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Traces on the Cone Algebra with Asymptotics

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Abstract

For every singular point on a manifold with conical singularities we construct a trace on the 'Cone Algebra with Asymptotics' introduced by B.-W. Schulze. Each of these traces is determined at the singularity; none of them therefore is induced by Wodzicki's noncommutative residue. On the ideal of operators with vanishing conormal symbol, however, we find another trace which coincides with the noncommutative residue in the interior. Moreover, it is shown that all these traces are essentially unique on a slightly extended version of the cone algebra.

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Introduction

Following upon work by Manin [8] and Adler [1], the noncommutative residue was discovered by Wodzicki in 1984 [12]. It is the unique trace on the algebra of all classical pseudodifferential operators (modulo the regularizing elements) on a compact manifold without boundary.

A trace on an algebra $\mathcal{A}$ here and in the following designates a linear map $\tau : \mathcal{A} \to \mathbb{C}$ which vanishes on commutators, i.e., $\tau[A, B] = \tau(AB - BA) = 0$ for all $A, B \in \mathcal{A}$. Clearly, scalar multiples of traces are traces, and the zero map is a trace, so uniqueness is to be understood in the sense that it is the only non-vanishing trace up to constant factors.

Guillemin independently discovered the noncommutative residue as an important ingredient in his so-called 'soft' proof of Weyl's formula on the asymptotic distribution of eigenvalues [7].

Meanwhile the noncommutative residue has found a wide range of applications in both mathematics and mathematical physics; it plays a prominent role in Connes' noncommutative geometry [2, 3].

While it can be shown that there is no (nonzero) trace on the algebra of all pseudodifferential operators on a manifold with boundary, a unique continuous trace was found for Boutet de Monvel's algebra of all classical pseudodifferential boundary value problems on a compact manifold with boundary by Fedosov, Golse, Leichtnam, and the author [6, 5]; this trace extends Wodzicki's residue. As a consequence one notices that the right choice of the operator algebra is indeed important.

Here, the underlying object is a manifold with conical singularities. A natural algebra to consider therefore is the 'Cone Algebra with Asymptotics' introduced by B.-W. Schulze [11]. The situation then is the following: For each conical singularity we obtain a trace. It vanishes on operators supported in the interior, so it is not induced by Wodzicki's residue. However, there is a natural ideal in the cone algebra, namely the operators with vanishing conormal symbol, where one finds an additional trace extending the one discovered by Wodzicki.

Uniqueness fails on the standard cone algebra due to the lack of non-smooth multipliers. On the other hand one can easily construct an extension of the cone algebra with a corresponding ideal for which the above traces are the only ones under the additional assumption that they are continuous and vanish on the ideal of the smoothing Mellin operators.

1 The Cone Algebra with Asymptotics

1.1 Basic notation. *A manifold with conical singularities of dimension $n + 1$ is a topological second countable Hausdorff space, B, with a finite subset Σ, the so called 'singularities', such that $B\backslash\Sigma$ is an $(n + 1)$-dimensional manifold and, for every $v \in \Sigma$, there is an open neighborhood U_v of v, a compact manifold without boundary X_v of dimension n, and a (maximal) system $\mathcal{F}_v \neq \emptyset$ of mappings with the following propertiy*

For all $\phi \in \mathcal{F}_v$, the mapping $\phi : U_v \to X_v \times [0,1)/X_v \times \{0\}$ is a homeomorphism with $\phi(v) = X_v \times \{0\}/X_v \times \{0\}$. It restricts to a diffeomorphism $U_v\backslash\{v\} \to X_v \times (0,1)$ Given $\phi_1, \phi_2 \in \mathcal{F}_v$, the restriction $\phi_1\phi_2^{-1} : X_v \times (0,1) \to X_v \times (0,1)$ extends to a diffeomorphism $X_v \times [0,1) \to X_v \times [0,1)$.

In this article, B is also assumed to be compact. $\mathbb{B}$, the stretched object associated with B, is the compact manifold with boundary constructed by replacing, for every singularity v, the neighborhood U_v by the cylinder $X_v \times [0,1)$ via gluing with any one of the diffeomorphisms ϕ.

