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ON THE NON-UNIQUENESS OF WEAK SOLUTION OF THE EULER EQUATIONS

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1. Introduction

Consider the incompressible Euler equations

$$\begin{cases} \frac{\partial u}{\partial t} + (u, \nabla)u + \nabla p = 0 \ , \\ \nabla u = 0 \ . \end{cases} \tag{1.1}$$

Here $u = u(x,t)$ is a velocity field of an ideal incompressible fluid, $p(x,t)$ is the pressure, $x \in \mathbb{R}^n$, $t \in \mathbb{R}$. In order to define the weak solution of equations (1.1), let us multiply both sides of the first equation (1.1) by a vector field $v(x,t) \in C_0^\infty(\mathbb{R}^{u+1}, \mathbb{R}^n)$, such that $\nabla \cdot v = 0$, and both sides of the second equation (1.1) by the field $\nabla\varphi$, where $\varphi(x,t) \in C_0^\infty(\mathbb{R}^{n+1}, \mathbb{R})$. After integration by parts, we get the following relations:

$$\begin{cases} \iint -\left[\left(u, \frac{\partial v}{\partial t}\right) + (u \otimes u, \nabla v)\right] dx dt = 0 \ , \\ \iint -(u, \nabla\varphi) dx dt = 0 \ . \end{cases} \tag{1.2}$$

These relations should be true for arbitrary test-functions $\varphi(x,t), v(x,t)$. The left hand sides of relations (1.2) make sense for arbitrary vector-function $u(x,t) \in L^2_{\text{loc}}$. This justifies the following

Definition 1.1. *Vector-function $u(x,t)$ is called a weak solution of the Euler equations if $u(x,t)$ satisfies the integral relations (1.2) for arbitrary test-functions $\varphi(x,t)$, $v(x,t)$.*

The nature of weak solutions of the Euler equations has been quite unclear, and only a few sparse results about them are known. Thus, it has been proved by P. Constantin,

W. E and E. Titi [CET] and by G. Eyink [E], that if the weak solution belongs to the Hölder class C^α, and $\alpha > \frac{1}{3}$, then the kinetic energy $W = \frac{1}{2}\int |u(x,t)|^2 dx$ is constant in time, which was the old hypothesis of L. Onsager [O]. Note that the Hölder-class solutions are not classical at all.

Another result is the existence of weak solutions in the 2-dimensional case, satisfying the initial condition $u(x,0) = u_0(x)$, where the vorticity $\nabla \times u_0(x)$ is a positive measure (J.-M. Delort [D]).

The following quite striking result has been obtained by V. Scheffer [S]. He has proved that there exists a weak solution $u(x,t) \in L^2(\mathbb{R}^2 \times \mathbb{R})$, such that $u(x,t) \equiv 0$ for $|x|^2 + |t|^2 > 1$. Thereby this solution is identically zero for $t < -1$; then "something happens", and the solution becomes non-zero; but for all $t > 1$ the solution disappears again!

The original proof of this outstanding result is long and complicated. Our aim is to present a simpler and more transparent construction, so that the nature of these strange solutions is more clear. Instead of the flows on the plane $\mathbb{R}^2$, we consider the flows on the 2-dimensional torus $\mathbb{T}^2$, and construct a weak solution $u(x,t) \in L^2(\mathbb{T}^2, \mathbb{R}^2)$, having compact support *in time*. This means, in particular, that the weak solution with given (zero) initial velocity is not unique, and that the kinetic energy is not constant in time. In fact, our solution, as well as the one constructed by V. Sheffer, is a discontinuous, unbounded L^2-function.

Our construction is quite different from the original one of V. Scheffer. It is rooted in the physical idea of the inverse energy cascade in the 2-dimensional turbulence [K].

