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RECENT EXISTENCE AND REGULARITY RESULTS FOR WAVE MAPS

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The setting. We consider maps u from $(m+1)$-dimensional Minkowski space to a compact, k-dimensional Riemannian manifold (N, g) with $\partial N = \emptyset$, the "target". By Nash's embedding theorem, we may assume that $N \subset \mathbb{R}^n$, isometrically, for some $n > k$. We denote as $T_pN \subset T_p\mathbb{R}^n \cong \mathbb{R}^n$ the tangent space of N at a point p, and we denote as $T_p^\perp N$ the orthogonal complement of T_pN with respect to the inner product $\langle \cdot, \cdot \rangle$ in $\mathbb{R}^n$. TN, $T^\perp N$ will denote, respectively, the corresponding tangent and normal bundles.

The space-time coordinates will be denoted as $z = (t, x) = (x^\alpha)_{0 \le \alpha \le m}$ and we denote as $\frac{\partial}{\partial x^\alpha} u = \partial_\alpha u = u_{x^\alpha}$ the partial derivative of u with respect to x^α, $0 \le \alpha \le m$. Also let $D = (\frac{\partial}{\partial t}, \nabla) = (\frac{\partial}{\partial x^\alpha})_{0 \le \alpha \le m}$ and let η be the Minkowski metric $\eta = (\eta_{\alpha\beta}) = (\eta^{\alpha\beta}) = (\eta_{\alpha\beta})^{-1} = \operatorname{diag}(-1, 1, \dots, 1)$. We raise and lower indeces with the metric. By convention, we tacitly sum over repeated indeces. Thus, for example, $\partial^\alpha = \eta^{\alpha\beta}\partial_\beta$. Moreover,

$$\Box = -\partial^\alpha \partial_\alpha = \frac{\partial^2}{\partial t^2} - \Delta$$

is the wave operator and

$$\frac{1}{2}\langle \partial^\alpha u, \partial_\alpha u \rangle = \frac{1}{2}\left(|\nabla u|^2 - |u_t|^2\right)$$

is the Lagrangean density of u.

Wave maps. A map u is a wave map if u is a stationary point for the action integral

$$\mathcal{A}(u; Q) = \frac{1}{2}\int_Q \langle \partial^\alpha u, \partial_\alpha u \rangle \, dz$$

with respect to compactly supported variations $u_\epsilon \colon \mathbb{R} \times \mathbb{R}^m \to N, |\epsilon| < \epsilon_0$, such that $u_\varepsilon = u$ outside a compact set in space-time and for $\varepsilon = 0$, in the sense that

$$\frac{d}{d\epsilon}\mathcal{A}(u_\epsilon; Q)|_{\varepsilon=0} = 0$$

for any $Q \subset\subset \mathbb{R} \times \mathbb{R}^m$ strictly containing the support of $u_\epsilon - u$.

Wave maps then satisfy the relation

$$\Box u \perp T_u N.$$

To understand this relation in more explicit terms, fix a point $z_0 \in \mathbb{R} \times \mathbb{R}^m$ and let $\nu_{k+1}, \dots, \nu_n$ be an orthonormal frame for $T_p^\perp N$, smoothly depending on

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$p \in N$ for p near $p_0 = u(z_0)$. Then we can find scalar functions $\lambda^l : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$, $k < l \leq n$, such that near $z = z_0$ there holds

$$\Box u = \lambda^l (\nu_l \circ u);$$

in fact,

$$\begin{aligned} \lambda^l &= \langle \Box u, \nu_l \circ u \rangle \\ &= -\partial^\alpha \langle \partial_\alpha u, \nu_l \circ u \rangle + \langle \partial_\alpha u, \partial^\alpha (\nu_l \circ u) \rangle \\ &= \langle \partial_\alpha u, d\nu_l(u) \cdot \partial^\alpha u \rangle = A^l(u)(\partial_\alpha u, \partial^\alpha u) \end{aligned}$$

is given by the second fundamental form A^l of N with respect to ν_l. Thus, the wave map equation takes the form

$$\Box u = A(u)(\partial_\alpha u, \partial^\alpha u) \perp T_u N, \tag{0.1}$$

where $A = A^l \nu_l$ is the second fundamental form of N.