For simplicity we assume that there is only one singularity with cross-section X. Let X be endowed with a Riemannian metric, write $X^\wedge = X \times \mathbb{R}_+$, and let $X^\wedge$ carry the canonical cylindrical metric. Near the singularity we shall employ geodesic coordinates (x, t), $x \in X, t \in [0, 1)$.

By $L^\mu(X)$ we denote the space of all classical pseudodifferential operators of order μ on X; by $L^\mu(X; \mathbb{R})$ the corresponding space of parameter-dependent classical elements with parameter space $\mathbb{R}$.

1.2 The Mellin transform. *For $\beta \in \mathbb{R}$, Γ_β denotes the vertical line $\{z \in \mathbb{C} : \operatorname{Re} z = \beta\}$. The Mellin transform Mu of a complex-valued $C_0^\infty(\mathbb{R}_+)$-function u is given by*

$$(Mu)(z) = \int_0^\infty t^{z-1} u(t)\, dt, \quad z \in \mathbb{C}. \tag{1.1}$$

This also makes sense for functions with values in a Fréchet space E. The fact that $Mu|_{\Gamma_{1/2-\gamma}}(z) = M_{t\to z}(t^{-\gamma}u)(z + \gamma)$ motivates the following definition of the weighted Mellin transform M_γ:

$$M_\gamma u(z) = M_{t\to z}(t^{-\gamma}u)(z + \gamma), \quad u \in C_0^\infty(\mathbb{R}_+, E).$$

The inverse of M_γ is given by

$$[M_\gamma^{-1}h](t) = \frac{1}{2\pi i} \int_{\Gamma_{1/2-\gamma}} t^{-z}h(z)dz.$$

1.3 Sobolev spaces and weighted Mellin Sobolev spaces.

(a) $H^s(\Omega)$, $s \in \mathbb{R}$, is the usual Sobolev space over a smooth compact manifold Ω with or without boundary.

(b) For $s \in \mathbb{N}$ and $\gamma \in \mathbb{R}$, the space $\mathcal{H}^{s,\gamma}(X^\wedge)$ is the set of all $u \in \mathcal{D}'(X^\wedge)$ such that $t^{\frac{n}{2}-\gamma}(t\partial_t)^k Du(x,t) \in L^2(X^\wedge)$ for all $k \leq s$ and all differential operators D of order $\leq s - k$ on X. Next we define $\mathcal{H}^{s,\gamma}(X^\wedge)$ for $s \geq 0$ by interpolation, then for $s \leq 0$ by duality: $\mathcal{H}^{s,\gamma}(X^\wedge) = [\mathcal{H}^{-s,-\gamma}(X^\wedge)]'$ with respect to the pairing

$$(u,v) = \frac{1}{2\pi i} \int_{\Gamma_{\frac{n+1}{2}}} (Mu(z), Mv(z))_{L^2(X)}\, dz.$$

(c) The following relations hold: $\mathcal{H}^{s,\gamma}(X^\wedge) \subseteq H^s_{loc}(X^\wedge)$; $\mathcal{H}^{s,\gamma}(X^\wedge) = t^\gamma \mathcal{H}^{s,0}(X^\wedge)$; $\mathcal{H}^{0,0}(X^\wedge) = t^{-n/2}L^2(X^\wedge)$.

(d) Fix a smooth function ω on $\mathbb{B}$, equal to 1 close to the boundary and supported close to the boundary. Given a distribution $u \in \mathcal{D}'(int\mathbb{B})$ write $u = u_1 + u_2$ with $u_1 = \omega u$ supported close to the boundary and $u_2 = (1 - \omega)u$ supported away from the boundary. We shall say that $u \in \mathcal{H}^{s,\gamma}(\mathbb{B})$, provided that $u_1 \in \mathcal{H}^{s,\gamma}(X^\wedge)$ and $u_2 \in H^s(\mathbb{B})$.