If the fluid is being pushed by an external force with small space scale (i.e. the force $f(x,t)$, whose Fourier-transform $\widetilde{f}(\xi,t)$, is concentrated in the domain of large $|\xi|$), then the energy is transported, via the non-linearity of the Euler equations, to different frequencies. But in 2-dimensional case the energy transport to the higher frequencies is forbidden by the vorticity conservation; the only possible direction of the energy flow is to the lower frequencies.

In particular, if the space scale of the force is infinitely-small, the simple dimension considerations show that it takes a finite time for the energy to reach the low-frequency range. We are constructing just the example of this inverse-cascade situation. The difficulty is that by no means can we solve the Euler equations in any nontrivial case. Therefore, we have to construct a very complicated hierarchical, finely tuned system of forces, imitating the inverse cascade. The forces are organized in such a manner, that at every step we have to solve a simple, linearized problem.

This is still very far from the physical postulate that the energy is transferred from the infinitely-high to the low frequencies for the generic forces.

"The force with infinite frequency" is still a non-defined mathematically object; it is zero from the usual viewpoint of distributions, but possibly makes sense as a Young measure, or some similar object.

In fact, we construct a sequence of solutions of non-homogeneous Euler equations with the right hand sides getting more and more oscillating, and prove that the limit is a weak solution of the homogeneous Euler equations. Thus, the weak solution, constructed in our work, as well as that of V. Sheffer is not in fact a solution; very strong external forces are present, but they are infinitely-fast oscillating in space, and therefore are undistinguishable from zero in the sense of distributions. The smooth test-functions are not "sensitive" enough to "feel" these forces. This is the fault of the sensors, not of the forces.

These examples show, that the usual definition of weak solution, given above, is not satisfactory. Though every possible candidate to be called weak solution should satisfy relations (1.2), for they express the local balance of mass and momentum (in fact, when deriving the Euler equations in the fluid mechanics, we start from the relations (1.2), or equivalent, and then, assuming sufficient regularity of the field $u(x,t)$, pass to the differential form of the Euler equations), we need some additional conditions. The "true" notion of weak solution remains still undefined.

2. The Idea of Construction

1. An incompressible vector field $u(x,t) \in L^2$ is called a weak solution of the non-homogeneous Euler equations with the external force $f(x,t) \in \mathcal{D}'$, if for every test-field $v(x,t) \in C_0^\infty$, $\nabla \cdot v \equiv 0$,

$$\iint -\Big[\Big(u, \frac{\partial v}{\partial a}\Big) + (u \otimes u, \nabla v)\Big] dxdt = \iint (f, v) dxdt \tag{2.1}$$

Our construction is based on the following simple

Lemma 2.1. *Let $u_i(x,t)$ be a weak solution of nonhomogeneous Euler equations with external forces $f_i(x,t)$, $i = 1,2,\dots$. Suppose that $u_i \to u$ strongly in L^2, while $f_i \to 0$ weakly in $\mathcal{D}'$, as $i \to \infty$. Then $u(x,t)$ is a weak solution of the Euler equations (2.1).*

Proof is clear from the definitions.

2. In our construction, we shall use the forces $f_i(x,t)$, having the special form

$$f_i(x,t) = \sum_{j=1}^{J_i} f_{ij}(x) \cdot \delta(t - t_{ij}) , \tag{2.2}$$

where $f_{ij}(x) \in C^\infty$. The weak solution $u_i(x,t)$ of the non-homogeneous equations (2.1) with such external force is a smooth solution of the homogeneous Euler equations (1.1) on every time interval $t_{i,j} < t < t_{i,j+1}$, satisfying condition $u_i(x,t) = 0$ for $t < t_{i,1}$, and the following jump condition:

$$u_i(x, t_{ij} + 0) - u_i(x, t_i, -0) = f_{ij}(x) . \tag{2.3}$$

Our construction starts from arbitrary smooth solution $u_0(x,t)$ of the Euler equations $(-\infty < t < \infty)$. Let us define the first term of our sequence as

$$u_1(x,t) = \begin{cases} u_0(x,t), & |t| < 1; \\ 0, & |t| > 1 . \end{cases} \tag{2.4}$$

