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## Inverse Scattering at Fixed Energy for Stratified Media

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In this report we describe work in progress on inverse scattering for the wave equation in a layered medium. We consider the wave equation in $R \times R^n, n \geq 3$, with a variable sound speed, $c(x)$,

$$\partial_t^2 u = c^2(x)\Delta u \tag{1}$$

as a perturbation of the wave equation with a sound speed, $c_0(x_n)$, which is a function of one variable,

$$\partial_t^2 u = c_0^2(x_n)\Delta u. \tag{2}$$

Thus the unperturbed wave equation could be used to model wave propagation in a medium composed of uniform layers with different physical properties. When one takes the scattering amplitude at fixed energy as the observed data, simple examples, e.g. infinitesimal perturbations of a homogeneous medium, show that it is not reasonable to expect to recover more than the Fourier transform of the perturbation restricted to a ball from this data. Hence one needs to assume that the perturbation will be determined by this restricted Fourier transform, and a natural way to do this is to assume exponential decay of the perturbation. We assume

$$|\partial_x^\alpha(c_0^{-2}(x_n) - c^{-2}(x))| < Ce^{-\mu|x|}, \tag{3}$$

for $|\alpha| \leq n$. We assume that the sound speed $c_0$ is n-times differentiable, and constant outside a bounded interval, i.e., $c_0(s) = c_+$ for $s > r$ and $c_0(s) = c_-$ for $s < -r$. For definiteness we assume $c_+ > c_-$. With these hypotheses we have

Theorem: $c_0(x_n)$ and the scattering amplitude at energy $k^2$ determine $c(x)$.

Our work is closely related to a recent paper of H. Isozaki [I] which proves the same result in the case that $c_0$ takes the constant value $c_+$ for $x_n > 0$ and the constant value $c_-$ for $x_n < 0$. This choice enables him to replace $n$ by 0 in the hypothesis (3), and use perturbations in $L^\infty$, but it does not include any problems with "guided waves". The work presented here and in [I] is partially based on [ER], but Isozaki uses a method based on the Mourre estimate to prove estimates which become elementary under the hypotheses we have used here.

### 1. The Scattering Amplitude

The statement of the theorem assumes that one knows what the scattering amplitude is for this problem. Actually the scattering amplitude here is more complicated than the familiar amplitude of the outgoing scattered wave in potential scattering. To define the scattering amplitude one begins with the "plane wave" solutions of (2):

$$u(t,x) = e^{-ikt}\Phi_\alpha(x,\delta,\lambda), \delta \in R^n, \lambda \in R.$$

Thus $c_0^2\Delta\Phi_\alpha + k^2\Phi_\alpha = 0$, and we must choose the set of $\Phi_\alpha$ to be a complete set of (generalized) eigenfunctions at eigenvalue $k^2$. For this we take $\Phi_\alpha(x,\delta,\lambda) = exp(ix' \cdot \delta)\phi_\alpha(x_n,\lambda)$, where

$$c_0^2(x_n)(-\delta^2 + \partial_{x_n}^2)\phi_\alpha = -k^2\phi_\alpha,$$

i.e.,

$$G\phi_\alpha = (-\partial_{x_n}^2 - \frac{k^2}{c_0^2(x_n)})\phi_\alpha = -\delta^2\phi_\alpha = \lambda\phi_\alpha$$

One should think of the operator $G$, defined by the first equality above, as a Schrödinger operator on the line with a potential taking the values $-k_+^2 = -k^2/c_+^2$ for $x_n > r$ and $-k_-^2 = -k^2/c_-^2$ for $x_n < -r$. We choose the $\phi_\alpha$ so that

$$\Psi : C_0^\infty(R) \to \oplus_\alpha L^2(I_\alpha), [\Psi_\alpha f](\lambda) = \int_R \phi_\alpha^*(s,\lambda)f(s)ds$$