1.4 Mellin symbols and Mellin operators.

Let $\mu \in \mathbb{Z}$, $\gamma \in \mathbb{R}$. By $L^\mu(X; \Gamma_{1/2-\gamma})$ denote the space of all parameter-dependent classical pseudodifferential operators on X with parameter space $\mathbb{R}$, using the canonical identification $\Gamma_{1/2-\gamma} \cong \mathbb{R}$. Given $f \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+, L^\mu(X; \Gamma_{1/2-\gamma}))$ we shall write $f = f(t, t', z)$, where z indicates the variable in $\Gamma_{1/2-\gamma}$. For t, t', z fixed, $f(t, t', z)$ is a pseudodifferential operator acting on sections of vector bundles over X. We define the Mellin operator $op_M^\gamma f$ on $C_0^\infty(X^\wedge) = C_0^\infty(\mathbb{R}_+, C^\infty(X))$ by

$$[op_M^\gamma f]u(t) = \frac{1}{2\pi i} \int_{\Gamma_{1/2-\gamma}} \int_0^\infty (t/t')^{-z} f(t, t', z)u(t')\frac{dt'}{t'}dz. \tag{1.1}$$

The function f is called a Mellin symbol for $op_M^\gamma f$. It is easy to see that $op_M^\gamma f : C_0^\infty(X^\wedge) \to C^\infty(X^\wedge)$ is continuous. If $f \in C^\infty(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+, L^\mu(X; \Gamma_{\frac{1}{2}-\gamma}))$, then there is a bounded extension

$$\omega_1 [op_M^\gamma f]\omega_2 : \mathcal{H}^{s,\gamma+\frac{n}{2}}(X^\wedge) \to \mathcal{H}^{s-\mu,\gamma+\frac{n}{2}}(X^\wedge)$$

for every choice of $s \in \mathbb{R}$ and $\omega_1, \omega_2 \in C_0^\infty(\overline{\mathbb{R}}_+)$.

1.5 Asymptotic types and Mellin Sobolev spaces with asymptotics.

In the following let us fix $\mu \in \mathbb{Z}$, $\gamma \in \mathbb{R}$, and the weight datum $g = (\gamma + n/2, \gamma + n/2, (-1, 0])$; the latter is a triple consisting of two reals (here both are equal to $\gamma + n/2$) and an interval, here $(-1, 0]$.

An asymptotic type associated with g is a finite set $P = \{(p_j, m_j, C_j) : j = 1, \ldots, J\}$, where $p_j \in \mathbb{C}$, $-1/2 - \gamma < Re\, p_j < 1/2 - \gamma$, $m_j \in \mathbb{N}_0$, and C_j is a finite-dimensional subspace of $C^\infty(X)$; J may depend on P. We let $\pi_\mathbb{C}P$ denote the set $\{p_1, \ldots, p_J\}$.

A Mellin asymptotic type *is a sequence* $P = \{(p_j, m_j, L_j) : j \in \mathbb{Z}\}$, *where* $Re\, p_j \to \pm\infty$ *as* $j \to \mp\infty$, $m_j \in \mathbb{N}_0$, *and* L_j *is a finite-dimensional subspace of finite rank operators in* $L^{-\infty}(X)$.

Given an asymptotic type $P = \{(p_j, m_j, C_j) : j = 1, \dots, J\}$ *associated with* g *and real numbers* s *and* γ, *we define the* weighted Mellin Sobolev space with asymptotics, $\mathcal{H}_P^{s,\gamma}(\mathbb{B})$ *as the set of all* $u \in \mathcal{H}^{s,\gamma}(\mathbb{B})$ *for which there are functions* $c_{jk} \in C_j$, $j = 1, \dots J$, $k = 0, \dots, m_j$, *such that for a suitable (then arbitrary) cut-off function* ω,

$$u - \sum_{j=1}^{J} \sum_{k=0}^{m_j} c_{jk}(x) t^{-p_j} \ln^k t\, \omega(t) \in \mathcal{H}^{s,\gamma+1-\epsilon}(\mathbb{B})$$

whenever $\epsilon > 0$.

A cut-off function *here and in the following is a function* ω *in* $C_0^\infty(\overline{\mathbb{R}}_+)$ *with* $\omega(t) \equiv 1$ *for* t *near zero. We also assume that* ω *has support in* $[0,1)$, *so that it may be considered as a function on the part of* $\mathbb{B}$ *identified with* $X \times [0,1)$.

