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Distribution of resonances for convex co-compact hyperbolic surfaces

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1 Introduction.

The purpose of this exposé is to indicate how the methods of Sjöstrand for proving geometric bounds for the density of resonances [17] apply to the case of convex co-compact hyperbolic surfaces. We present the result of [27] which shows that the exponent in the Weyl estimate for the number of resonances in subconic neighbourhoods of the continuous spectrum is related to the dimension of the limit set of the corresponding Kleinian group. We also review some other results on the distribution of resonances for hyperbolic surfaces.

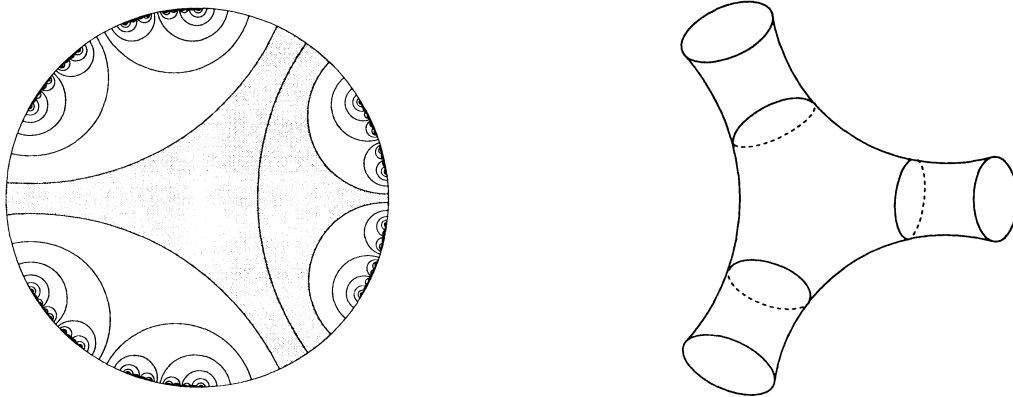


Figure 1: Tessellation by the Schottky group generated by inversions in three symmetrically placed circles each cutting the unit circle in an 110° angle, with the fundamental domain of its subgroup of direct isometries and the associated Riemann surface. The dimension of the limit set is $\delta = 0.70055063\dots$

Some of the reasons for studying the relation of the density of resonances for hyperbolic surfaces to the dimension of the limit set are

- In the case of hyperbolic surfaces we know optimal global lower bounds on the density of resonances (see Theorem 3 below and [6]) so showing that the density is lower near the continuous spectrum is particularly meaningful.

- Hyperbolic surfaces constitute a natural and well understood class of systems with hyperbolic dynamics and for which some computations can be made – see Fig.1¹. It is harder to construct potentials with hyperbolic flow for the corresponding Hamiltonian (see Sect.2 and [17]).
- For constant curvature surfaces the resonances coincide with the *nontrivial* poles of the meromorphic continuation of the corresponding dynamical Zeta function (see [14] and Sect.3), that is with the dynamical resonances (see [15]). The question of relating the density of zeros of the dynamical zeta function to the entropy can then be asked in greater generality².
- The structure of infinity is of course dramatically different in the hyperbolic case than in the euclidean case and that creates a new challenge in the study of resonances. The resolution of this in the two dimensional case is based on very special techniques while the results should be true for all negatively curved conformally compact analytic manifolds [10]. In particular the curvature should not need to be constant.

2 Semi-classical scattering in Euclidean space.

We will recall briefly the relevant result of Sjöstrand [17]. Let $P = -h^2\Delta + V(x)$ be a Schrödinger operator on \mathbb{R}^n with an analytic potential satisfying the (very) general assumptions of Helffer and Sjöstrand [7]. The resonances of P are defined as the poles of the meromorphic continuation of $(P - z)^{-1}$ on suitably defined spaces adapted to V – see [7]. We remark that a larger class of operators could also be considered.

If we are interested in resonances close to a fixed energy level $E_0 \in (0, \infty)$ then the dynamics of the Hamilton flow of the symbol of $P - E_0$, $p(x, \xi) = \xi^2 + V(x) - E_0$, near the energy surface $p^{-1}(0)$ plays a crucial rôle. The Hamilton flow is the flow of the Hamilton vector field $H_p = \partial_\xi p \partial_x - \partial_x p \partial_\xi$ and we assume that on the energy surfaces close to $p^{-1}(E_0)$ that flow is *hyperbolic* which is stronger than the assumption made in [17] (see Sect.3 for the definition of a hyperbolic Hamiltonian flow in the geometric situation).

