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NONDEGENERACY CONDITIONS AND NORMAL FORMS FOR REAL HYPERSURFACES IN COMPLEX SPACE

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1. INTRODUCTION

Let M be a real-analytic hypersurface in $\mathbb{C}^N$ and let p_0 be a distinguished point on M. We shall address the problem of finding a normal form for M near $p_0 \in M$. We require that the normal form is "sufficiently unique" (we shall be more precise below) so that it carries useful information about the biholomorphic equivalence class of (M, p_0), i.e. those germs (M', p_0') of real-analytic hypersurfaces M' at $p_0' \in M'$ for which there is a germ at p_0 of a biholomorphic transformation $Z' = H(Z)$ such that $H(M) \subset M'$ and $H(p_0) = p_0'$.

We shall begin (in §2) by recalling the classical normal form for Levi nondegenerate hypersurfaces due to Chern–Moser [CM] (see also Cartan [C1–2] and Tanaka [T1–2]). Our aim is to find similar results for hypersurfaces that are Levi degenerate. In order to isolate the right class of hypersurfaces to consider, we will need to discuss (in §3) some fairly recent nondegeneracy conditions for real hypersurfaces, *holomorphic nondegeneracy* and *finite nondegeneracy*. The former was introduced by Stanton [S1] in connection with the study of infinitesimal CR automorphisms of real hypersurfaces, and the latter, which can be viewed as a generalization of Levi nondegeneracy, by Baouendi–Huang–Rothschild [BHR] in connection with a regularity problem for CR mappings. Both notions were further discussed and used by the author together with Baouendi and Rothschild in [BER1–3]. We shall present an intrinsic formulation of the definition of finite nondegeneracy and also present a sequence of invariants that distinguishes different ways that finite nondegeneracy can occur (Definition 3.25). After the preliminaries on nondegeneracy, we shall describe (in §4) a formal normal form for a certain class of finitely nondegenerate hypersurfaces in $\mathbb{C}^3$. This normal form was obtained by the author in [E].

First, however, let us introduce some notation. We denote by $\rho(Z, \bar{Z}) = 0$ a defining equation of M near p_0; thus, $\rho(Z, \bar{Z})$ is a real-analytic function near p_0 with $\rho(p_0, \bar{p}_0) = 0$ and $d\rho(p_0, \bar{p}_0) \neq 0$. We denote by $T_p^c(M)$ the complex tangent space of M at $p \in M$ and by $T_p^{0,1}(M)$ the CR tangent space of M at $p \in M$, i.e.

$$T_p^c(M) = T_p(M) \cap J_p(T_p(M)), \quad T_p^{0,1}(M) = \mathbb{C}T_p(M) \cap T^{0,1}(\mathbb{C}^N), \tag{1.1}$$

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where $J_p\colon T_p(\mathbb{C}^N) \to T_p(\mathbb{C}^N)$ is the complex structure map and $T_p^{0,1}(\mathbb{C}^N)$ is the usual tangent space of $(0,1)$-vectors in $\mathbb{C}T_p(\mathbb{C}^N)$. Thus, $T_p^c(M)$ is the subspace of the real tangent space that is invariant under the complex structure map J_p and $T_p^{0,1}(M)$ is the $-i$ eigenspace of J_p in $\mathbb{C}T_p^c(M)$. The CR tangent spaces all have complex dimension $n = N-1$ and form a subbundle $T^{0,1}(M)$ of the complexified tangent bundle $\mathbb{C}T(M)$. We refer to sections of $T^{0,1}(M)$ as CR vector fields, and denote by $L_1, \ldots, L_n$ a local basis for the real-analytic CR vector fields near p_0. Studying normal forms for the real hypersurface M in $\mathbb{C}^N$ is equivalent to studying normal forms for a basis of the real-analytic sections of the CR bundle $T^{0,1}(M)$ on the (abstract) real-analytic manifold M.

We may choose coordinates (z,w), with $z = (z_1, \ldots, z_n)$, vanishing at p_0 such that the defining equation of M at p_0 can be written

$$\operatorname{Im} w = \phi(z, \bar{z}, \operatorname{Re} w), \tag{1.2}$$

where $\phi(z,\bar{z},s)$ is a real-valued, real-analytic function near 0 with $\phi(0,0,0) = 0$ and $d\phi(0,0,0) = 0$. We can take $(z,s) \in \mathbb{C}^n \times \mathbb{R}$ to be local coordinates on M near p_0 via the mapping $(z,s) \mapsto (z, s + i\phi(z,\bar{z},s))$. In these coordinates, the following vector fields constitute a basis for the CR vector fields near p_0

$$L_j = \frac{\partial}{\partial \bar{z}_j} - \frac{i\phi_{\bar{z}_j}(z,\bar{z},s)}{1 + i\phi_s(z,\bar{z},s)} \frac{\partial}{\partial s}, \quad j = 1, \ldots, n, \tag{1.3}$$

where e.g. $\phi_s = \partial\phi/\partial s$.

2. Levi nondegenerate hypersurfaces

The hypersurface M, given by (1.2) near $p_0 = 0$ above, is called Levi nondegenerate at p_0 if the $n \times n$ matrix

$$\left(\frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k}(0,0,0)\right)_{1 \le j,k \le n}, \tag{2.1}$$

called the (extrinsic) Levi form of M at $p_0 = 0$, is nondegenerate. This can also be expressed intrinsically on the manifold M as follows. We define the (intrinsic) Levi form $\mathcal{L}_p\colon T_p^{0,1}(M) \to \mathbb{C}T_p(M)/\mathbb{C}T_p^c(M)$ at $p \in M$ by

$$\mathcal{L}_p(L_p) = \frac{1}{2i}\pi_p\left([L, \bar{L}]_p\right), \tag{2.2}$$

where π_p is the projection $\mathbb{C}T_p(M) \to \mathbb{C}T_p(M)/\mathbb{C}T_p^c(M)$ and L is a CR vector field that equals L_p at p. Then, M is Levi nondegenerate at p_0 if and only if the corresponding bilinear form is nondegenerate. We shall reformulate and generalize this notion in the next section.