1.6 Meromorphic Mellin symbols.

$M_O^\mu(X)$ *is the space of all entire functions* $h : \mathbb{C} \to L^\mu(X)$ *for which the restrictions* $h|_{\Gamma_\beta}$ *are elements of* $L^\mu(X; \Gamma_\beta)$, *uniformly for* β *in compact intervals. It is naturally a Fréchet space.*

Given a Mellin asymptotic type P, *let* $M_P^\mu(X)$ *denote the space of all holomorphic functions* $h : \mathbb{C} \setminus \pi_\mathbb{C}P \to L^\mu(X)$ *with the following properties:*

(i) *In a neighborhood of* $p_j \in \pi_\mathbb{C}P$ *we can write*

$$h(z) = \sum_{k=0}^{m_j} \nu_{jk}(z - p_j)^{-k-1} + h_0(z)$$

with suitable $\nu_{jk} \in L_j$ *and a function* h_0 *which is holomorphic near* p_j.

(ii) *For each interval* $[c_1, c_2]$ *we find elements* ν_{jk} *in* L_j *such that*

$$h(\beta + i\tau) - \sum_{\{j : Re\, p_j \in [c_1, c_2]\}} \sum_{k=0}^{m_j} \nu_{jk} M_{t \to z}(t^{-p_j} \ln^k t\, \omega(t))(\beta + i\tau) \in L^\mu(X; \mathbb{R}_\tau),$$

uniformly for $\beta \in [c_1, c_2]$. *Here* ω *is an arbitrary cut-off function.*

Again we have a natural Fréchet topology. As usual, we let $M_P^{-\infty}(X) = \bigcap_\mu M_P^\mu(X)$ *be the space of smoothing Mellin symbols. One can then decompose* $M_P^\mu(X) = M_O^\mu(X) + M_P^{-\infty}(X)$.

We call the elements of $C^\infty(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+, M_O^\mu(X))$ *holomorphic Mellin symbols of order* μ, *those of* $C^\infty(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+, M_P^\mu(X))$ *meromorphic Mellin symbols with asymptotic type* P.

1.7 The 'Cone Algebra with Asymptotics'.

(a) $C_G(\mathbb{B}, g)$ *denotes the space of all operators* $G : C_0^\infty(int\,\mathbb{B}) \to \mathcal{D}'(int\,\mathbb{B})$ *which have continuous extensions*

$$G : \mathcal{H}^{-k, \gamma + \frac{n}{2}}(\mathbb{B}) \quad \to \quad \mathcal{H}_{Q_1}^{\infty, \gamma + \frac{n}{2}}(\mathbb{B}) \quad and$$

$$G^* : \mathcal{H}^{-k, -\gamma - \frac{n}{2}}(\mathbb{B}) \quad \to \quad \mathcal{H}_{Q_2}^{\infty, -\gamma - \frac{n}{2}}(\mathbb{B})$$

for arbitrary $k \in \mathbb{N}$ *and suitable asymptotic types* Q_1 *and* Q_2, *associated with* g *and independent of* k. *Here* G^* *is the formal adjoint with respect to the pairing in 1.3(b). In*

fact it is sufficient to require these mapping properties for $k = 0$; continuity on the larger spaces can then be deduced. The operators in $C_G(\mathbb{B}, g)$ are called Green operators.

(b) $C_{M+G}(\mathbb{B}, g)$ is the space of all operators $A : C_0^\infty(int\mathbb{B}) \to \mathcal{D}'(int\mathbb{B})$ that can be written

$$A = \omega_1 [op_M^{\gamma_0} h_0] \omega_2 + G$$

with $\gamma - 1 \le \gamma_0 \le \gamma$, $h_0 \in M_{P_0}^{-\infty}(X)$, $\pi_\mathbb{C} P_0 \cap \Gamma_{1/2-\gamma_0} = \emptyset$, cut-off functions ω_1, ω_2, and $G \in C_G(\mathbb{B}, g)$. The definition makes sense, since the difference induced by changing the cut-off functions results in a Green operator. These operators are the smoothing Mellin *operators.*

(c) $C^\mu(\mathbb{B}, g)$ is the collection of all operators $A : C_0^\infty(int\mathbb{B}) \to \mathcal{D}'(int\mathbb{B})$ of the form $A = A_M + P + R$, where

A_M is a Mellin operator supported close to the boundary, i.e., $A_M = \omega_1 [op_M^\gamma h] \omega_2$ for a holomorphic Mellin symbol h of order μ and suitable cut-off functions ω_1, ω_2;

P is a pseudodifferential operator of order μ supported in the interior, i.e., there are functions ϕ_1, ϕ_2 vanishing in a neighborhood of $\partial\mathbb{B}$ with $P = \phi_1 P \phi_2$; finally

R is an operator in $C_{M+G}(\mathbb{B}, g)$.