Example 1. If we take $n = 2$, $V = \frac{2}{3}x_1^3 - 2x_1x_2^2$ and $E_0 = 2$ then the assumptions listed above are satisfied³.

With $p(x, \xi) = \xi^2 + V(x) - E_0$ and a fixed small $\epsilon_0 > 0$ we define, as in [1],

$$\begin{aligned}\Gamma_+ &= \{\rho \in p^{-1}([-\epsilon_0, \epsilon_0]) : \exp tH_p(\rho) \rightarrow \infty \text{ as } t \rightarrow -\infty\}, \\ \Gamma_- &= \{\rho \in p^{-1}([-\epsilon_0, \epsilon_0]) : \exp tH_p(\rho) \rightarrow \infty \text{ as } t \rightarrow \infty\},\end{aligned}$$

and then the *trapped set* :

$$K = \Gamma_- \cap \Gamma_+ \in T^*\mathbb{R}^n. \tag{2.1}$$

¹The example of the Schottky group used there and the dimension of the corresponding limit set were kindly provided by Curt McMullen. I would also like to thank Laurent Guillopé for his help with this and the other pictures.

²It is well possible that in place of our microlocal argument there exists a direct proof using the Zeta function.

³The hyperbolicity of the flow follows from a computer assisted argument, see [17].

In some heuristic sense K plays, in scattering, the rôle of the full cotangent bundle of a compact domain. Hence the dimension of K is important and the natural definition of dimension turns out to be the Minkowski dimension: for L a compact set in \mathbb{R}^k we put

$$\dim L = k - \sup \{ \mu : \overline{\lim}_{\epsilon \rightarrow 0} \epsilon^{-\mu} (\text{vol} \{x \in \mathbb{R}^k : d(x, L) < \epsilon\}) < \infty \}. \quad (2.2)$$

The set is called to be of *pure dimension* if the sup in the definition (2.2) is achieved. We can now quote the following consequence of Theorem 5.7 of [17]:

Theorem 1. *Let P be a Schrödinger operator with an analytic potential satisfying the assumptions listed above. Let m be equal to $\dim K$ when K is of pure dimension and otherwise let $m > \dim K$. Then there exists a constant $C_0 > 0$ such that for $0 < h \leq 1/C_0$, $C_0 h \leq \delta \leq 1/C_0$, the number of resonances of P in*

$$\left(E_0 - \frac{1}{2}\epsilon_0, E_0 + \frac{1}{2}\epsilon_0 \right) - i[0, \delta) \quad (2.3)$$

is

$$\mathcal{O} \left(\delta^{-\frac{m}{2}} \left(\frac{\delta}{h} \right)^n \right). \quad (2.4)$$

When $\delta = C_0 h$ the theorem is shown to be optimal by the results of Gérard and Sjöstrand [1]: in rectangles (2.3) with the above δ they described the precise location of resonances generated by single hyperbolic trajectory. In that case $m = 2$. We restrict ourselves to recalling

Example 2. Let $V(x) = x_1^2 - \sum_{k=2}^n x_k^2$ and $E_0 > 0$. Then the number of resonances in (2.3) with $\delta = Ch$, C sufficiently large is given by $(\alpha(\epsilon_0, C) + o(1))h^{-1}$.

We also recall that Sjöstrand [19] recently obtained optimal lower bounds, h^{-n}/C , in (2.3) with $\delta = 1/C$ and V , E_0 satisfying some natural assumptions (for instance for V satisfying some general assumptions and $V(x) \rightarrow 0$ as $x \rightarrow \infty$, $E_0 > 0$, it is required that $E_0 \in \text{sing supp}_a \{x : V(x) \geq \bullet\}$). It would be interesting to see the connection between the assumptions of Theorem 1 above and those of [19]. As in the hyperbolic case, the optimal lower bounds in larger regions make the finer upper bounds in smaller regions much more interesting.