Let us describe the classical Chern–Moser normal form for a Levi nondegenerate hypersurface. First of all, one can choose the coordinates (z,w) so that the equation for M at $p_0 = 0$ is

$$\operatorname{Im} w = \langle z, z \rangle + F(z, \bar{z}, \operatorname{Re} w), \tag{2.3}$$

where $\langle z, z\rangle$ is the Hermitean form

$$\langle z, z\rangle = \sum_{j=1}^{q} |z_j|^2 - \sum_{j=q+1}^{n} |z_j|^2. \tag{2.4}$$

If we require $n/2 \leq q \leq n$, then the number q is an invariant of M corresponding to the number of eigenvalues of a fixed sign the Levi form of M has at p_0. The function $F(z, \bar{z}, s)$ in (2.3) is a real-valued, real-analytic function which is $O(3)$ in the weighted coordinate system in which z carries the weight one and s the weight two, i.e. the Taylor series of $F(z, \bar{z}, s)$ contains only terms of weighted degree at least 3. Let us denote by $\mathcal{F}$ the space of such real-valued, real-analytic functions. We decompose elements of $\mathcal{F}$ according to $(z, \bar{z})$-type, i.e.

$$F(z, \bar{z}, s) = \sum_{k,l} F_{kl}(z, \bar{z}, s) \tag{2.5}$$

where $F_{kl}(z, \bar{z}, s)$ has type (k, l). The latter means that, for every $\lambda, \mu \in \mathbb{C}$,

$$F_{kl}(\lambda z, \mu \bar{z}, s) = \lambda^k \mu^l F_{kl}(z, \bar{z}, s). \tag{2.6}$$

The reality of $F(z, \bar{z}, s)$ is reflected by the fact that

$$F_{kl}(z, \bar{z}, s) = \overline{F_{lk}(z, \bar{z}, s)}, \tag{2.7}$$

for all k, l. We define the contraction $\operatorname{tr}\colon \mathcal{F}_{kl} \to \mathcal{F}_{k-1,l-1}$, where $\mathcal{F}_{kl}$ has the obvious meaning, as follows. We write $F_{kl}(z, \bar{z}, s)$ as

$$F_{kl}(z, \bar{z}, s) = \sum a^{\alpha_1 \ldots \alpha_k, \beta_1 \ldots \beta_l} z_{\alpha_1} \ldots z_{\alpha_k} \bar{z}_{\beta_1} \ldots \bar{z}_{\beta_l}, \tag{2.8}$$

where such a decomposition is made unique by requiring that the coefficients are invariant under permutations of the α's and of the β's, and define

$$\operatorname{tr} F_{kl} = \sum b^{\alpha_1 \ldots \alpha_{k-1}, \beta_1 \ldots \beta_{l-1}} z_{\alpha_1} \ldots z_{\alpha_{k-1}} \bar{z}_{\beta_1} \ldots \bar{z}_{\beta_{l-1}}, \tag{2.9}$$

where

$$b^{\alpha_1 \ldots \alpha_{k-1}, \beta_1 \ldots \beta_{l-1}} = \sum_{j=1}^{q} a^{\alpha_1 \ldots \alpha_{k-1} j, \beta_1 \ldots \beta_{l-1} j} - \sum_{j=q+1}^{n} a^{\alpha_1 \ldots \alpha_{k-1} j, \beta_1 \ldots \beta_{l-1} j}. \tag{2.10}$$

The space of normal forms $\mathcal{N} \subset \mathcal{F}$ is defined by

$$\mathcal{N} = \left\{ N \in \mathcal{F} \colon N = \sum_{\min(k,l) \geq 2} N_{kl},\ \operatorname{tr} N_{22} = 0,\ \operatorname{tr}^2 N_{32} = 0,\ \operatorname{tr}^3 N_{33} = 0 \right\}, \tag{2.11}$$

where $\operatorname{tr}^2 = \operatorname{tr} \circ \operatorname{tr}$, etc. Any biholomorphic transformation $(z', w') = H(z, w)$,

$$z' = \tilde{f}(z, w), \quad w' = \tilde{g}(z, w), \tag{2.12}$$

preserving the origin and the form (2.3) of M, i.e. the image $M' = H(M)$ is also of the form (2.3) for a possibly different $F \in \mathcal{F}$, can be uniquely factored as

$$H = T \circ R, \tag{2.13}$$

where R is an element of a certain group $\mathcal{R}$ of rational transformation (see [CM]) that corresponds to the action of the isotropy subgroup of $SU(q+1, n-q+1)$ on $\mathbb{CP}^n$ and where T is a transformation of the form

$$z' = z + f(z,w), \quad w' = w + g(z,w) \tag{2.14}$$

such that $f(z,w)$ is $O(2)$ (here, z has weight one and w has weight two), $g(z,w)$ is $O(3)$, and the following vanish

$$\frac{\partial f}{\partial w}(0,0), \quad \mathrm{Re}\, \frac{\partial^2 g}{\partial w^2}(0,0). \tag{2.15}$$

We denote the space of such biholomorphic transformations T by $\mathcal{G}_0$.

Theorem 2.16 (Chern–Moser [CM]). *Let M be a real-analytic hypersurface of the form (2.3). Then, given any $R \in \mathcal{R}$, there is a unique biholomorphic transformation $H = T \circ R$, with $T \in \mathcal{G}_0$, that transforms $(M,0)$ to normal form, i.e. such that the image $M' = H(M)$ is of the form*

$$\mathrm{Im}\, w = \langle z, z\rangle + N(z, \bar{z}, \mathrm{Re}\, w) \tag{2.17}$$

with $N \in \mathcal{N}$.