$C(\mathbb{B}, g) = \bigcup_{\mu \in \mathbb{Z}} C^\mu(\mathbb{B}, g)$, is called the cone algebra with asymptotics. *It is indeed an algebra, with ideals $C_G(\mathbb{B}, g)$ and $C_{M+G}(\mathbb{B}, g)$, as the following theorem shows.*

1.8 Theorem. *(a) $C_G(\mathbb{B}, g) \hookrightarrow C_{M+G}(\mathbb{B}, g) \hookrightarrow C^\mu(\mathbb{B}, g)$.*

(b) The composition of operators induces continuous maps

$$C^{\tilde\mu}(\mathbb{B}, g) \times C^\mu(\mathbb{B}, g) \to C^{\mu+\tilde\mu}(\mathbb{B}, g),$$
$$C^{\tilde\mu}(\mathbb{B}, g) \times C_{M+G}(\mathbb{B}, g) \to C_{M+G}(\mathbb{B}, g),$$
$$C_{M+G}(\mathbb{B}, g) \times C^\mu(\mathbb{B}, g) \to C_{M+G}(\mathbb{B}, g),$$
$$C^{\tilde\mu}(\mathbb{B}, g) \times C_G(\mathbb{B}, g) \to C_G(\mathbb{B}, g), \quad \text{and}$$
$$C_G(\mathbb{B}, g) \times C^\mu(\mathbb{B}, g) \to C_G(\mathbb{B}, g).$$

2 The Extended Cone Algebra

Fix $\gamma \in \mathbb{R}$, the weight datum $g = (\gamma + \frac{n}{2}, \gamma + \frac{n}{2}, (-1, 0])$, and $\mu \in \mathbb{Z}$.

2.1 Definition. *Choose a smooth nonnegative function $\underline{t}$ on $\mathbb{B}$, strictly positive in the interior and coinciding with the geodesic distance t to the boundary near $\partial\mathbb{B}$, i.e., for $t < 1$. Let $C^\mu(\mathbb{B}, g)^+$ be the vector space generated by the set of all linear combinations of operators of the form $\underline{t}^m A$, where $A \in C^\mu(\mathbb{B}, g)$ and $Re\, m \ge 0$. Multiplication by $\underline{t}^m$ is a continuous action on all spaces $\mathcal{H}^{s,\gamma}(\mathbb{B})$, so the composition of the operators makes sense. We call $C(\mathbb{B}, g)^+ = \bigcup_\mu C^\mu(\mathbb{B}, g)^+$ the* extended cone algebra.

Correspondingly let $C_{M+G}(\mathbb{B}, g)^+$ denote the space generated by operators of the form $\underline{t}^m R$, with $R \in C_{M+G}(\mathbb{B}, g)$ and $Re\, m \ge 0$.

We next introduce $C^\mu(\mathbb{B}, g)_0^+$. It consists of those elements in $C^\mu(\mathbb{B}, g)^+$ that can be written in the form $\underline{t}^\epsilon A$ with suitable $\epsilon > 0$ and $A \in C^\mu(\mathbb{B}, g)^+$. Analogously, $C_{M+G}(\mathbb{B}, g)_0^+$ is the space of all $\underline{t}^\epsilon R$ with $R \in C_{M+G}(\mathbb{B}, g)^+$.

The intersection $C^\mu(\mathbb{B}, g)_0^+ \cap C^\mu(\mathbb{B}, g)$ clearly consists of the elements in the cone algebra with vanishing conormal symbol, i.e., where $h(0) + h_0 = 0$ in the notation of 1.7.

It follows from the definition that $\underline{t}^m G$ and $G\underline{t}^m$ are elements of $C_G(\mathbb{B}, g)$ whenever $Re\, m \geq 0$ and $G \in C_G(\mathbb{B}, g)$.