3 Review of scattering on convex co-compact hyperbolic surfaces.

The convex co-compact surfaces can be defined as the infinite volume finite geometry quotients, $X = \Gamma \backslash \mathbb{H}^2$, for which the group Γ has only hyperbolic elements. The limit set of Γ , $\Lambda(\Gamma) \subset \partial \mathbb{H}^2$, is classically defined as the set of accumulation points of orbits of hyperbolic elements. As we will recall below it is naturally related to the dynamically defined *trapped set*, K of X , $K \subset T^*X \setminus 0$, [12, 23]. The convex co-compact condition can also be formulated as saying that the projection of the convex hull of the limit set from \mathbb{H}^2 to $\Gamma \backslash \mathbb{H}^2$ is compact. Resonances of X are equivalently given as either the poles of the meromorphic continuation of the resolvent, $R_X(s) = (\Delta_X - s(1-s))^{-1}$, the scattering matrix of X , or of the Eisenstein series – see [13],[6].

The surface X can then be decomposed as follows

$$X = N \cup Y_1^0 \cup \dots \cup Y_n^0 \quad (3.1)$$

where $\partial N = \partial Y_1^0 \cup \dots \cup \partial Y_n^0$ and

$$Y_j^0 \simeq [0, \infty)_r \times (\mathbb{R}/\ell_j \mathbb{Z})_y, \quad g|_{Y_j^0} \simeq dr^2 + \cosh^2 r dy^2. \quad (3.2)$$

We note that each boundary component of N is a geodesic on X . The compact set N is called the Fenchel-Nielsen region and it is the projection of the convex hull of $\Lambda(\Gamma)$ from \mathbb{H}^2 to X (and hence the name *convex co-compact*).

To recall the basic properties of the geodesic flow on X we will also denote by $g \in C^\infty(T^*X; \mathbb{R})$, the quadratic form dual to the metric. In particular, g is the symbol of the Laplace-Beltrami operator of (X, g) , Δ_X . The geodesic flow lifted to the cotangent bundle is given by

$$\exp tH_g : T^*X \longrightarrow T^*X,$$

where, as usual, H_g is the Hamilton vector field of g : $\omega(\bullet, H_g) = dg$ with ω , the canonical symplectic form on T^*X . The flow is hyperbolic (Anosov) in the following strong sense:

$$\begin{aligned} \exists C \forall m \in S^*X \exists E_m^\pm \quad E_m^0 \subset T_m(S^*X), \dim, E_m^\pm = \dim E_m^0 = 1, E_m^+ + E_m^- + E_m^0 = T_m(S^*X), \\ d(\exp tH_g)_m(E_m^\bullet) = E_{\exp tH_g(m)}^\bullet, H_g(m) \in E_m^0, \\ \|d(\exp tH_g)_m(v_\pm)\| \leq C \exp(\pm Ct) \|v_\pm\|, v_\pm \in E_m^\pm. \end{aligned} \quad (3.3)$$

We also note that if $\pi : T^*X \rightarrow X$ is the natural projection then

$$\pi(m) \notin \bar{N} \implies \pi(\exp tH_g(m)) \notin \bar{N} \quad \text{for either all } t > 0 \text{ or for all } t < 0. \quad (3.4)$$

The trapped set, K , defined by

$$\begin{aligned} K &= \Gamma_+ \cap \Gamma_- \\ \Gamma_\pm &= \{m \in T^*X \setminus 0 : \exists T(m) \forall \pm t > T(m) \exp tH_g(m) \in N\} \end{aligned} \quad (3.5)$$

is, in view of (3.4), contained in $T^*N \setminus 0$. Using the Patterson-Sullivan measure on $\Lambda(\Gamma)$ (see [12],[23]) Sullivan constructed an invariant measure on K and his construction shows that

$$\dim K = 2 \dim \Lambda(\Gamma) + 2, \quad (3.6)$$

where \dim denotes the Hausdorff dimension. As shown in [23] the Hausdorff dimension coincides in this case with the Minkowski dimension used in [17] and recalled above in (2.2) Since K is homogeneous we have

$$\dim K = \dim \tilde{K} + 1, \quad \tilde{K} = K \cap S^*X, \quad S^*X = \{m : g(m) = 1\},$$

and \tilde{K} is compact. Hence, using local coordinates, the definition (2.2) is applicable to \tilde{K} . In addition, $\Lambda(\Gamma)$ and \tilde{K} are of *pure dimension*, that is sup in the definition (2.2) is achieved or, in other words, that $\mathcal{H}^\delta(\Lambda(\Gamma)) < \infty$, $\delta = \dim \Lambda(\Gamma)$ – see [23].