Examples. i) For $N = S^k \subset \mathbb{R}^{k+1}$ equation (0.1) translates into the particularly simple equation

$$\Box u = \big(|\nabla u|^2 - |u_t|^2 \big) u.$$

Indeed, since $u \perp T_u S^k$ it suffices to check that

$$\langle \Box u, u \rangle = -\partial^\alpha \langle \partial_\alpha u, u \rangle + \langle \partial_\alpha u, \partial^\alpha u \rangle = |\nabla u|^2 - |u_t|^2.$$

ii) Suppose $\gamma : \mathbb{R} \to N$ is a geodesic parametrized by arc-length and $u = \gamma \circ v$ for some map $v : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$. Compute

$$-\Box u = \partial^\alpha \big(\gamma'(v) \partial_\alpha v \big) = \gamma''(v) \partial_\alpha v \partial^\alpha v - \gamma'(v) \Box v.$$

Note that γ' is parallel along γ; that is, $\gamma''(s) \perp T_{\gamma(s)} N$ for all $s \in \mathbb{R}$. Thus, u satisfies (0.1) if and only if v solves the linear, homogeneous wave equation $\Box v = 0$.

Basic questions. In view of the hyperbolic nature of equation (0.1), it is natural to ask whether the Cauchy problem for equation (0.1) for (sufficiently) smooth initial data

$$(u, u_t)\,|_{t=0} = (u_0, u_1) : \mathbb{R}^m \to TN \tag{0.2}$$

always admits a unique smooth solution for small time $|t| < T$. That is, we consider data $u_0 : \mathbb{R}^m \to N, u_1 : \mathbb{R}^m \to \mathbb{R}^n$ such that $u_1(x) \in T_{u_0(x)} N$ for almost every $x \in \mathbb{R}^m$.

The smoothness hypothesis on the solution and the data may be rather weak. In fact, for a function $u \in L^2_{\text{loc}}(\mathbb{R} \times \mathbb{R}^m; N)$ it is possible to interpret equation (0.1) in the sense of distributions provided $Du \in L^2_{\text{loc}}(\mathbb{R} \times \mathbb{R}^m)$. More generally, we may consider initial data (u_0, u_1) in Sobolev spaces $H^s \times H^{s-1}(\mathbb{R}^m; TN), s \geq 1$, and solutions u of class H^s, that is, such that $(u, u_t) \in L^\infty\big(\mathbb{R}; H^s \times H^{s-1}(\mathbb{R}^m; TN)\big)$.

Then we may ask for which s the initial value problem (0.1), (0.2) with data $(u_0, u_1) \in H^s \times H^{s-1}(\mathbb{R}^m; TN)$ admits a unique local solution of class H^s ("local well-posedness in H^s") and for which s this solution may be extended for all time and also preserves higher regularity properties of the data ("global well-posedness" and regularity).

A dimensional analysis tells us what we may hope for. Assigning scaling dimensions 1 to each coordinate x^α, 0 to the function u, the H^s-energy in m space dimensions has dimension $m - 2s$; that is, if $s > \frac{m}{2}$, no concentration discontinuities on length scales $L \to 0$ are possible if the H^s-energy of u remains bounded. We refer to this case as sub-critical, in contrast to the critical and supercritical cases $s = \frac{m}{2}, s < \frac{m}{2}$, respectively.

By a fixed point argument, using only classical energy estimates (for u and derivatives), for a general hyperbolic equation $\Box u = f(u, Du)$ with a smooth function f it is not hard to establish local well-posedness of the Cauchy problem in H^s, if $s > \frac{m}{2} + 1$.

Using, however, the special geometric, analytic, and algebraic structure properties of the wave map system, this result can be improved drastically.

Geometric structure. Orthogonality $\Box u \perp T_uN$ immediately implies the conservation law

$$0 = \langle \Box u, u_t \rangle = \frac{1}{2}\frac{d}{dt}|Du|^2 - \operatorname{div}\langle \nabla u, u_t \rangle.$$

Integrating over $\mathbb{R}^m$, if $Du(t)$ has spatially compact support, we obtain the energy identity

$$E(u(t)) := \frac{1}{2}||Du(t)||^2_{L^2(\mathbb{R}^m)} = \text{const}. \tag{0.3}$$

Similarly, we can argue for higher derivatives. Suppose $(u_0, u_1) \in H^2 \times H^1(\mathbb{R}^m; TN)$. Let ∂ be any first order spatial derivative. Differentiating equation (0.1), we obtain

$$\Box(\partial u) = \partial[A(u)(\partial_\alpha u, \partial^\alpha u)] = dA(u)(\partial u, \partial_\alpha u, \partial^\alpha u) + 2A(u)(\partial_\alpha \partial u, \partial^\alpha u)$$

with data

$$(\partial u, \partial u_t)\,|_{t=0} = (\partial u_0, \partial u_1) \in H^1 \times L^2(\mathbb{R}^m; \mathbb{R}^n).$$