extends to a unitary spectral representation for $G$ in $L^2(R)$. The spectrum of $G$ is continuous with multiplicity two for $\lambda > -k_+^2$. The corresponding generalized eigenfunctions are $\phi_\alpha(s,\lambda)$ with $\alpha = 1,2$ on the interval $[-k_+^2, \infty) = I_\alpha, \alpha = 1,2$. The spectrum of $G$ is continuous with multiplicity one for $-k_+^2 > \lambda > -k_-^2$, and the generalized eigenfunctions for this are $\phi_3(s,\lambda)$ on the interval $[-k_-^2, -k_+^2) = I_3$. Below $-k_-^2$ there can be finite number of simple eigenvalues $\{\lambda_1, ..., \lambda_N\}$. These give rise to the "guided waves", and we parametrize them by their normalized eigenfunctions $\phi_\alpha(s,\lambda_\alpha), \alpha = 4, ..., N+3$. For these $I_\alpha = \{\lambda_\alpha\}$ and $L^2(I_\alpha)$ becomes $C$. For simplicity in notation it is convenient to assume that the $\phi_\alpha$ are extended to be identically zero for $\lambda$ outside the intervals where they are eigenfunctions.

With the plane waves described above the scattering amplitude is defined in the conventional way. One considers the distorted plane waves, i.e. the solutions of (1) of the form $u(t,x) = exp(-ikt)(\Phi_\alpha(x,\delta,-\delta^2) + v(x))$ where $v$ is outgoing. The outgoing solution $v$ is defined by the limiting amplitude principle as $v = \lim_{\epsilon\to 0_+} v_\epsilon$, where $v_\epsilon$ is the $L^2$ solution to

$$(-c^2\Delta - k^2 - i\epsilon)v_\epsilon = -(-c^2\Delta - k^2)\Phi_\alpha,$$

or

$$(-\Delta - \frac{k^2 + i\epsilon}{c^2})v_\epsilon = -q\Phi_\alpha,$$

with

$$q = k^2(c^{-2} - c_0^{-2}).$$

The validity of the limiting amplitude principle here, i.e. the existence of the limit of $v_\epsilon$ in suitable spaces, is proven in [dBP], [BDG] and [BdMM]. Setting $g = -q(v + \Phi_\alpha)$ and

$$h = \hat{g} = \int_{R^{n-1}} e^{-i\xi\cdot x'} g(x', x_n)dx',$$

we have

$$v = \lim_{\epsilon \to 0_+} (-\Delta - \frac{k^2}{c_0^2} - i\epsilon)^{-1} g = \lim_{\epsilon \to 0_+} \int_{R^{n-1}} e^{-i\xi \cdot x'} [(G + \xi^2 - i\epsilon)^{-1} h(\xi, \cdot)](x_n) dx'.$$

This gives us the following version of the Lipman-Schwinger equation

$$\begin{aligned} h(\xi, x_n) + (2\pi)^{1-n} \int_{R^{n-1}} \hat{q}(\xi - \eta, x_n)[(G + \eta^2 - i0)^{-1} h(\eta, \cdot)](x_n) d\eta \\ = -\hat{q}(\xi - \delta, x_n)\phi_\alpha(x_n, \lambda). \end{aligned} \tag{4}$$

Note that we have suppressed some of the variables in $h$. We really have $h(\xi, x_n; \delta, \lambda, \alpha)$ and are primarily interested in its restriction to $\lambda = -\delta^2$. However, to simplify notation we will often suppress some variables, usually in blocks separated by semi-colons.

The scattering amplitude for our problem is defined to be the union over all pairs $(\alpha, \beta)$ of the functions $[\Psi_\alpha h(\xi, \cdot; \delta, -\delta^2, \beta)](-\xi^2), \xi, \delta \in R^{n-1}$. To reconcile this definition with one's intuition that the scattering amplitude ought to have something to do with the asymptotics of the scattered wave $v$ as $|x|$ goes to infinity, we include a short digression.

Consider the representation of $v$ in terms of the spectral representation $\Psi$,

$$v = \lim_{\epsilon \to 0_+} (2\pi)^{1-n} \int_{R^{n-1}} e^{-i\eta \cdot x'} \sum_\alpha \Psi_\alpha^* [(\lambda + \eta^2 - i\epsilon)^{-1} [\Psi_\alpha h(\eta, \cdot)](\lambda)](x_n) d\eta$$

$$v = \sum_\alpha (2\pi)^{1-n} \int_{R^{n-1}} d\eta \int_{I_\alpha} e^{ix' \cdot \eta} \frac{\phi_\alpha(x_n, \lambda)[\Psi_\alpha h(\eta, \cdot)](\lambda)}{\lambda + \eta^2 - i0} d\lambda.$$