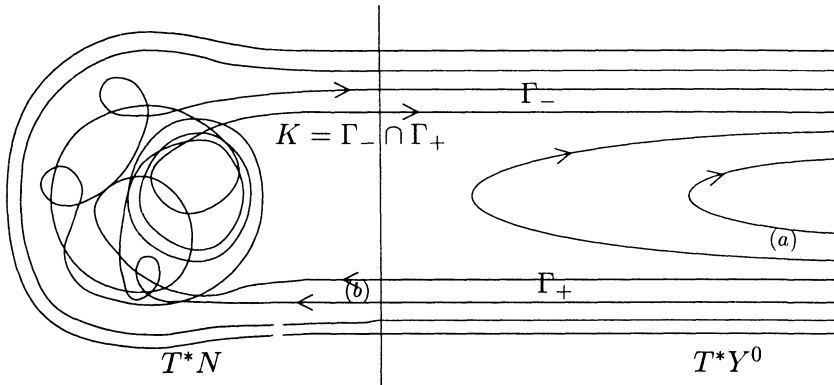


Figure 2: Schematic representation of the dynamics in T^*X (a) trajectories near infinity: bounded hyperbolic energy, large cylindrical energy. (b) escaping trajectory close to the trapped set K : bounded hyperbolic and cylindrical energy.

The dynamical structure of X is shown schematically in Fig. 2: by hyperbolic energy we mean the energy given by the Hamiltonian $g \in C^\infty(T^*X; \mathbb{R})$ and by the cylindrical energy, the energy given by the cylindrical metric on Y_j^0 , $\rho^2 + \eta^2$.

The dimension of $\Lambda(\Gamma)$, δ is also equal to the entropy of X which appears in the asymptotics of the counting function of the lengths of primitive closed geodesics on X , $\pi_X(l)$:

$$\pi_X(l) = \frac{e^{\delta l}}{\delta l} (1 + o(1)) , \quad l \longrightarrow \infty \quad (3.7)$$

see [2, 3, 9]. The dynamical zeta function of X is defined for $\text{Re } s \gg \frac{1}{2}$ as

$$Z_X(s) = \prod_{\{\gamma\}} \prod_{n=0}^{\infty} (1 - \exp(-(s+n)l(\gamma))) \quad (3.8)$$

where $\{\gamma\}$ denotes the conjugacy class of a primitive hyperbolic $\gamma \in \Gamma$ and $l(\gamma)$ is the minimal displacement length of $\{\gamma\}$, that is the length of the primitive closed geodesic corresponding to $\{\gamma\}$.

We will now review briefly some basic scattering theory of X referring to [13], [3] and [6] for detailed presentations.

Let Δ_X be the Laplace-Beltrami operator of (X, g) acting on functions. As a positive, unbounded self-adjoint operator on $L^2(X, d\text{vol}_g)$ with the domain $H^2(X, d\text{vol}_g)$, its spectrum, $\text{spec}(\Delta_X)$ is given by

$$\text{spec}(\Delta_X) = \{E_1, \dots, E_k\} \cup \left[\frac{1}{4}, \infty\right) , \quad 0 < E_1 < E_2 \leq \dots \leq E_k \leq \frac{1}{4}.$$

The first summand constitutes the pure point spectrum and the second the absolutely continuous one. Hence the resolvent of the Laplacian has the following mapping property:

$$R_X(s) \stackrel{\text{def}}{=} (\Delta_X - s(1-s))^{-1} : L^2(X, d\text{vol}_g) \longrightarrow H^2(X, d\text{vol}_g) , \quad \text{Re } s > \frac{1}{2} ,$$

$$s(1-s) \notin \text{spec}(\Delta_X) .$$

By shrinking the source space and enlarging the target space the operator, R_X continues meromorphically to all of \mathbb{C} :

$$R_X(s) : L_{\text{comp}}^2(X, d\text{vol}_g) \longrightarrow H_{\text{loc}}^2(X, d\text{vol}_g),$$

with poles of finite rank. Following [11] (see also [26]) we define the multiplicity of a pole of R_X as the rank of the full polar part of R_X :

$$m_s(R_X) = \epsilon(s) \dim \sum_{j=1}^k \text{Im} A_j, \quad \epsilon(s) = \begin{cases} 2 & s = \frac{1}{2} \\ 1 & s \neq \frac{1}{2} \end{cases}$$

$$R_X(s') = \sum_{j=1}^k \frac{A_j}{(s' - s)^j} + H(s'), \quad H(s') \text{ is holomorphic in } s' \text{ near } s,$$