Note that, since $\langle u_t, A(u)(\cdot, \cdot)\rangle = 0$ by orthogonality, we have

$$\langle \partial u_t, A(u)(\partial_\alpha \partial u, \partial^\alpha u)\rangle = -\langle u_t, dA(u)(\partial u, \partial_\alpha \partial u, \partial^\alpha u)\rangle.$$

Hence we obtain

$$\begin{aligned} \frac{d}{dt}E(\partial u(t)) &= \int_{\{t\}\times\mathbb{R}^m} \langle \Box(\partial u), \partial u_t\rangle\, dx \\ &\leq C||dA(u)||_{L^\infty} \cdot \int_{\mathbb{R}^m} |Du(t)|^3 |D^2 u(t)|\, dx. \end{aligned}$$

Since N is compact, dA is uniformly bounded on N. Moreover, by Sobolev's embedding, we can estimate

$$\int_{\mathbb{R}^m} |Du(t)|^3 |D^2 u(t)|\, dx \leq C||Du(t)||^{4-\alpha}_{L^2} ||D^2 u(t)||^\alpha_{L^2},$$

where $\alpha = 2, 3$, or 4 if $m = 1, 2$, or 3, respectively.

Thus, by (0.3) we arrive at a Gronwall type inequality

$$\frac{d}{dt}||D^2 u(t)||^2_{L^2} \leq C||D^2 u(t)||^\alpha_{L^2}.$$

A local-in-time H^2-bound follows. If $m = 1$, we have $\alpha = 2$, and we even obtain global unique H^2-solutions. We summarize these facts in the following result.

Theorem 0.1. *Suppose $m \leq 3$. Then for any data $(u_0, u_1) \in H^2 \times H^1(\mathbb{R}^m; TN)$ there exists a unique local solution u of class H^2. If $m = 1$, the solution extends uniquely for all time. If $(u_0, u_1) \in H^s, s > 2$, then so is u.*

For $m = 1$, the above result is due to Gu [11] and Ginibre-Velo [10]; in [17], Shatah gave a very elegant and concise proof. Finally, Yi Zhou [21] showed that the initial value problem is globally well-posed even in the energy space H^1.

For $m = 2, 3$ the above result also was obtained by Klainerman-Machedon [13] by a completely different technique. The above proof was first given in [20]; proof of Theorem 3.3. See also Choquet-Bruhat [2] for early results on wave maps.

Analytic structure. As illustrated best by the wave map system for maps to the sphere, equation (0.1) also exhibits the special analytic structure of "null forms" in the sense of Klainerman-Machedon [13].

As a simple model, consider solutions $u\colon \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ of the equation

$$\Box u = |\nabla u|^2 - |u_t|^2 \text{ on } \mathbb{R} \times \mathbb{R}^m \tag{0.4}$$

with initial data $u_{|t=0} = 0, u_{t|t=0} = u_1 \in H^{s-1}(\mathbb{R}^m)$.

Letting $v = e^u$, we compute

$$\Box v = e^u(\Box u - |\nabla u|^2 + |u_t|^2) = 0$$

with $v_{|t=0} = 1, v_{t|t=0} = u_1 \in H^{s-1}(\mathbb{R}^m)$.

By exact dependence of the solution v on its data in $H^s \times H^{s-1}(\mathbb{R}^m)$, we have $v \in C^0(\mathbb{R}; H^s(\mathbb{R}^m))$. On the other hand, a necessary condition for v to arise as $v = e^u$ from a (local) solution u to (0.4) is $v > 0$ (for short time), which requires $H^s(\mathbb{R}^m) \hookrightarrow L^\infty(\mathbb{R}^m)$, that is, $s > \frac{m}{2}$.

In remarkable agreement with this classical example, Klainerman-Machedon [14] establish the following result.

Theorem 0.2. *The initial value problem (1), (2) is locally well-posed for data* $(u_0, u_1) \in H^s \times H^{s-1}(\mathbb{R}^m; TN)$ *with* $s > \frac{m}{2}$.

This result underscores the importance of the critical case $s = \frac{m}{2}$, in particular, the case $s = 1$ in $m = 2$ space dimensions. Progress on this issue can be made by taking into account a third structure property of the wave map system.