Here $d\lambda$ becomes the Dirac measure at $\lambda_j$ when $I_\alpha$ is $\{\lambda_j\}$. When one computes the asymptotics of $v(x)$ as $|x|$ goes to infinity from this representation, there are two types of terms. First, when $I_\alpha$ is a nondegenerate interval with left endpoint $-k_\alpha^2$, one introduces $\eta_n > 0$ by $\lambda = -k_\alpha^2 + \eta_n^2$ and gets a term of the form

$$v_\alpha = \int_{R^n} e^{ix' \cdot \eta} \frac{\phi_\alpha(x_n, -k_\alpha^2 + \eta_n^2)[\Psi_\alpha h(\eta, \cdot)](-k_\alpha^2 + \eta_n^2)\chi(\eta_n)}{\eta^2 + \eta_n^2 - i0} d\eta d\eta_n.$$

When $I_\alpha = \{\lambda_j\}$, one gets a term of the form

$$v_\alpha = \int_{R^{n-1}} e^{ix' \cdot \eta} \frac{\phi_j(x_n, \lambda_j)[\Psi_j h(\eta, \cdot)](\lambda_j)}{\eta^2 + \lambda_j - i0} d\eta.$$

This corresponds to a guided wave. Terms of the first type contribute terms to the asymptotics of $v(r\theta), r = |x|, \theta = x/|x|$, of the form $exp(ik_+ r) r^{(1-n)/2} a_+(\theta)$ for $\theta_n > 0$, and of the form $exp(ik_- r) r^{(1-n)/2} a_-(\theta)$ for $\theta_n < 0$. The coefficents $a_+(\theta)$ qnd $a_-(\theta)$ are multiples of $[\Psi_{1,2} h(\eta, \cdot)](-\eta^2)$ for $\theta_n > 0$ and for $\theta_n < -(1 - (k_+/k_-)^2)^{1/2}$, respectively. For $0 > \theta_n > -(1 - (k_+/k_-)^2)^{1/2}$ the cofficient $a_-(\theta)$ is a multiple of $[\Psi_3 h(\eta, \cdot)](-\eta^2)$. Terms of the second type contribute $exp(i\sqrt{-\lambda_j}) r^{(2-n)/2} a_j(\theta)$ when $\theta_n = 0$. The coefficient $a_j(\theta)$ is a multiple of $[\Psi_j h(\sqrt{-\lambda_j}\theta', \cdot)](\lambda_j)$.

2. Faddeev's Scattering Amplitude.

Equation (4) relates the scattering amplitude with the perturbation $\hat{q}$, but it has no obvious use in solving the inverse problem because of the complicated integral term in the equation. In [F] Faddeev found a way to replace (4) with an equation involving several parameters that could be exploited in the inverse problem. In our setting Faddeev's method consists in replacing $(G+\eta^2-i0)^{-1}$ in (4) by $(G+\eta^2+i0(\eta\cdot\nu-\sigma))^{-1}$, where $\nu \in S^{n-2}$ and $\sigma \in R$. We denote the solution of the resulting equation, assuming that a unique solution exists, by $h^*_\nu(\xi, x_n; \delta, \lambda, \alpha; \sigma)$, but again we will usually suppress most of these variables, and will always suppress the subscript $\nu$ which remains fixed in the rest of this discussion. The most important property of $h^*$ is that there is an integral equation relating $\Psi h^*$ and $\Psi h$, namely

$$\begin{aligned} &[\Psi_\alpha h^*(\xi,\cdot;\delta,-\delta^2,\gamma;\sigma)](-\xi^2) \\ +\sum_\beta 2\pi i \int_{\eta\cdot\nu>\sigma} &[\Psi_\alpha h(\xi,\cdot;\eta,-\eta^2,\beta)](-\xi^2)[\Psi_\beta h^*(\eta,\cdot;\delta,-\delta^2,\gamma;\sigma)](-\eta^2)d\eta \\ &= [\Psi_\alpha h(\xi,\cdot;\delta,-\delta^2,\gamma)](-\xi^2). \end{aligned} \tag{5}$$