By Lemma 2.3 of [6] this multiplicity is equal to the rank of the residue of R_X as a function of $s(1-s)$:

$$m_s(R_X) = \text{rank} \int_{\gamma_{s,\epsilon}} R_X(s')(1-2s') ds', \quad s \neq \frac{1}{2}, \quad \gamma_{s,\epsilon}(t) = s + \epsilon e^{it}, \quad t \in [0, 2\pi). \quad (3.9)$$

The poles of R_X are called *resonances* and their set, with multiplicities given by $m_s(R_X)$ is going to be denoted by \mathcal{R}_X . For the equivalence with other definitions we refer to [6].

Patterson and Perry [14] recently showed, in the generality including all even dimensions, that the non-trivial zeros of the dynamical zeta function, Z_X , given by (3.8) and continued meromorphically to \mathbb{C} (that continuation was proved in [3] for more general groups and with an order estimate in [14]) coincide with the resonances with multiplicities given by $m_s(R_X) - m_{1-s}(R_X)$, $\text{Re } s < 1/2$.

4 Distribution of resonances.

Perhaps the first result on the distribution of resonances for convex co-compact surfaces is the theorem of Patterson [12], later extended by Sullivan [23]:

Theorem 2. *For a convex co-compact surface, $X = \Gamma \backslash \mathbb{H}^2$, the maximal half plane free of poles of R_X is given by*

$$\{s : \text{Re } s > \delta\}, \quad \delta = \dim \Lambda(\Gamma),$$

where $\Lambda(\Gamma)$ is the limit set of Γ . In particular, if $\delta > \frac{1}{2}$ then the lowest eigenvalue of the Laplacian, Δ_X , is given by $\delta(1-\delta)$.

More recently Guillopé and the author [4],[6] obtained estimates on the number of resonances in discs for a larger class of surfaces with hyperbolic ends. In fact, the case of surfaces with hyperbolic ends is the only higher dimensional, non-degenerate, non-radial case in which optimal lower bound on the number of resonances is known and that provided additional motivation. This work is largely based on recent contributions to euclidean scattering theory by Melrose, Sjöstrand, Vodev and the author – see [25] and references given there. In the case discussed here we get

Theorem 3. *If $X = \Gamma \backslash \mathbb{H}^2$ is a convex co-compact surface and if \mathcal{R}_X denotes the set of resonances of X included according to their multiplicities, then for some $C > 0$ and for all $r > C$*

$$r^2/C \leq \#\{s \in \mathcal{R}_X : |s| \leq r\} \leq Cr^2.$$

As mentioned in the introduction the knowledge of the global lower bound motivates the study of finer upper bounds near the continuous spectrum similar to the semi-classical ones reviewed in Sect.2. In that direction we have [27]:

Theorem 4. *If $X = \Gamma \backslash \mathbb{H}^2$ is a convex co-compact surface and if \mathcal{R}_X denotes the set of resonances of X included according to their multiplicities, then for C sufficiently large, any α , $0 \leq \alpha \leq 1$, and $a, b > 0$, we have*

$$\#\left\{s \in \mathcal{R}_X : \frac{1}{2} - \operatorname{Re} s < a|\operatorname{Im} s|^\alpha + b, |s| \leq r\right\} = \mathcal{O}\left(r^{1+\alpha+\delta(1-\alpha)}\right), \quad \delta = \dim \Lambda(\Gamma) \quad (4.1)$$

where $\Lambda(\Gamma)$ is the limit set of Γ .

As proposed by Sullivan, X could be viewed from the dynamical point of view as a hyperbolic manifold of dimension $1+\delta$. If the resonances in a strip are considered as analogous to eigenvalues of compact surfaces then their Weyl estimate given by Theorem 1 above is $\mathcal{O}(r^{1+\delta})$. This is opposed to the usual estimate $\mathcal{O}(r^2) = \mathcal{O}(r^{\dim X})$ and is a quantum reflection of the dynamical point of view.