Algebraic structure. As an illustration, first consider the case of a homogeneous target space $N = G/H$, where G is a Lie group and H is a discrete subgroup of G.

Then there exist Killing vector fields Y_i spanning T_pN at any point $p \in N$ and (0.1) is equivalent to the system of equations

$$0 = \langle \Box u, Y_i \circ u \rangle = -\partial^\alpha \langle \partial_\alpha u, Y_i \circ u \rangle + \langle \partial_\alpha u, dY_i(u) \cdot \partial^\alpha u \rangle$$

for all i. Since Y_i is Killing, the last term vanishes and we obtain the first order Hodge system

$$-\partial^\alpha \langle \partial_\alpha u, Y_i \circ u \rangle = 0 \tag{0.5}$$

for all i, equivalent to (0.1). This form of (0.1) immediately implies the following weak compactness result. Suppose (u^L) is a sequence of wave maps such that $u^L \to u$ in $L^2, Du^L \rightharpoonup Du$ weakly in L^2, locally, as $L \to \infty$. Then u again is a (weak) wave map.

Coupling this observation with a suitable scheme for obtaining approximate solutions to (0.1), Shatah [17] (for $N = S^k$), Yi Zhou [22] (for $m = 2$), and Freire [7] (for the general case) then obtain the following result.

Theorem 0.3. *Suppose $N = G/H$ is homogeneous. Then for any $(u_0, u_1) \in H^1 \times L^2(\mathbb{R}^m; TN)$ there exists a global weak solution u of* (0.1), (0.2) *of class H^1.*

In the case of a general target manifold, the algebraic structure giving rise to a Hodge system analogous to (0.5) was uncovered independently by Christodoulou-Tahvildar-Zadeh [3] and Hélein [12]. With no loss of generality (as shown by these authors) we may assume that TN is parallelizable; that is, there exists a smooth orthonormal frame field $\bar{e}_1, \dots, \bar{e}_k$ for TN. Given a (weak) wave map $u: \mathbb{R} \times \mathbb{R}^m \to N$, we then obtain a frame for the pull-back bundle $u^{-1}TN$ by letting

$$e_i(z) = R_{ij}(z)\bar{e}_j\big(u(z)\big) \text{ for } z = (t,x) \in \mathbb{R} \times \mathbb{R}^m,$$

where

$$R = (R_{ij}): \mathbb{R} \times \mathbb{R}^m \to SO(k).$$

Denote $\theta_i = \langle du, e_i \rangle = \theta_{i,\alpha} dx^\alpha, \omega_{ij} = \langle de_i, e_j \rangle = \omega_{ij,\alpha} dx^\alpha, 1 \le i,j \le k$.

Then (0.1) is equivalent to the system of equations

$$\begin{aligned} 0 = \langle \Box u, e_i \rangle &= -\partial^\alpha \langle \partial_\alpha u, e_i \rangle + \langle \partial_\alpha u, \partial^\alpha e_i \rangle \\ &= -\partial^\alpha \theta_{i,\alpha} + \omega_{ij}^\alpha \cdot \theta_{j,\alpha} =: \delta_\eta \theta_i + \omega_{ij} \cdot_\eta \theta_j \end{aligned} \tag{0.6}$$

for $1 \le i \le k$. Note that (0.6) is a first order Hodge system analogous to (0.5); however, (0.6) differs from (0.5) by a quadratic expression.

Using the Hodge structure (0.6), in joint work with A. Freire and S. Müller [8], [9] we obtain weak compactness of wave maps in $m = 2$ space dimensions.

Theorem 0.4. *Let $m = 2$. Suppose (u^L) is a sequence of wave maps such that $u^L \to u$ in L^2 and $Du^L \rightharpoonup Du$ weakly in L^2, locally on $\mathbb{R} \times \mathbb{R}^m$, as $L \to \infty$. Then u is a (weak) wave map.*

The proof makes contact with the work of Evans [5] and Bethuel [1] on the partial regularity of stationary harmonic maps. In particular, we also use special compensation properties of Jacobians ([4]) and $\mathcal{H}^1 - BMO$ duality ([6]).

The crucial determinant structure for the nonlinear term in (0.6) is achieved by localizing the equation to a compact domain which we then regard as contained in the fundamental domain of a torus $T^3 = \mathbb{R}^3/\mathbb{Z}^3$.