As a corollary, one obtains a bound on the dimension of the stability group of M at a Levi-nondegenerate point p_0, i.e. the group of biholomorphic transformations preserving the germ (M, p_0). We denote the stability group by $\mathrm{Aut}(M, p_0)$ and the bound we get is the following

$$\dim_{\mathbb{R}} \mathrm{Aut}(M, p_0) \leq \dim_{\mathbb{R}} \mathcal{R} = (n+1)^2 + 1. \tag{2.18}$$

Furthermore, one obtains the following description of the biholomorphic equivalence class of (M, p_0): *a germ (M', p_0') is biholomorphically equivalent to (M, p_0) if and only if for two, possibly different, choices of normalization $R, R' \in \mathcal{R}$ the two germs (M, p_0) and (M', p_0') can be brought to the same normal form via the unique transformations $H = T \circ R$ and $H' = T' \circ R'$, where $T, T' \in \mathcal{G}_0$ are provided by Theorem* 2.16, *respectively.*

3. Holomorphic nondegeneracy and finite nondegeneracy

Let us consider real-analytic hypersurfaces M that are Levi degenerate at a distinguished point $p_0 \in M$. We observe that if our aim is to obtain a theorem about normal forms similar in spirit to Theorem 2.16 above, then we must restrict our investigations to some proper subclass of hypersurfaces. The reason is the following: an important feature of Theorem 2.16 is the fact that the transformation to normal form is unique modulo the finite dimensional group $\mathcal{R}$, and any transformation to a normal form of a hypersurface M at a point $p_0 \in M$ can only be unique up to composition with elements of the stability group $\mathrm{Aut}(M, p_0)$. Thus, if $\mathrm{Aut}(M, p_0)$ is infinite dimensional, then another approach to normal forms must be taken. With this in mind, we proceed with the following definition (which will be given in an intrinsic form in Definition 3.16).

Definition 3.1. Let M be a real hypersurface in $\mathbb{C}^N$ defined near the point $p_0 \in M$ by $\rho(Z, \bar{Z}) = 0$, and let $L_1, \dots, L_n$ be a local basis for the CR vector fields on M near p_0. We say that M is *finitely nondegenerate* at p_0 if

$$\text{span} \left\{ L^J \left(\frac{\partial \rho}{\partial Z} \right) (p_0, \bar{p}_0) \colon |J| \leq k \right\} = \mathbb{C}^N, \tag{3.2}$$

for some integer k.

Here, we use the notation $L^J = L_{J_1} \dots L_{J_k}$ and $|J| = k$ for any integer valued k-vector $J \in \{1, 2, \dots, n\}^k$, and also

$$\frac{\partial \rho}{\partial Z} = \left(\frac{\partial \rho}{\partial Z_1}, \dots, \frac{\partial \rho}{\partial Z_N} \right). \tag{3.3}$$

This definition is independent of the defining function $\rho(Z, \bar{Z})$, the basis for the CR vector fields $L_1, \dots, L_n$, and the coordinates Z (see [BHR]). Moreover, the smallest integer $k = k_0$ for which (3.2) holds is an invariant and we say, more precisely, that M is *k_0-nondegenerate* at p_0. It is an easy exercise to show that M is Levi nondegenerate at p_0 if and only if it is 1-nondegenerate at p_0. The notion of finite nondegeneracy is connected with the following; by a holomorphic vector field, we mean a vector field of the form $\sum a_j(Z)\partial/\partial Z_j$, where the coefficients are holomorphic functions.

Definition 3.4. A real hypersurface M is said to be *holomorphically nondegenerate* at p_0 if there is *no* germ at p_0 of a holomorphic vector field that is tangent to M near p_0.

The relationship between these two notions and some of the basic properties regarding them can be summarized in the following proposition. The proof can be found in [BER1] (see also [BR] and [BHR]).

Proposition 3.5. *Let $M \subset \mathbb{C}^N$ be a connected real-analytic hypersurface. The following are equivalent.*

(i) *There exists $p_1 \in M$ such that M is holomorphically nondegenerate at p_1.*
(ii) *M is holomorphically nondegenerate at every point $p \in M$.*
(iii) *There exists $p_1 \in M$ such that M is finitely nondegenerate at p_1.*
(iv) *There exists a proper real-analytic subset V of M and an integer $\ell = \ell(M)$, with $1 \leq \ell \leq N - 1$, such that M is ℓ-nondegenerate at every $p \in M \setminus V$.*

We say that a connected real-analytic hypersurface is holomorphically nondegenerate if it is so at one point (and hence at all points). If M is holomorphically nondegenerate, then the number $\ell(M)$ provided by Proposition 3.5 (iv) is called the *Levi number* of M.

Stanton [S1–2] proved that the space of infinitesimal CR automorphisms of a holomorphically degenerate hypersurface is infinite dimensional. From this, it easily follows that if M is holomorphically degenerate, then $\mathrm{Aut}(M, p_0)$ is infinite dimensional for any $p_0 \in M$. In contrast, the following theorem was proved in [BER3].

Theorem 3.6 ([BER3]). *Let M be a real-analytic hypersurface in $\mathbb{C}^N$ that is k_0-nondegenerate at $p_0 \in M$. Then the stability group $\mathrm{Aut}(M, p_0)$ is a finite dimensional Lie group whose dimension is bounded by a number that only depends on k_0 and N.*

Thus, in view of the preceding discussion, it seems reasonable to try to obtain a normal form, in the spirit if Theorem 2.16, for the class of (M, p_0) where, for some fixed integer k_0, M is k_0-nondegenerate at p_0. In the next section, we begin this program by obtaining a formal, i.e. not necessarily convergent, normal form for real-analytic hypersurfaces M in $\mathbb{C}^3$ at points where M is 2-nondegenerate and has one non-zero eigenvalue of the Levi form; such points are generic on holomorphically nondegenerate real-analytic hypersurfaces in $\mathbb{C}^3$ that are everywhere Levi degenerate (see e.g. [E] for examples of such). Since the normal form that we obtain is only formal, the following theorem is needed to obtain information about the biholomorphic equivalence class of hypersurfaces M in $\mathbb{C}^3$ for which our normal form is valid.

Theorem 3.7 ([BER3]). *Let M and M' be real analytic hypersurfaces in $\mathbb{C}^N$ that are finitely nondegenerate at $p_0 \in M$ and $p_0' \in M'$, respectively. If $H(Z)$ is a formal equivalence, i.e. a formal invertible transformation, between (M, p_0) and (M', p_0'), then $H(Z)$ is convergent, i.e. there is a biholomorphic equivalence between (M, p_0) and (M', p_0') whose power series coincides with $H(Z)$.*

We shall conclude this section by giving an intrinsic description of finite nondegeneracy (Definition 3.16), similar to the one given for Levi nondegeneracy in §2. We shall also introduce a sequence of invariants (Definition 3.25) that is connected with the "data" of k-nondegeneracy of a CR manifold of hypersurface type.