2.2 Proposition. *One obtains the composition laws of Theorem 1.8 for the corresponding extended algebras. Moreover, $C(\mathbb{B}, g)_0^+$ is an ideal in the sense that composition maps*

$$C^{\tilde\mu}(\mathbb{B}, g)_0^+ \times C^\mu(\mathbb{B}, g)^+ \quad \to \quad C^{\mu+\tilde\mu}(\mathbb{B}, g)_0^+ \quad and$$
$$C^{\tilde\mu}(\mathbb{B}, g)^+ \times C^\mu(\mathbb{B}, g)_0^+ \quad \to \quad C^{\mu+\tilde\mu}(\mathbb{B}, g)_0^+.$$

A Trace on the Extended Cone Algebra

We shall change the notation slightly. Given a Mellin symbol $h \in C^\infty(\overline{\mathbb{R}}_+, M_O^\mu(X))$ we shall denote by $h(t)$ the operator in $M_O^\mu(X)$. This is a parameter-dependent classical pseudodifferential operator along each line Γ_β in $\mathbb{C}$. We write $h(t)(x, \xi, \beta + i\tau)$ for its local symbol, and $h_k(x, \xi, \beta + i\tau)$ for the component which is homogeneous of degree k in (ξ, τ). We start with a simple observation.

2.3 Lemma. *Let $f = \sum_{m \in C} t^m f_m$, where C is a finite subset of $\{Re\, z \geq 0\}$ and $f_m \in C_0^\infty(\overline{\mathbb{R}}_+)$. Then the mapping*

$$f \mapsto f_0(0)$$

is well-defined, i.e. independent of the representation of f.

We understand the notation $\sum_{m \in C}$ as the summation over all different elements of C.

2.4 Definition. *It is immediate from the definition that an operator $A \in C^\mu(\mathbb{B}, g)^+$ can be written in the form*

$$A = \sum_{m \in C} \omega_1 t^m [op_M^\gamma h_m] \omega_2 + P + R$$

with a finite subset C of $\{Re\, z \geq 0\}$, cut-off functions near zero, ω_1, ω_2, Mellin symbols $h_m \in C^\infty(\overline{\mathbb{R}}_+, M_O^\mu(X))$, a pseudodifferential operator P of order μ, supported away from the boundary, and $R \in C_{M+G}(\mathbb{B}, g)^+$.

For the summation $\sum_{m \in C} \omega_1 t^m [op_M^\gamma h_m] \omega_2$, the operator $h_0(0)$ is well-defined as an element of, say, $L^\mu(X; \Gamma_0)$ according to Lemma 2.3. By $h_0(0)_{-n-1}(x, \xi, i\tau)$, $\tau \in \mathbb{R}, \xi \in \mathbb{R}^n$, denote the homogeneous component of degree $-n-1$ of its complete local symbol in a coordinate neighborhood.

In the representation of A, the elements h_m are not unique. Any other choice, however, differs by a linear combination of elements of the form $t^m \omega_1 [op_M^\gamma \tilde h_m] \omega_2$ with $Re\, m \geq 0$ and $\tilde h_m \in M_O^{-\infty}(X)$. The contribution to $h_0(0)_{-n-1}$ is not affected by this ambiguity.

So let $S = \{\xi \in \mathbb{R}^n : |\xi| = 1\}$ and $\sigma_\xi = \sum_{j=1}^n (-1)^j \xi_j d\xi_1 \wedge \ldots \wedge \widehat{d\xi_j} \wedge \ldots \wedge d\xi_n$. The hat indicates that $d\xi_j$ is omitted. It is well-known that σ_ξ induces the surface measure on S. We therefore define

$$res_x(A) = \int_S \int_{-\infty}^\infty Tr\, h_0(0)_{-n-1}(x, \xi, i\tau)\, d\tau\, \sigma_\xi\, dx_1 \wedge \ldots \wedge dx_n,$$

where x belongs to a coordinate neighborhood and "Tr" denotes the (finite dimensional) trace on the symbol. Note that in view of the holomorphy of h_0 we might integrate over another line Γ_β instead of Γ_0.

2.5 Lemma. *$res_x(A)$ is a density on X.*

Proof. Since t is global, any change of coordinates is of the form $(x, t) \mapsto (\chi(x), t)$, where χ is a change of coordinates for X. So the lemma follows essentially as in the standard case, cf. [5, Theorem 1.4]. ◁

Lemma 2.5 shows that the following definition makes sense.

