{2}} := c_L = \frac{1}{\sqrt{2\pi i \alpha L}} e^{i\langle M_L, \mu\rangle}$$

and that the higher wave invariants $a_{T_L, -\frac{1}{2}+k}$ are given by c_L times polynomials in L and in the derivatives of $h, h^{-1}, h_{-1}, \ldots$ at $\xi = 0$. For instance, the subprincipal wave invariant in dimension 2 is given given in terms of universal coefficients C_{ijk} by:

$$a_{T_L, -\frac{1}{2}} = c_L[C_{004}\partial_\xi^4 h(\xi)|_{\xi=0} + C_{010}L^2 h_{-1}(0) + C_{002}\partial_\xi^2 h(\xi)^{-\frac{1}{2}}|_{\xi=0}].$$

The key observation now is that the different terms decouple under iteration of the closed geodesics. That is:

Proposition 18 *We have*

$$a_{L,-\frac{1}{2}+k} = c_L P_k(L, h^{(2)}(0), \ldots, h^{(2k+4)}(0), h_{-1}(0), \ldots, h_{-1}^{(2k)}(0), \ldots, h_{-k}(0), h_{-k}^2(0), h_{-k-1}(0))$$

where P_k is a polynomial with the following properties:
(i) It involves only the first $2k+4$ Taylor coefficients of h at 0, the first $2k$ of h_{-1},..., the first $2k+2-2n$ of h_{-n} ..., the first 2 of h_{-k} and the 0th of h_{-k-1}.
(ii) It is of degree 1 in the variables $h_{-k-1}(0)$, $h_{-k}^{(2)}(0), \ldots, h^{(2k+4)}(0)$, and each occurs in precisely one term.
(iii) The L-order of the monomials containing these terms is respectively $L^{k+2}, L^{k+1}, \ldots, L^0$.

By using the joint $\rho, L \to \infty$ asymptotics one can recover the complete Taylor expansions of all the h_j's ¿from the wave invariants of the meridian torus and its iterates. A key role is played by the fact that each h_j is even under reflection thru the I_1-axis, stemming from the fact that the Laplacian is invariant under complex conjugation. Hence all of the odd Taylor coefficients of h_j vanish and the even ones are separated by different powers of L.

6. Metric from normal form.

The final step is to try to determine the metric from the normal form. At this time the only results pertain to the cases of surfaces of revolution and axisymmetric plane domains [CV.1]. We restrict our discussion to surfaces of revolution.

6.1 Simple surfaces of revolution

To complete the proof of the Theorem 3, we need to show that $\hat{H}$ determines a metric in $\mathcal{R}$. The proof is basically to write down explicit expressions for H and H_{-1} in terms of the metric (i.e. in terms of $a(r)$) and then to invert the expressions to determine $a(r)$. The first step is therefore to construct an initial part of the quantum normal form explicitly from the metric.

We begin from the fact that $H(I_1, I_2)$ is a known function. Set $H = E$ and solve for I_2. We know that $I_2 = |I_1| + \int(E - \frac{I_1}{a(r)^2})dr$ and hence the function $\int(E - \frac{I_1}{a(r)^2})dr$ is a spectral invariant. We may write the integral in the form

$$\int_{\mathbb{R}} (E - x)_+^{\frac{1}{2}} d\mu(x)$$

where μ is the distribution function of $\frac{1}{a^2}$, i.e.

$$\mu(x) := |\{r : \frac{1}{a(r)^2} \leq x\}|$$

with $|\cdot|$ the Lebesgue measure. The above integral is an Abel transform and as is well known it is invertible. Hence

$$d\mu(x) = \sum_{r:\frac{1}{a(r)^2}=x} |\frac{d}{dr}\frac{1}{a(r)^2}|^{-1} dx$$

is a spectral invariant. It follows that the function

$$J(x) := \sum_{r:a(r)=x} \frac{1}{|a'(r)|}$$

is known from the spectrum. By the simplicity assumption on a, there are just two solutions of $a(r) = x$; the smaller will be written $r_-(x)$ and the larger, $r_+(x)$. Thus, the function

$$J(x) = \frac{1}{|a'(r_-(x))|} + \frac{1}{|a'(r_+(x))|}$$

is a spectral invariant.

Clearly, $J(x)$ does not determine $a(r)$. Hence we must go into the H_{-1} expression. For various universal constants C_1, C_2, C_3 it takes the form

$$[C_1\partial_E^2 \int \frac{(a')^2}{a^6}(E-\frac{1}{a^2})_+^{-\frac{1}{2}}dr + C_2\partial_E \int \frac{(a')^2}{a^4}(E-\frac{1}{a^2})_+^{-\frac{1}{2}}dr$$

$$+C_3 \int \frac{(a')^2}{a^2}(E-\frac{1}{a^2})_+^{-\frac{1}{2}}dr]|_{E=\frac{H(I_1,I_2)}{I_1^2}}.$$

By a change of variables, we may rewrite this in the form

$$[C_1\partial_E^2 \int K(x)x^{\frac{3}{2}}(E-x)_+^{-\frac{1}{2}}dx + C_2\partial_E \int K(x)x^{\frac{1}{2}}(E-x)_+^{-\frac{1}{2}}dx$$

$$+C_3 \int K(x)x^{-\frac{1}{2}}(E-x)_+^{-\frac{1}{2}}dx]|_{E=\frac{H(I_1,I_2)}{I_1^2}}$$

where

$$K(x) = |a'(r_-(x))| + |a'(r_+(x))|.$$

All values of E which occur as ratios $\frac{H(I_1,I_2)}{I_1^2}$ give spectral invariants.

We now claim that K is a spectral invariant. To determine it we rewrite the above in terms of the fractional integral operators

$$I_\alpha f(E) = f * \frac{x_+^{\alpha-1}}{\Gamma(\alpha)}(E) = \frac{1}{\Gamma(\alpha)}\int_0^E f(y)(E-y)^{\alpha-1}dy$$

on the half-line $\mathbb{R}^+$. These operators satisfy

$$I_\alpha \circ I_\beta = I_{\alpha+\beta}, \qquad I_{-k} = (\frac{d}{dx})^k.$$

Thus the above equals $\mathcal{L}(K)$ where $\mathcal{L}$ is the fractional integral operator

$$\mathcal{L} := C_1 I_{-3/2}x^{\frac{3}{2}} + C_2 I_{-\frac{1}{2}}x^{\frac{1}{2}} + C_3 I_{\frac{1}{2}}x^{-\frac{1}{2}}.$$

To solve for K we apply $I_{-\frac{1}{2}}$ to $\mathcal{L}K$ to get

$$C_1'\frac{d}{dx}^2(x^{\frac{3}{2}}K(x)) + C_2'\frac{d}{dx}(x^{\frac{1}{2}}K) + C_3'x^{-\frac{1}{2}}K = I_{-\frac{1}{2}}\mathcal{L}K.$$

This equation determines K up to a solution f of the associated homogeneous equation, essentially an Euler equation. But also $K=0$ on $[0, a(r_o)^{-2}]$ Since no homogeneous solution can have this property, K is uniquely determined by this boundary condition.

Given two real numbers a, b one knows that from $a+b$ and $\frac{1}{a}+\frac{1}{b}$ one can determine the pair (a,b). Hence $a'(r)$ is determined from the spectrum. Since $a(0)=0$ this determines a and hence the surface. $\square$

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