Let θ denote a real 1-form on M near p_0 such that the annihilator of θ_p in $\mathbb{C}T_p(M)$ equals $\mathbb{C}T_p^c(M) = T_p^{0,1}(M) \oplus \overline{T_p^{0,1}(M)}$, i.e.

$$\theta_p^\perp = \mathbb{C}T_p^c(M), \tag{3.8}$$

at every point $p \in M$ in a neighborhood of p_0. The span of θ_p in $\mathbb{C}T_p^*(M)$, denoted by T_p^0, coincides with *the characteristic set* of the CR vector fields at p; hence, θ is sometimes called a *characteristic form.* We denote the annihilator of $T_p^{0,1}(M)$ by T_p'. Let us denote by $\omega_1, \ldots, \omega_n$ a system of 1-forms on M near p_0 such that $\omega_{1,p}, \ldots, \omega_{n,p}$ forms a dual basis to $L_{1,p}, \ldots, L_{n,p}$ at every $p \in M$ near p_0. We denote the span of $\omega_{1,p}, \ldots, \omega_{n,p}$ by T_p'' and we observe that the covectors $\overline{\omega}_{1,p}, \ldots, \overline{\omega}_{n,p}, \theta_p$ form a basis for T_p'. We define the linear operator $\mathcal{T}_j$, for $j = 1, \ldots, n$, mapping sections of T' to sections of T' as follows

$$\mathcal{T}_j\omega = \frac{1}{2i} L_j \lrcorner d\omega, \tag{3.9}$$

where the symbol $\lrcorner$ stands for the usual contraction operator. We leave to the reader the easy verification that $\mathcal{T}_j$ maps sections of T' to sections of T'. As above, for $J \in \{1, 2, \dots, n\}^k$, we shall use the notation

$$\mathcal{T}^J = \mathcal{T}_{J_1} \circ \dots \circ \mathcal{T}_{J_k}, \tag{3.10}$$

and we shall denote by $|J|$ the dimension k of J. We use the convention that $\mathcal{T}^J$, when $|J| = 0$, is the identity. Let us remark here that the operators $\mathcal{T}_j$ and $\mathcal{T}_k$ commute if the vector fields L_j and L_k do. This is straightforward to verify by applying the Jacobi identity for commutators of vector fields combined with the following useful identity for vector fields X, Y and a 1-form ω

$$\langle X \wedge Y, d\omega \rangle = X\left(\langle Y, \omega \rangle\right) - Y\left(\langle X, \omega \rangle\right) - \langle [X, Y], \omega \rangle .$$

Note that if $\rho(Z, \bar{Z}) = 0$ is the defining equation of M near p_0, then we may choose as our characteristic form $\theta = i(\partial - \bar{\partial})\rho$ which, since $\partial\rho = -\bar{\partial}\rho$ on M, can also be written $\theta = 2i\partial\rho$.

Proposition 3.11. *With notation as above, choose $\theta = i(\partial - \bar{\partial})\rho$. Then M is k_0-nondegenerate at p_0 if and only if*

$$\text{span}\,\{(\mathcal{T}^J\theta)_{p_0} \colon |J| \le k_0\} = T'_{p_0}. \tag{3.12}$$

Proof. Note that in the coordinates Z of the ambient space, we have

$$\theta = i(\partial - \bar{\partial})\rho = i \sum_{j=1}^{N} \left(\frac{\partial \rho}{\partial Z_j} dZ_j - \frac{\partial \rho}{\partial \bar{Z}_j} d\bar{Z}_j \right). \tag{3.13}$$

Using the fact that the vector field L_k is a $(0,1)$-vector field, i.e. is of the form

$$L_k = \sum_{j=1}^{N} a_j^k(Z, \bar{Z}) \frac{\partial}{\partial \bar{Z}_j}, \tag{3.13}$$

it is easy to calculate

$$\mathcal{T}_k\theta = \sum_{j=1}^{N} L_k \left(\frac{\partial \rho}{\partial Z_j} \right) dZ_j. \tag{3.14}$$

Repeating this argument, we obtain

$$\mathcal{T}^J\theta = \sum_{j=1}^{N} L^J \left(\frac{\partial \rho}{\partial Z_j} \right) dZ_j. \tag{3.15}$$

Since we have $(\mathcal{T}^J\theta)_{p_0} \in T'_{p_0}$ and since the dimension of T'_{p_0} equals $n + 1 = N$, the conclusion of the proposition follows from Definition (3.1). $\square$

Remark. Since M is Levi nondegenerate at p_0 if and only if it is 1-nondegenerate, we obtain the following reformulation of Levi nondegeneracy (whose validity is also easy to verify directly): M is Levi nondegenerate at p_0 if and only if the covectors $\theta_{p_0}, (\mathcal{T}_1\theta)_{p_0}, \ldots, (\mathcal{T}_n\theta)_{p_0}$ span T'_{p_0}.

Let us formulate an intrinsic definition of finite nondegeneracy for abstract CR structures of hypersurface type.

Definition 3.16. Let M be a smooth manifold and $\mathcal{V}$ a CR bundle of hypersurface type on M. Let θ be a real 1-form on M near $p_0 \in M$ such that $\theta_p^\perp = \mathcal{V}_p \oplus \overline{\mathcal{V}}_p$, let $L_1, \ldots, L_n$ be a basis for the CR vector fields on M near p_0, and let $\mathcal{T}_j$ be the corresponding linear operators (cf. (3.9)) on sections of T', where $T'_p = \mathcal{V}_p^\perp$. Then the CR bundle is said to be finitely nondegenerate at p_0 if

$$\mathit{span}\,\{(\mathcal{T}^J\theta)_{p_0} \colon |J| \le k\} = T'_{p_0}, \tag{3.16}$$

for some integer k. If k_0 is the smallest integer for which (3.16) holds then the CR bundle is said to be k_0-nondegenerate at p_0.

Let us verify that this definition is independent of the choice of θ and the choice of basis $L_1, \ldots, L_n$. We do this by proving the following two propositions that also reveal a sequence of invariants that is associated with k_0-nondegeneracy.