On T^3 (following Hélein [12]) we then impose the Coulomb gauge condition (with respect to the *Euclidean* background metric) by choosing, for each L, a "gauge" $R^L \in H^1(T^3; SO(k))$ such that

$$\sum_i \int_{T^3} |De_i^L|^2 \, dz = \min_R \sum_i \int_{T^3} |D(R_{ij}(\bar{e}_j \circ u^L))|^2 \, dz.$$

In this gauge, we have

$$\partial_\alpha \omega_{ij,\alpha} = \delta_{eucl} \omega_{ij} = 0,$$

and (e_i^L) is bounded in $H^{1,2}(T^3)$ with

$$\sum_i \int_Q |De_i^L|^2 \, dz \le \sum_i \int_Q |D(\bar{e}_i \circ u^L)|^2 \, dz \le CE\big(u^L(0)\big) \le C.$$

Hence we may assume that $e_i^L \rightharpoonup e_i$ weakly in $H^{1,2}(T^3)$ and

$$\theta_i^L = \langle du^L, e_i^L \rangle = \theta_{i,\alpha}^L \, dx^\alpha \rightharpoonup \theta_i = \langle du, e_i \rangle,$$
$$\omega_{ij}^L = \langle de_i^L, e_j^L \rangle = \omega_{ij,\alpha}^L \, dx^\alpha \rightharpoonup \omega_{ij} = \langle de_i, e_j \rangle$$

weakly in L^2 as $L \to \infty$.

Using the Hodge $*$-operator (with respect to η), we may express

$$\omega_{ij}^L \cdot_\eta \theta_j^L dz = \omega_{ij}^L \wedge (*_\eta \theta_j^L).$$

By Hodge decomposition (with respect to the Euclidean metric on T^3), moreover, we have

$$*_\eta \theta_j^L = da_j^L + \delta_{eucl} b_j^L + c_j^L,$$

where $a_j^L \rightharpoonup a_j, b_j^L \rightharpoonup b_j, c_j^L \to c_j$ in $H^1(T^3)$ as $L \to \infty$. The harmonic forms c_j^L are constant multiples of the basis vectors $dx^\alpha \wedge dx^\beta$; hence $c_j^L \to c_j$ smoothly, as $L \to \infty$, and $\omega_{ij}^L \cdot_\eta c_j^L \rightharpoonup \omega_{ij} \cdot_\eta c_j$ in $\mathcal{D}'$. Using the Coulomb gauge condition, and letting $\beta_j^L = *b_j^L$, the second term may be re-written

$$\omega_{ij}^L \wedge \delta_{eucl} b_j^L = \delta_{eucl}(\omega_{ij}^L \beta_j^L) \, dz,$$

which tends to the desired distributional limit. Similarly, for the third term we have

$$\omega_{ij}^L \wedge da_j^L = -d(\omega_{ij}^L \wedge a_j^L) + d\omega_{ij}^L \wedge a_j^L.$$

Again, it is easy to pass to the limit $L \to \infty$ in the divergence term. The last term, finally, possesses a determinent structure

$$d\omega_{ij}^L \wedge a_j^L = de_i^L \wedge de_j^L \wedge a_j^L.$$

Using the Hardy space estimates for Jacobians of [4] and $\mathcal{H}^1 - BMO$ duality of [6] we are able to show that, as $L \to \infty$,

$$de_i^L \wedge de_j^L \wedge a_j^L \rightharpoonup de_i \wedge de_j \wedge a_j + \nu \text{ in } D',$$

and to characterize the defect measure ν in a way analogous to P.L. Lions' [15] concentration-compactness principle. In particular, from energy estimates we derive that the H^1-capacity of the support of ν vanishes. But, passing to the limit $L \to \infty$ in (0.6), on the other hand we have

$$0 = \delta_\eta \theta_i^L + \omega_{ij}^L \cdot_\eta \theta_j^L \rightharpoonup \delta_\eta \theta_i + \omega_{ij} \cdot_\eta \theta_j + \nu \text{ in } \mathcal{D}';$$

that is,

$$\nu = -\delta_\eta \theta_i - \omega_{ij} \cdot_\eta \theta_j \in H^{-1} + L^1(T^3),$$

and hence $\nu = 0$.

Finally, in joint work with S. Müller [16] we couple the above weak compactness argument with the viscuous approximation method suggested by Yi Zhou [22] to obtain

Theorem 0.5. *Let $m = 2$. Then for any $(u_0, u_1) \in H^1 \times L^2(\mathbb{R}^m; TN)$ there exists a global weak solution to the Cauchy problem* (0.1), (0.2).

It remains to question whether this solution is unique and regular for smooth data.

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