This is a weak solution of the Euler equations with the force

$$f_1(x,t) = u_0(x,-1)\delta(t+1) - u_0(x,1)\delta(t-1) , \tag{2.5}$$

and

$$t_{1,1} = -1 , \quad t_{1,2} = 1 . \tag{2.6}$$

3. Now we are going to describe the inductive rule for passage from u_i to u_{i+1}. Suppose that at the moment t_0 the solution $u(x,t)$ has a jump. It is smooth and satisfies the Euler equations for $t < t_0$ and $t > t_0$, but $u(x, t_0 + 0) - u(x, t_0 - 0) = f(x)$ (we omit the indices i, j). Thus, $u(x,t)$ is a weak solution with the force $f(x) \cdot \delta(t - t_0)$.

We shall replace this force by a sum of finite number of δ-like pulses

$$g(x,t) = \sum_{i=0}^{I} g_i(x) \cdot \delta(t - t_i) , \tag{2.7}$$

so that the weak solution $v(x,t)$ of (2.1) with the force $g(x,t)$ satisfies the conditions

$$v(x,t_0-0)=u(x,t_0-0); v(x,t_0+T+0)=u(x,t_0+0)\ , \qquad (2.8)$$

where $T = t_I - t_0$. Thus, we shift the solution $u(x,t)$ for $t > t_0$ by T, and inserted something in the interval $(t_0, t_0 + T)$.

The most important property of the pulses $g_i(x)$ is that each of them is either *small* (in the sup-norm), or is an oscillating vector field with a high frequency. In both cases $g_i(x)$ are "weakly close to zero". The nonlinearity of the Euler equations should transform these high-frequency pulses into a smooth field $u(x,t)$.

Let us apply this operation to the function $u_i(x,t)$ at every moment $t_{i,j}$ of discontinuity. If $T_{i,j}$ is the time delay at the j-th moment, then

$$u_i(x,t)=u_{i+1}\Big(x,t+\sum_{t_{ij}<t}T_{ij}\Big)\ . \qquad (2.9)$$

4. At every step of our construction, we take the new much shorter time delays $T_{i+1,j}$ and much higher frequencies of the oscillating forces $f_{i+1,j}$.

The function $u(x,t) = \lim_{t\to\infty} u_i(x,t)$ is a smooth solution of the Euler equations (1.1) on the complement to some perfect set $\mathfrak{M}$ on the t-axis, and zero outside some finite time interval. The external force is concentrated on $\mathfrak{M}$, but it has zero space scale, and therefore is undistinguishable from zero as a distribution. (The rigorous sense of the last sentence is that the sequence $\{u_i(x,t)\}$ satisfies conditions of Lemma 2.1).

Now we shall describe the construction of the forces $f_{ij}(x)$ and the velocity fields $u_i(x,t)$.

3. Asymptotic Solution for Modulated Kolmogorov Flow

The main building block of our construction is a special type of flow, called modulated Kolmogorov flow.

Every velocity field $v(x)$ on $\mathbb{T}^2$, such that $\nabla\cdot v = 0$, and $\int_{\mathbb{T}^2} v(x)dx = 0$, can be defined using the stream function $\psi(x)$:

$$v(x)=J\nabla\psi(x)\ , \qquad (3.1)$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is an operator of rotation by $\frac{\pi}{2}$.