Proposition 3.17. *With notation as above, let $\tilde{L}_1, \ldots, \tilde{L}_n$ be another basis for the CR vector fields near $p_0 \in M$. Then, for any $J \in \{1, \ldots, n\}^k$ and any section ω of T', there exist $J^1 \in \{1, \ldots, n\}^{k_1}, \ldots, J^\nu \in \{1, \ldots, n\}^{k_\nu}$, with $k_j \le k$ for $j = 1, \ldots, \nu$, and smooth functions $a_1, \ldots, a_\nu$ near p_0 such that*

$$\widetilde{\mathcal{T}}^J\omega = \sum_{j=1}^{\nu} a_j \mathcal{T}^{J_j}\omega, \tag{3.18}$$

where $\widetilde{\mathcal{T}}_k$ is the linear operator on sections of T' that corresponds the CR vector field $\tilde{L}_k$ (see (3.9)).

Proof. Since there are smooth functions $b_1, \ldots, b_n$ such that

$$\tilde{L}_k = \sum_{j=1}^{n} b_j^k L_j \tag{3.19}$$

and the contraction operator is linear, (3.18) certainly holds for all J with $|J| = 1$. We proceed by induction on $|J|$. Assume (3.18) has been proved for $|J| \le \mu$. Then, for any J with $|J| \le \mu$ and any $k \in \{1, \ldots, n\}$, we have

$$\begin{aligned}
\widetilde{\mathcal{T}}_k\widetilde{\mathcal{T}}^J\omega &= \widetilde{\mathcal{T}}_k\left(\sum_j a_j\mathcal{T}^{J_j}\omega\right) \\
&= \sum_j \tilde{L}_k \lrcorner d\left(a_j\mathcal{T}^{J_j}\omega\right) \\
&= \sum_j \left(\tilde{L}_k \lrcorner \left(da_j \wedge \mathcal{T}^{J_j}\omega\right) + a_j\left(\sum_l b_l^k \mathcal{T}_l\mathcal{T}^{J_j}\omega\right)\right).
\end{aligned} \tag{3.20}$$

Now, the identity (3.18), for $|J| \leq \mu + 1$, follows from the identity

$$\begin{aligned} \tilde{L}_k \lrcorner \left(da_j \wedge \mathcal{T}^{J_j}\omega\right) &= \left(\tilde{L}_k \lrcorner da_j\right) \mathcal{T}^{J_j}\omega - da_j \left(\tilde{L}_k \lrcorner \mathcal{T}^{J_j}\omega\right) \\ &= \left(\tilde{L}_k \lrcorner da_j\right) \mathcal{T}^{J_j}\omega, \end{aligned} \tag{3.21}$$

where the last equality follows from the fact that $\mathcal{T}^{J_j}\omega$ is a section of T' and $\tilde{L}_k$ is a CR vector field. This concludes the proof of the proposition. □

Proposition 3.22. *With notation as above, let a be any smooth function and ω any section of T' near $p_0 \in M$. Then, for any $J \in \{1, \dots, n\}^k$, there exist $J^1 \in \{1, \dots, n\}^{k_1}, \dots, J^\nu \in \{1, \dots, n\}^{k_\nu}$, with $k_j \leq k$ for $j = 1, \dots, \nu$, and smooth functions $a_0, a_1, \dots, a_\nu$ near p_0 such that*

$$\mathcal{T}^J(a\omega) = a_0\omega + \sum_{j=1}^{\nu} a_j \mathcal{T}^{J_j}\omega. \tag{3.23}$$

Proof. We observe that

$$\begin{aligned} T_j(a\omega) &= L_j \lrcorner d(aw) \\ &= L_j \lrcorner \left(da \wedge \omega + ad\omega\right) \\ &= (L_j \lrcorner da)\omega - (L_j \lrcorner \omega)da + aT_j\omega \\ &= (L_j \lrcorner da)\omega + aT_j\omega, \end{aligned} \tag{3.24}$$

since $L_j \lrcorner \omega = 0$ as above. Thus, (3.23) holds for $|J| \leq 1$. Proposition 3.22 now follows from a simple inductive argument. We leave the details to the reader. □

Propositions 3.17 and 3.22 imply in particular that Definition 3.16 is independent of the choice of basis $L_1, \dots, L_n$ and the choice of characteristic form θ. Indeed, they imply that the span on the left hand side of (3.16), for every integer $k \geq 0$, is independent of the choice of basis and the choice of characteristic form. Thus, the following sequences are invariants of the k_0-nondegenerate CR structure.

Definition 3.25. Suppose M is k_0-nondegenerate at p_0. With notation as above, define μ_1, $1 \leq \mu_1 \leq k_0$, to be the smallest integer such that

$$E_1 = span \left\{(T^J\theta)_{p_0} \colon |J| \leq \mu_1\right\} \tag{3.26}$$

contains $E_0 = T^0_{p_0}$ as a *proper* subspace. Define λ_1, with $1 \leq \lambda_1 \leq n$, by

$$\lambda_1 = \dim E_1 - \dim E_0 = \dim E_1 - 1. \tag{3.27}$$

Then, define μ_j, λ_j, and E_j inductively as follows. Assume μ_l, λ_l, and E_l have been defined for $1 \leq l \leq j-1$, and let μ_j be the smallest integer $\mu_{j-1} < \mu_j \leq k_0$ such that

$$E_j = span \left\{(T^J\theta)_{p_0} \colon |J| \leq \mu_j\right\} \tag{3.28}$$

contains E_{j-1} as a *proper* subspace. Define λ_j, with $1 \leq \lambda_j \leq n$, by

$$\lambda_j = \dim E_j - \dim E_{j-1}.$$

We stop when $E_j = T'_{p_0}$ and denote this value of j by j_0.

Thus, for any CR manifold M of hypersurface type that is k_0-nondegenerate at $p_0 \in M$, we have defined two invariant sequences $(\mu_j)_{1\leq j\leq j_0}$,

$$1 \leq \mu_1 < \mu_2 < \ldots < \mu_{j_0} = k_0, \tag{3.28}$$

and $(\lambda_j)_{1\leq j\leq j_0}$,

$$\sum_{j=1}^{j_0} \lambda_j = n. \tag{3.29}$$

Observe, e.g., that if $\mu_1 = 1$, then λ_1 denotes the rank of the Levi form. Also, when M is k_0-nondegenerate at $p_0 \in M$, then it is also of finite (commutator) type in the sense of Kohn and Bloom–Graham. Indeed, if m denotes the type of M at p_0, then $m - 1 \leq \mu_1$, as is easy to verify.