2.6 Definition. *For $A \in C^\mu(\mathbb{B}, g)^+$ with a representation as in Definition 2.4 let*

$$resA = \int_X res_x A = \int_X \int_S \int_{-\infty}^\infty Tr\, h_0(0)_{-n-1}(x, \xi, i\tau)\, d\tau\, \sigma_\xi\, dx_1 \wedge \ldots \wedge dx_n.$$

We shall abbreviate $dx = dx_1 \wedge \ldots \wedge dx_n$.

2.7 Theorem. *$res : C(\mathbb{B}, g)^+ \to \mathbb{C}$ is a trace.*

Proof. Let $A, B \in C^\mu(\mathbb{B}, g)^+$. We have to show that $res[A, B] = res(AB - BA) = 0$. Using a partition of unity on X and the linearity of 'res', it is no restriction to assume that both are supported by a single coordinate neighborhood of X. By linearity we may also assume that

$$A = \omega_1 t^m [op_M^\gamma g_m] \omega_2 + \ldots \quad \text{and} \quad B = \omega_1 t^{\widetilde{m}} [op_M^\gamma h_{\widetilde{m}}] \omega_2 + \ldots$$

each have a single Mellin term. Applying [5, Lemma 3.1.10], the Mellin symbol of AB has the asymptotic expansion

$$t^{m+\widetilde{m}} \sum_{k=0}^\infty \frac{1}{k!} \partial_\tau^k g_m(t)(x, \xi, i\tau - \widetilde{m}) \# (-t\partial_t)^k h_{\widetilde{m}}(t)(x, \xi, i\tau).$$

Here $\#$ denotes the symbol composition with respect to x and ξ. A nonvanishing contribution to the residue requires $\widetilde{m} = -m \in i\mathbb{R}$, say $m = i\mu = -\widetilde{m}$. Moreover, only the terms with $k = 0$ can influence the result due to the powers of t in the above formula. We can therefore focus on the terms

$$g_{i\mu}(0)(x, \xi, i\tau + i\mu) \# h_{-i\mu}(0)(x, \xi, i\tau).$$

We may apply the corresponding consideration to BA and conclude that

$$
\begin{aligned}
res\, [A, B] &= \int_X \int_S \int_{-\infty}^\infty Tr\, [g_{i\mu}(0)(x, \xi, i\tau + i\mu) \# h_{-i\mu}(0)(x, \xi, i\tau)]_{-n-1} \\
&\quad - Tr\, [h_{-i\mu}(0)(x, \xi, i\tau - i\mu) \# g_{i\mu}(0)(x, \xi, i\tau)]_{-n-1}\, d\tau\, \sigma_\xi\, dx \\
&= \int_X \int_S \int_{-\infty}^\infty Tr\, [g_{i\mu}(0)(x, \xi, i\tau + i\mu) \# h_{-i\mu}(0)(x, \xi, i\tau) \\
&\quad - Tr\, [h_{-i\mu}(0)(x, \xi, i\tau) \# g_{i\mu}(0)(x, \xi, i\tau + i\mu)]_{-n-1}\, d\tau\, \sigma_\xi\, dx,
\end{aligned}
$$

using the translation invariance of the integral with respect to τ. The Leibniz product in the integrand has the asymptotic expansion

$$\sum_{|\alpha| \geq 0} \frac{1}{\alpha!} \left\{ \partial_\xi^\alpha g_{i\mu}(0)(i\tau + i\mu) D_x^\alpha h_{-i\mu}(0)(i\tau) - \partial_\xi^\alpha h_{-i\mu}(0)(i\tau) D_x^\alpha g_{i\mu}(0)(i\tau + i\mu) \right\},$$

where we have omitted the variables x and ξ for better legibility. We may now proceed similarly as in [5, Theorem 1.4] to obtain the assertion. ◁

Traces on the Cylinder Algebra

For the proof of the uniqueness of res on the extended cone algebra and the definition of yet another trace on the smaller ideal $C(\mathbb{B}, g)_0^+$ it will be more convenient to work on the cylinder $X^\wedge = X \times \mathbb{R}_+$. We shall employ the "cylinder algebra" $C^\mu(X^\wedge, g)^+$, i.e., the algebra generated by the Mellin operators supported close to the boundary $\partial \mathbb{B}$.