The *Kolmogorov flow* is a flow, defined by the stream function

$$\psi(x) = k^{-1}b(x)\sin k(a,x) \qquad (3.2)$$

where $a \in \mathbb{Z}^2$, $k \in \mathbb{Z}$, $b(x)$ is a given smooth function, independent of k. We are interested in the asymptotic behaviour of the flow with such initial condition for large k; in fact, we are going to construct the asymptotic solution of the Euler equations with the initial velocity (3.1), (3.2), if $k \to \infty$. The Kolmogorov flow is known to be desperately unstable, so our asymptotics should be valid on the small time intervals (depending on k). The asymptotic solution we are constructing has the form

$$v(x,t) = v_0(x) + (t-t_0)v_1(x) + \dots \quad , \qquad (3.3)$$

where every term v_j may be found explicitly from the Euler equations. Let

$$v^N(x.t) = v_0(x) + (t-t_0)v_1(x) + \dots + (t-t_0)^N v_n(x). \qquad (3.4)$$

Let us write the Euler equations (1.1) in the explicit form

$$\frac{\partial v}{\partial t} = A(v,v) \ , \qquad (3.5)$$

where

$$A(v,w) = -P(v,\nabla)w \ , \qquad (3.6)$$

and P is an orthogonal projector in $L^2(\mathbb{T}^2, \mathbb{R}^2)$ onto the subspace of divergence free vector fields. Then v^N satisfies the non-homogeneous Euler equations

$$\frac{\partial v^N}{\partial t} - A(v^N, v^N) = R_N \ . \qquad (3.7)$$

Theorem 3.1. *Suppose that the functions $b(x), w(x)$ are trigonometric polinomials, independent of k. Then there exist constants $C_1, \dots, C_N$, independent of k, such that*

$$\text{(i)} \qquad |\widetilde{v}_p(\xi)| \le C_p k^{[\frac{p}{2}]+(j+1)\alpha} \ ; \qquad (3.8)$$

$$\text{(ii)} \qquad |\widetilde{R}_N(\xi)| \le C_N k^{[\frac{N+1}{2}]+(N+1)\alpha} \ . \qquad (3.9)$$

Thus, the series (3.3) is asymptotic in the domain $|t-t_0| \le k^{2\alpha}$; this means, that for every s, M there exist N and $C_{s,M}$, such that

$$\|R_N\|_s \le C_{s,M} k^{-M} \qquad (3.10)$$

for all t in the interval $|t - t^0| \le k^{-2\alpha}$.

4. The First-order Term of the Asymptotic Solution

1. Let us find the explicit expression for the low-frequency part of the term $v_1(x)$ in the asymptotic solution (3.30). This term is the most important for our construction.

Suppose that

$$v_0(x) = k^\alpha v(x) + w(x) \ , \tag{4.1}$$

where

$$v(x) = J\nabla\psi(x) \ , \tag{4.2}$$

$$\psi(x) = k^{-1}b(x)\sin k(a,x) \ . \tag{4.3}$$

Then the straightforward computation shows that

$$v_1(x) = G(x) + H(x) \ ; \tag{4.4}$$

$$G(x) = \sum_{0\le j,l\le 2} PG_{j,l}(x)k^{j\alpha-l} \ ; \tag{4.5}$$

$$H(x) = \sum_{0\le j,l\le 2}\sum_{1\le m\le 2} P\Big[H'_{j,l,m}(x)\sin mk(a,x)k^{j\alpha-l}$$

$$+H''_{j,l,m}(x)\cos mk(a,x)k^{j\alpha-l}\Big] \ . \tag{4.6}$$

Here $G_{j,l}, H'_{j,l,m}, H''_{j,l,m}$ are smooth functions, independent of k; they are obtained from the functions $b(x), w(x)$ by some quadratic differential operators of order not more than 2.

The most interesting and important for our purposes is the term $PG_{2,0}(x)$, the main non-oscillating part of $v_1(x)$. After simple calculations we find that

$$PG_{2,0} = \frac{1}{2}P[(XB)X], \tag{4.7}$$

where $B(x) = b^2(x)$; $X = Ja$ is a constant vector field with components $(a_2, -a_1)$; we identify the field X with the differential operator $a_2\frac{\partial}{\partial x_1} - a_1\frac{\partial}{\partial x_2}$; P is the orthogonal projector in $L^2(\mathbb{T}^2, \mathbb{R}^2)$ onto the subspace of divergence free vector fields.