In the next section, we shall consider 2-nondegenerate hypersurfaces in $\mathbb{C}^3$. In this case, there are only two possible sequences, namely (i) $j_0 = 2$, $(\mu_j) = (1,2)$, $(\lambda_j) = (1,1)$, and (ii) $j_0 = 1$, $(\mu_j) = 2$, $(\lambda_j) = 2$. However, we shall see that there are still other invariants attached to the lowest order terms in a normal form for such a hypersurface.

4. Real hypersurfaces in $\mathbb{C}^3$

Let us begin with some motivation. Suppose M is a (connected) real-analytic hypersurface in $\mathbb{C}^{n+1}$ and p_0 is a distinguished point on M. In view of Proposition 3.5, such a hypersurface is either holomorphically degenerate (at all points) or it is ℓ-nondegenerate, for some $1 \leq \ell \leq n$, outside a proper real-analytic variety $V \subset M$. There are plenty of examples, when $n \geq 2$, of real-analytic hypersurfaces that are holomorphically nondegenerate and for which $\ell \geq 2$ (see e.g. [E]).

Now, if M is holomorphically degenerate then, since $\mathrm{Aut}(M, p_0)$ in this case is infinite dimensional (cf. §3), there is no hope of obtaining any "finite dimensional" uniqueness in a transformation to normal form. On the other hand, at a generic point p near p_0 such a hypersurface can be locally transformed to $\tilde{M} \times \mathbb{C}^\nu$, where $1 \leq \nu \leq n$ and $\tilde{M}$ is a real hypersurface in $\mathbb{C}^{n+1-\nu}$, in view of the Frobenius theorem. Thus, the study of the biholomorphic equivalence class of M at such a point p can essentially be reduced to that of $\tilde{M}$ in $\mathbb{C}^{n+1-\nu}$.

If M instead is 1-nondegenerate (which is the same as Levi nondegenerate) at a point $p \in M$, then its biholomorphic equivalence class can be described using the Chern–Moser normal form.

Thus, in $\mathbb{C}^2$ real-analytic hypersurfaces are completely understood (in the sense that a normal form is known) at generic points, whereas in $\mathbb{C}^{n+1}$, with $n \geq 2$, there are real-analytic hypersurfaces for which a normal form is not known at any point. With this as our motivation, we shall consider real-analytic hypersurfaces M in $\mathbb{C}^3$ at 2-nondegenerate points $p_0 \in M$.

We have the following theorem from [E] describing a partial normal form for M, analogous to (2.3) in the Levi nondegenerate case.

Theorem 4.1 ([E]). *Let M be a real-analytic hypersurface in $\mathbb{C}^3$ and assume that M is 2-nondegenerate at $p_0 \in M$. Then (M, p_0) is biholomorphically equivalent to $(M', 0)$, where M' is a real-analytic hypersurface of one the following model forms.*

(i) *If the Levi form of M at p_0 has precisely one non-zero eigenvalue, then M' is one of the following:*

$$\text{(A.i.1)} \qquad \mathrm{Im}\ w = |z_1|^2 + |z_2|^2(z_2 + \bar{z}_2) + \gamma(z_1^2\bar{z}_2 + \bar{z}_1^2 z_2) + O(|z|^4 + |\mathrm{Re}\ w||z|^2),$$

where $\gamma = 0, 1$;

$$\text{(A.i.2)} \qquad \mathrm{Im}\ w = |z_1|^2 + (z_1^2\bar{z}_2 + \bar{z}_1^2 z_2) + O(|z|^4 + |\mathrm{Re}\ w||z|^2);$$

$$\text{(A.i.3)} \qquad \mathrm{Im}\ w = |z_1|^2 + |z_2|^2(z_1 + \bar{z}_1) + O(|z|^4 + |\mathrm{Re}\ w||z|^2).$$

(ii) *If the Levi form of M at p_0 is 0, i.e. both eigenvalues of the Levi form are zero, then M' is one of the following:*

$$\text{(A.ii.1)} \qquad \mathrm{Im}\ w = |z_1|^2(z_2 + \bar{z}_2) + r(z_1^2\bar{z}_2 + \bar{z}_1^2 z_2) + O(|z|^4 + |\mathrm{Re}\ w||z|^2),$$

where $r > 0$;

$$\text{(A.ii.2)} \quad \mathrm{Im}\ w = |z_1|^2(z_2+\bar{z}_2)+(z_1^2\bar{z}_2+\bar{z}_1^2 z_2)+i|z_1|^2(z_1-\bar{z}_1)+O(|z|^4+|\mathrm{Re}\ w||z|^2);$$

$$\text{(A.ii.3)} \quad \mathrm{Im}\ w = |z_1|^2(z_2+\bar{z}_2)+(z_1\bar{z}_2^2+\bar{z}_1 z_2^2)+|z_2|^2(\lambda z_2+\bar{\lambda}\bar{z}_2)+O(|z|^4+|\mathrm{Re}\ w||z|^2),$$

where $\lambda \in \mathbb{C}$, $\lambda \neq 0$;

$$\text{(A.ii.4)} \qquad \begin{aligned} \mathrm{Im}\ w =& |z_1|^2(z_1 + \bar{z}_1) + |z_2|^2(z_2 + \bar{z}_2) + (\mu z_1^2\bar{z}_2 + \bar{\mu}\bar{z}_1^2 z_2) + \\ & (\nu z_1\bar{z}_2^2 + \bar{\nu}\bar{z}_1 z_2^2) + O(|z|^4 + |\mathrm{Re}\ w||z|^2), \end{aligned}$$

where $\mu, \nu \in \mathbb{C}$, $\mu\nu \neq 1$.

$$\text{(A.ii.5)} \qquad \begin{aligned} \mathrm{Im}\ w =& |z_1|^2(\eta z_1 + \bar{\eta}\bar{z}_1) + (z_1^2\bar{z}_2 + \bar{z}_1^2 z_2) + \\ & (z_1\bar{z}_2^2 + \bar{z}_1 z_2^2) + O(|z|^4 + |\mathrm{Re}\ w||z|^2), \end{aligned}$$

where $\eta \in \mathbb{C}$.