Here $d\eta$ is replaced by $(-4\lambda_j)^{-1/2}$ times the surface measure on the sphere of radius $\sqrt{-\lambda_j}$ when $I_\alpha = \{\lambda_j\}$. Thus the scattering amplitude determines $[\Psi h^*(\xi,\cdot)](-\xi^2)$. One can show easily that (5), considered as an equation for $\Psi h^*$ has a unique solution in the space of continuous functions if and only if the equation for $h^*$ is uniquely solvable. To put that equation in a more convenient form we introduce $h_*(\xi, x_n; \delta, \lambda, \alpha; \sigma) = h^*(\xi + \sigma\nu, x_n; \delta + \sigma\nu, \lambda, \alpha; \sigma)$. Then $h_*$ is a solution of

$$\begin{aligned} h_*(\xi, x_n) + (2\pi)^{1-n}\int_{R^{n-1}} \hat{q}(\xi-\eta, x_n)[(G+(\eta+\sigma\nu)^2+i0\eta\cdot\nu)^{-1}h_*(\eta,\cdot)](x_n)d\eta \\ = -\hat{q}(\xi-\delta, x_n)\phi_\alpha(x_n,\lambda). \end{aligned} \tag{6}$$

3. The Strategy for Recovering the Perturbation

Equation (6) contains the parameter $\sigma$, but there is still no *large* parameter in the equation that could be used to simplify the equation in the limit. For that we need to use analytic continuation in $\sigma$. If we replace $(G+(\eta+\sigma\nu)^2+i0\eta\cdot\nu)^{-1}$ in (6) by $(G+(\eta+i\tau\nu)^2)^{-1}$ and denote the solution of the resulting equation by $h_*(\xi, x_n; i\tau)$, $\tau > 0$, then at least formally $h_*(\xi, x_n; \sigma)$ at $\sigma = 0$ will be the limit of $h_*(\xi, x_n; i\tau)$ as $\tau$ goes to zero. Our strategy will be to show that $h_*(\xi, x_n; i\tau)$ extends to a meromorphic function in neighborhood of the positive imaginary axis in such a way that $h_*(\xi, x_n; z)$ is the analytic continuation of $h_*(\xi, x_n; \sigma)$. We will solve the equation for $h_*(\xi, x_n; z)$ in a Banach space of functions analytic in $\xi$ on $|Im\{\xi\}| < \epsilon$ which decay exponentially in $x_n$. This analyticity and the exponential decay of $h_*$ in $x_n$ will lead to the analyticity of

$$\Gamma(s) = [\Psi_\alpha h_*(\xi(s),\cdot;\delta(s),-(\delta(s)+z(s)\nu)^2,\beta;z(s))](-(\xi(s)+z(s)\nu)^2).$$

Here $(\xi(s), \delta(s), z(s))$ is analytic in $s$ near the real axis, and $\xi(s)$ and $\delta(s)$ are real for $s$ real. For $s < s_0$ the function $z(s)$ is real-valued, and $z(s)$ is on the positive imaginary axis for $s >> 0$. Since $\Gamma(s)$ agrees with

$$[\Psi_\alpha h^*(\xi(s), \cdot; \delta(s), -(\delta(s))^2, \beta; z(s)](-(\xi(s))^2)$$

for $s < s_0$, it follows by analyticity that $\Gamma(s)$ is determined by the scattering amplitude for all $s$. One shows that the integral term in the equation for $h_*(i\tau)$ goes to zero as $\tau$ goes to infinity. Hence $\Gamma(s)$ is asymptotic to

$$\Gamma_0(s) = [\Psi_\alpha \hat{q}(\xi(s) - \delta(s), \cdot)\phi_\beta(\cdot, -(\delta(s) + z(s)\nu)^2)](-(\xi(s) + z(s)\nu)^2).$$

The family of curves $(\xi(s), \delta(s), z(s))$ for which one can apply this argument is large enough that one easily recovers the Fourier transform of $q$ on an open set in $R^n$ from the asymptotics of $\Gamma_0(s)$ as $s$ goes to infinity. This completes the recovery of $q$ from the scattering amplitude. This approach is used in both [ER] and [I].