2. Let us consider the inverse problem: for a given field $V(x)$, such that $\nabla \cdot V = 0$, $\int V dx = 0$, find an amplitude $b(x)$ and the wave vector a of the Kolmogorov flow, such that

$$\frac{1}{2} P[(X \cdot B)X] = V \ , \tag{4.8}$$

where $B(x) = b^2(x)$, and $X = Ja$. In general, this is impossible, but the following is true.

Theorem 4.1. *There exist two vectors X_1, X_2, such that for every smooth vector field $V(x)$, satisfying $\nabla \cdot V = 0$, $\int V dx = 0$, there exist smooth positive functions $B_1(x), B_2(x)$, such that*

$$V = \sum_{j=1}^{2} \frac{1}{2} P[(X_j B_j) X_j] \ ; \tag{4.9}$$

Moreover, there exist two pseudo-differential operators Φ_1, Φ_2 of order (-1) with symbols depending only on ξ, such that

$$B_j = \Phi_j V_\xi + B_j^0 \ . \tag{4.15}$$

Now we have all the tools ready for the construction.

5. The Construction

Suppose that $u(x,t)$ is a weak solution of the Euler equations with the right-hand-side $\delta(t - t_0) f(x)$; $u(x,t)$ is smooth for $t \neq t_0$, and if $u_\pm(x) = u(x, t_0 \pm 0)$, then $u_+(x) - u_-(x) = f(x)$. We are going to construct a sequence of pulses, imposed at the time interval $t_0 \le t \le t_0 + T$ in such a manner that it produces the same flow (with the time delay T) and consists of the pulses, such that part of them are small in the sup-norm, and others are oscillating with high frequency.

This sequence contains 4 "strong" pulses, having the amplitudes $f_1(x), f_1'(x), f_2(x), f_2'(x)$, where

$$f_j(x) = J \nabla \psi_j \ ; \tag{5.1}$$

$$\psi_j(x) = k^{-1+\alpha} b_j(x) \sin k(a_j, x) \ ; \tag{5.2}$$

$$b_j(x) = (B_j(x) + 2 \sup_{x \in \mathbb{T}^2} |B_j(x)|)^{\frac{1}{2}} \ ; \tag{5.3}$$

$$B_j(x) = (\Phi_j f)(x) \quad (j = 1, 2) \tag{5.4}$$

(Here $X_j = Ja_j, j = 1,2$ are two vectors, whose existence is claimed in Theorem 4.1.)

The pulses f_j' follow the pulses f_j with the time interval $k^{-2\alpha}$ between them; f_j' kills the high-frequency part of the flow field, generated by f_j.

We should add a finite number of "weak" pulses, intended to keep the flow close to the asymptotic solution v^N for given sufficiently large N, and a few weak pulses, truncating the Fourier transform of the velocity field and thus converting it into a trigonometric polinomial. As a result, we obtain at $t = t_0 + 2k^{-2\alpha} + \tau = t_0 + T$ the velocity field $u_+(x)$.

If the flow field $u(x,t)$ is discontinuous at the moments $t_1, \ldots, t_I$, we may apply the same operation at every moment t_i and obtain a new flow, which we denote as Λu. The operation Λ depends on some parameters; the most important of them are k, N.

The idea of our construction is to iterate this operation with growing k and N. Let us define

$$u_{i+1} = \Lambda(k_i, N_i, \ldots) u_i \ . \tag{5.5}$$

Theorem 5.1. *If k_i, N_i are growing sufficiently fast, as $i \to \infty$, then the sequences $\{u_i\}, \{f_i\}$ satisfy conditions of Lemma 2.1, and thus, $u_i \to u$, and u is a weak solution of homogeneous Euler equations, having compact support in time.*

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