Moreover, all of these models can be taken in regular form (see below) and are mutually non-equivalent, provided that we in (A.ii.4) *arrange so that $|\mu| \geq |\nu|$ and* $\arg \mu \geq \arg \nu$, *where* $\arg \mu, \arg \nu \in [0, 2\pi)$, *if* $|\mu| = |\nu|$.

In [E] a complete, formal, normal form is obtained for real hypersurfaces M that are of one of the forms (A.i.1–3). It is also proved that if M is everywhere Levi degenerate and 2-nondegenerate at p_0, then M is equivalent to a hypersurface M' of the form (A.i.2). For that reason and for the sake of brevity, we choose to present only the normal form for a hypersurface M of the form (A.i.2) here, and refer the reader to [E] for the other cases.

We subject a germ $(M,0)$, of the type (A.i.2), to a formal invertible transformation

$$z=\tilde{f}(z',w') \quad , \quad w=\tilde{g}(z',w'), \tag{4.2}$$

where $\tilde{f}=(\tilde{f}^1,\tilde{f}^2)$, such that the form (A.i.2) is preserved. We assign the weight one to the variables $z=(z_1,z_2)$, the weight two to w, and say that a polynomial $p_\nu(z,w)$ is weighted homogeneous of degree ν if, for all $t>0$,

$$p_\nu(tz,t^2w)=t^\nu p_\nu(z,w). \tag{4.3}$$

We shall write $O(\nu)$ for terms of weighted degree greater than or equal to ν. Similarly, we speak of weighted homogeneity of degree ν and $O(\nu)$ for polynomials and power series in $(z,\bar{z},\operatorname{Re} w)$, where $\bar{z}$ is assigned the weight one and $\operatorname{Re} w$ the weight two. The following is proved in [E].

Proposition 4.4. *A transformation* (4.2) *preserving regular form (see below) also preserves the form* (A.i.2) *if and only if the mapping is of the form*

$$\begin{aligned}
\tilde{f}^1(z,w)&=C^{1/2}e^{it}z_1+Dw-e^{2it}(2i\bar{D}+C^{1/2}\bar{A}e^{it})z_1^2+O(3)\\
\tilde{f}^2(z,w)&=Az_1+e^{2it}z_2+O(2)\\
\tilde{g}(z,w)&=Cw+2i\bar{D}C^{1/2}e^{it}z_1w+O(4),
\end{aligned} \tag{4.5}$$

where $C>0$, $t\in\mathbb{R}$, $A,D\in\mathbb{C}$.

We shall consider formal mappings (4.2) of the following form

$$(\tilde{f}(z,w),\tilde{g}(z,w))=(T\circ P)(z,w). \tag{4.6}$$

Here, $P(z,w)=(P_1(z,w),P_2(z,w),P_3(z,w))$ is a polynomial mapping with

$$\begin{aligned}
P_1(z,w)&=C^{1/2}e^{it}z_1+Dw-e^{2it}(2i\bar{D}+C^{1/2}\bar{A}e^{it})z_1^2+q_1(z,w)\\
P_2(z,w)&=Az_1+e^{2it}z_2+q_2(z,w)\\
P_3(z,w)&=Cw+2i\bar{D}C^{1/2}e^{it}z_1w,
\end{aligned} \tag{4.7}$$

where A,C,D,t are as in Proposition 4.4, q_1, q_2 are weighted homogeneous polynomials such that q_1 is $O(3)$ and q_2 is $O(2)$, and $T(z,w)$ is a formal mapping

$$T(z,w)=(z+f(z,w),w+g(z,w)), \tag{4.8}$$

where $f=(f^1,f^2)$, and g are formal power series in (z,w) such that f^1 is $O(3)$, f^2 is $O(2)$, and g is $O(4)$. Moreover, we require that the polynomials q_1, q_2 in (4.7) are of the form

$$\begin{aligned}
q_1(z,w)=&B_1z_1w+B_2z_2w+\sum_{|\beta|=3}C_\beta z^\beta+\sum_{|\alpha|=2}D_\alpha z^\alpha w+\\
&E_1z_1w^2+E_2z_2w^2+\sum_{|\beta|=3}F_\beta z^\beta w+Rz_1w^3\\
q_2(z,w)=&G_1z_1w+G_2z_2w+H_1z_1w^2+H_2z_2w^2,
\end{aligned} \tag{4.9}$$

for $B_k, C_\alpha, D_\beta, E_k, F_\beta, G_k, H_k \in \mathbb{C}$ and $R \in \mathbb{R}$, and that the formal series f^1, f^2 are such that the constant terms in the following formal series vanish (the indices j, k below range over $\{1,2\}$, α range over multi-indices with $|\alpha| = 2$, and β range over multi-indices with $|\beta| = 3$)

$$\tag{4.10} \begin{cases} \dfrac{\partial^2 f^j}{\partial z_k \partial w}, \dfrac{\partial^3 f^1}{\partial z^\beta}, \dfrac{\partial^3 f^1}{\partial z^\alpha \partial w}, \dfrac{\partial^3 f^j}{\partial z_k \partial w^2} \\ \dfrac{\partial^4 f^1}{\partial z^\beta \partial w}, \operatorname{Re} \dfrac{\partial^4 f^1}{\partial z_1 \partial w^3}. \end{cases}$$

Any transformation preserving the form (A.i.2) can be factored uniquely according to (4.6) into such a P and such a T. We say that a choice of P, as described above, is a choice of *normalization* for the transformations preserving (A.i.2).