4. Estimates

To make the procedure described in the preceding section work we clearly need some estimates. The starting point for the argument is the equation for $h_*(i\tau)$, i.e.,

$$[h_*(i\tau) + A(i\tau)h_*(i\tau)](\xi, x_n) = \hat{q}(\xi - \delta, x_n)\phi_\alpha(x_n, \lambda) \tag{7}$$

where

$$[A(i\tau)f](\xi, x_n) = (2\pi)^{1-n} \int_{R^{n-1}} \hat{q}(\xi - \eta, x_n)[(G + (\eta + i\tau\nu)^2)^{-1} f(\eta, \cdot)](x_n) d\eta. \tag{8}$$

Our hypothesis (3) implies that the inhomogeneous terms in (7) belong to the Banach space

$$B = \{f(\xi, x_n) \in C(\{|Im\{\xi\}| \leq \mu/3\} \times R) : \text{ f is analytic in } |Im\{\xi\}| < \mu/3$$

$$\text{and } sup(1 + |\xi|)^n e^{\mu|x_n|/3} |f(\xi, x_n)| < \infty\},$$

and we will solve the equation for $h_*$ in this space. The operator $A(i\tau)$ is compact on this space, and the results on the analytic continuation of $h_*$ needed here follow from the extension of $A$ to an analytic compact operator valued function on a domain of the form

$$D_\epsilon = \{z \in C : Im\{z\} > 0, |Re\{z\}| < \epsilon\}.$$

Since $(G - wI)^{-1}$ is analytic for $Im\{w\} \neq 0$, one sees that $(G + (\eta + z\nu)^2)^{-1}$ will be analytic in $D_\epsilon$ when $|\eta \cdot \nu| > 2\epsilon$. Thus, using a cutoff in $\eta_\nu = \eta \cdot \nu$, we can split $A$ into $A_1 + A_2$, where $A_1$ has a direct analytic extension to $D_\epsilon$ and the integration in $A_2$ is restricted to $|\eta_\nu| < 3\epsilon$. The analytic extension of $A_2$ is less direct. The resolvent $(G - wI)^{-1}$ is, of course, analytic for $Re\{w\} < \inf\{\lambda_j\}$, and, since we are working on a space of exponentially decreasing functions, it has extensions across the real axis for $Re\{w\} > -k_+^2$. However, when $Re\{-(\eta+z\nu)^2\}$ falls between $\inf\{\lambda_j\}$ and $-k_+^2$, we must define $A_2$ by deforming the contour of integration in $|r| = |\eta - \eta_\nu \nu|$ into the upper half-plane for $\eta_\nu > 0$ and into the lower half-plane for $\eta_\nu < 0$, as in [ER]. This extends $A$ analytically to $D_\epsilon$ and one sees that the limit of $A(\sigma + i\tau)$ as $\tau$ goes to zero, is the integral operator in (6) so that $h_*(\sigma + i\tau)$, if it exists, will be an analytic continuation of the solution $h_*(\sigma)$ of (6), if *it* exists.

As noted above there are some existence problems here. Fortunately, since $A(z)$ is an analytic compact operator valued function on $D_\epsilon$, they will all be solved if we can show that $(I + A(z_0))^{-1}$ exists for one $z_0$ in $D_\epsilon$. We prove this by showing that $||A(i\tau)||$ goes to zero as $\tau$ goes to infinity – which we also need to know for the recovery procedure. In the setting here this requires only the following estimates on the kernel $g(s,t;w)$ of $(G - wI)^{-1}$. Choose $M$ sufficiently large that $-k_+^2$ and the eigenvalues of $G$ are contained in $|w| < M - 1$. Then for $|w| > M$ one has $|g(s,t,w)| < C|w|^{-1/2}$. For $|w| < M$ one has

$$|g(s,t,w)| < C\sum_j |w - \lambda_j|^{-1} + C|w + k_-^2|^{-1/2}. \tag{9}$$

The second term in (9) is exceptional, occuring only when there is a "half-bound state" for the Schrödinger operator $G$ at $-k_-^2$. With these estimates one sees that $||A(i\tau)||$ will go to zero as $\tau$ goes to infinity, provided

$$sup_\xi \int_{R^{n-1}} (1 + |\xi - \eta|)^{-n} |(\eta + i\tau\nu)^2|^\gamma d\eta \tag{10}$$

goes to zero as $\tau$ goes to infinity for $\gamma = 1, 1/2$. Since

$$|(\eta + i\tau\nu)^2| = ((|\eta|^2 - \tau^2)^2 + 4\tau^2\eta_\nu^2)^{1/2}$$

and hence

$$|(\eta + i\tau\nu)^2| > \tau((|\eta| - |\tau|)^2 + 4\eta_\nu^2)^{1/2},$$

one can show that the supremum in (10) is bounded by $C\tau^{-1/2}$. For $n > 3$ this is the estimate in [ER,pp.214-6] and for $n = 3$ it is simpler.

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