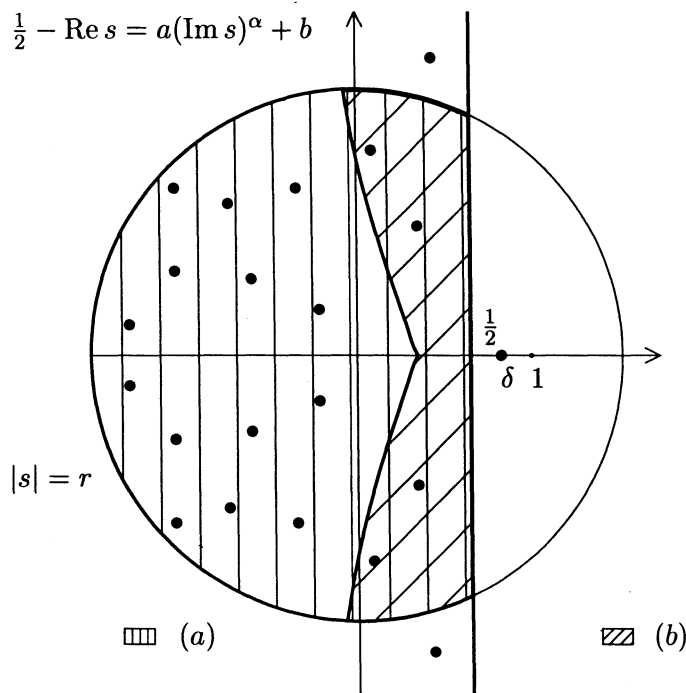


Figure 3: Density of resonances in different regions: (a) the number of resonances is in $[r^2/C, Cr^2]$; (b) the number of resonances is $\mathcal{O}(r^{1+\alpha+(1-\alpha)\delta})$.

In view of the recent results of Patterson and Perry [14] reviewed briefly in Sect.3, Theorem 4 provides an upper bound on the number of zeros of the dynamical zeta function in a neighbourhood of $\operatorname{Re} s = \frac{1}{2}$. It is perhaps possible that Theorem 4 could follow directly from the asymptotics of $l(\gamma)$ given (3.7) and the analysis of the meromorphic continuation of Z_X given by (3.8). Considering the difficulties involved in proving the existence of this meromorphic continuation that seems rather unlikely. What is the relation between the density of zeros of the dynamical zeta function and the entropy (the exponent δ in (3.7)) could however be asked in greater generality.

The most subtle part of the proof of Theorem 4 comes almost directly from [17] (with some modifications from [20],[22],[18]). A new difficulty is present however because of the more complicated structure of infinity which prevents an application of the general results of Helffer and Sjöstrand [7]. This difficulty appears quite serious for general analytic conformally compact analytic manifolds (in the sense of Mazzeo and Melrose [10]). In the special case of dimension two a detailed analysis of a neighbourhood of infinity based on separation of variables and a delicate "gluing" argument allow nevertheless a solution to this problem.

Theorem 4 is known to be optimal essentially in the same case as Theorem 1 recalled above. We have the following example of Epstein and Guillopé (see [2],[4]):

Example 3. For $l \geq 0$ let $X = \langle z \mapsto e^l z \rangle \backslash \mathbb{H}^2$, that is, X is a hyperbolic cylinder with one closed hyperbolic orbit constituting its trapped set. The limit set consists of two points and $\delta = 0$. The resonances lie on a lattice and the number of resonances in a set of the form (4.1) is $(C(a, b, \alpha) + o(1))r^{1+\alpha}$.

Finally we would like to describe some other lower bounds for the number of resonances of hyperbolic surfaces. A basic bound general was derived in [21]:

$$\sum_{\substack{|\operatorname{Im} s| < r \\ \operatorname{Re}(s - \frac{1}{2}) < \rho \log(s)}} e^{-(d-\epsilon)(\frac{1}{2} - \operatorname{Re} s)} > \left(\frac{1}{2\pi \sinh(d/2)} \sum_{ml(\gamma)=d} l - o_\epsilon(1) \right) r. \quad (4.2)$$

This bound implies that the density of resonances in logarithmic neighbourhoods of $\operatorname{Re} s = \frac{1}{2}$ is at least r . An easy modification of the argument of Ikawa [8] shows also that there are infinitely many resonances in some strip $\frac{1}{2} > \operatorname{Re} s > \frac{1}{2} - \alpha$. A little more is true: for large enough strips the density of resonances cannot be less than $r^{1-\epsilon}$, $\epsilon > 0$.

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