We write the equation of M near 0 as follows

$$\tag{4.11} \operatorname{Im} w = |z_1|^2 + z_1^2 \bar{z}_2 + \bar{z}_1^2 z_2 + F(z, \bar{z}, \operatorname{Re} w).$$

Here, $F(z, \bar{z}, \operatorname{Re} w)$ is a real-valued, real-analytic function that is $O(4)$. In what follows, we shall consider $F(z, \bar{z}, s)$ as a formal power series

$$\tag{4.12} F(z, \bar{z}, s) = \sum_{\alpha, \beta, k} c^k_{\alpha\beta} z^\alpha \bar{z}^\beta s^k$$

consisting only of terms of weighted degree greater than 3 (here, s is assigned the weight two) and subjected to the reality condition

$$\tag{4.13} c^k_{\alpha\beta} = \overline{c^k_{\beta\alpha}}.$$

We shall denote by $\mathcal{F}$ the space of all such power series. In order to describe the normal form we shall need to decompose such a power series $F(z, \bar{z}, s)$ according to $(z, \bar{z})$-type

$$\tag{4.14} F(z, \bar{z}, s) = \sum_{k,l} F_{kl}(z, \bar{z}, s),$$

as in §2. In what follows, F_{kl}, H_{kl}, and N_{kl} denote formal power series of type (k, l). We define the space of normal forms $\mathcal{N}^2 \subset \mathcal{F}$ for the type (A.i.2) as follows: First, $N(z, \bar{z}, s)$ is in *regular form* which can be expressed (see e.g. [E] for another description and further discussion) by

$$\tag{4.15} N(z, \bar{z}, s) = \sum_{\min(k,l) \geq 1} N_{kl}(z, \bar{z}, s).$$

Moreover, the non-zero terms N_{kl} satisfy the following conditions.

$$\tag{4.16} \begin{aligned} &N_{33} \in \mathcal{N}^2_{33} \quad , \quad N_{43} \in \mathcal{N}^2_{43} \\ &N_{53} \in \mathcal{N}^2_{53} \quad , \quad N_{44} \in \mathcal{N}^2_{44} \\ &N_{54} \in \mathcal{N}^2_{54} \quad , \quad N_{55} \in \mathcal{N}^2_{55} \\ &N_{k1} \in \mathcal{N}^2_{k1} \quad , \quad N_{k2} \in \mathcal{N}^2_{k2}, \quad k = 1, 2, 3 \ldots, \end{aligned}$$

where

$$
\begin{aligned}
&\mathcal{N}^2_{11} = \left\{F_{11}\colon F_{11} = z_2H_{01} + \overline{z_2H_{01}}\right\} \\
&\mathcal{N}^2_{21} = \{F_{21}\colon F_{21} = \bar z_2H_{20} + \bar z_1z_2H_{10}\} \\
&\mathcal{N}^2_{31} = \left\{F_{31}\colon F_{31} = z_2H_{21} + z_1^3\bar z_2H_{00}\right\} \\
&\mathcal{N}^2_{22} = \{F_{22}\colon F_{22} = z_2\bar z_2H_{11}\} \\
&\mathcal{N}^2_{33} = \left\{F_{33}\colon F_{33} = z_2H_{23} + \overline{z_2H_{23}}\right\} \\
&\mathcal{N}^2_{43} = \left\{F_{43}\colon F_{43} = \bar z_2H_{42} + \bar z_1^3z_2^3H_{10}\right\} \\
&\mathcal{N}^2_{53} = \left\{F_{53}\colon F_{53} = \bar z_2H_{52} + \bar z_1^3z_2^4H_{10}\right\} \\
&\mathcal{N}^2_{44} = \left\{F_{44}\colon F_{44} = z_2^2H_{24} + \overline{z_2^2H_{24}} + z_1^3z_2\bar z_1^3\bar z_2H_{00}\right\} \\
&\mathcal{N}^2_{54} = \left\{F_{54}\colon F_{54} = z_2H_{44} + z_1^5\bar z_2^2H_{02}\right\} \\
&\mathcal{N}^2_{55} = \left\{F_{55}\colon F_{55} = z_2H_{45} + \overline{z_2H_{45}}\right\} \\
&\mathcal{N}^2_{k1} = \{F_{k1}\colon F_{k1} = \bar z_2H_{k0}\}, \quad k = 4,5,\ldots \\
&\mathcal{N}^2_{k2} = \{F_{k2}\colon F_{k2} = \bar z_2H_{k1}\}, \quad k = 3,4,\ldots .
\end{aligned} \tag{4.17}
$$

We are now in a position to state the theorem on normal forms for (A.i.2), which is a special case of Theorem B in [E].

Theorem 4.18 ([E]). *Let M be a real-analytic hypersurface in $\mathbb{C}^3$ given near $0 \in M$ in the form* (A.i.2) *as defined in Theorem* 4.1. *Then, given any choice of normalization (i.e. a choice of P as described above), there is a unique formal transformation* (4.2) *with this normalization that transforms the defining equation* (4.11) *of $(M, 0)$ to*

$$\operatorname{Im} w' = |z_1'|^2 + z_1^2\bar z_2 + \bar z_1^2 z_2 + N(z', \bar z', \operatorname{Re} w'), \tag{4.19}$$

where $N(z, \bar z, s) \in \mathcal{N}^2$.

Theorem 4.18 implies a bound on the dimension of the stability group of a real hypersurface M at a point $p_0 \in M$ where M is of the form (A.i.2), namely $\dim\operatorname{Aut}(M, p_0) \le 45$. Theorem B of [E] implies that the stability group at a point where M is of form (A.i.1) with $\gamma = 0$ satisfies $\dim\operatorname{Aut}(M, p_0) \le 17$, whereas if M is of the form (A.i.1) with $\gamma = 1$ or of the form (A.i.3), then $\dim\operatorname{Aut}(M, p_0) \le 19$. This improves the bound that can be deduced from the results in [BER3], which is $\dim\operatorname{Aut}(M, p_0) \le 102$. On the other hand, the latter bound is valid for all points where $M \in \mathbb{C}^3$ is 2-nondegenerate, i.e. also for points where M is of any of the forms (A.ii.1–5).

Combining Theorem 4.18 with Theorem 3.7 ([BER3]) and Theorem 4.1, we obtain the following the biholomorphic equivalence classes: *Let M and M' be real-analytic hypersurfaces in $\mathbb{C}^3$ that are of the form* (A.i.2) *at $p_0 \in M$ and $p_0' \in M'$, respectively. Then (M, p_0) and (M', p_0') are biholomorphically equivalent if and only if, for two (possibly different) choices of normalization as described in Theorem* 4.18, *(M, p_0) and (M', p_0') can be brought to the same normal form.*

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