2.8 Definition. *By $C^\mu(X^\wedge, g)^+$ denote the algebra of all operators generated by the elements $\omega_1 t^m [op_M^\gamma h_m] \omega_2$ with arbitrary cut-off functions ω_1, ω_2, $\operatorname{Re} m \geq 0$, and $h_m \in C^\infty(\overline{\mathbb{R}}_+, M_P^\mu(X))$ for arbitrary Mellin asymptotic types. By $C_{M+G}(X^\wedge, g)^+$ denote the subspace generated by all operators whose Mellin symbol is smoothing, i.e., where $h_m \in C^\infty(\overline{\mathbb{R}}_+, M_P^{-\infty}(X))$ for a suitable asymptotic type P. We let $C(X^\wedge, g)^+ = \bigcup_\mu C^\mu(X^\wedge, g)^+$.*

Just as before we define $C^\mu(X^\wedge, g)_0^+$ as the subspace of all those elements in $C^\mu(X^\wedge, g)^+$, where $\operatorname{Re} m > 0$ for all m involved. Finally, $C_{M+G}(X^\wedge, g)_0^+$ denotes the subspace of operators that in addition have smoothing Mellin symbols.

Indeed, it follows as before – see Proposition 2.2 – that $C(X^\wedge, g)^+$ is an algebra. The spaces $C(X^\wedge, g)_0^+$, $C_{M+G}(X^\wedge, g)^+$, and $C_{M+G}(X^\wedge, g)_0^+$ clearly form ideals.

With the same considerations as before we obtain the following theorem:

2.9 Theorem. *The trace " res" also yields a trace on $C(X^\wedge, g)^+$. It vanishes on $C_{M+G}(X^\wedge, g)^+$ and on $C(X^\wedge, g)_0^+$.*

On the subalgebra $C(X^\wedge, g)_0^+$ we find another trace.

2.10 Definition. *Let $A = \sum_{m \in \mathcal{C}} \omega_1 t^m [op_M^\gamma h_m] \omega_2 + R$ with a finite subset $\mathcal{C}$ of $\{\operatorname{Re} z > 0\}$, cut-off functions ω_1, ω_2 satisfying additionally $\omega_1 \omega_2 = \omega_1$, $h_m \in C^\infty(\overline{\mathbb{R}}_+, M_O^\mu(X))$, and $R \in C_{M+G}(X^\wedge, g)_0^+$. Let $h(t) = \omega_1(t) \sum_{m \in \mathcal{C}} t^m h_m(t)$ and define*

$$res_{x,t}^0 A = \int_{-\infty}^\infty \int_S Tr\, h(t)_{-n-1}(x, \xi, i\tau) \sigma_\xi \, d\tau dx_1 \wedge \dots dx_n \wedge \frac{dt}{t}.$$

The condition $\omega_1 \omega_2 = \omega_1$ is necessary for this to make sense. One next establishes

2.11 Lemma. *Let $\sigma_{\xi,\tau} = (-1)^{n+1} \tau d\xi_1 \wedge \dots \wedge d\xi_n + \sigma_\xi \wedge d\tau$ be the n-form constructed in analogy to the construction of σ_ξ. Then*

$$res_{x,t}^0 A = \int_{\{|\xi,\tau|=1\}} Tr\, h(t)_{-n-1}(x, \xi, i\tau) \sigma_{\xi,\tau} dx_1 \wedge \dots dx_n \wedge \frac{dt}{t},$$

and $res_{x,t}^0 A$ is a density on $X^\wedge$.

2.12 Definition. *In the notation of Definition 2.10 we can therefore let*

$$res^0 A = \int_{X^\wedge} res_{x,t}^0 A.$$

This makes sense, since all m have positive real parts. Moreover we get:

2.13 Theorem. *res^0 is a trace on $C(X^\wedge, g)_0^+$. It vanishes on $C_{M+G}(X^\wedge, g)_0^+$.*

We shall now show that there are no other traces. As a preparation we need the following results.

2.14 Lemma. Let $f \in M_O^\mu(X)$. For $\varepsilon > 0$ let

$$f_\varepsilon(z) = \frac{f(z + \varepsilon) - f(z)}{\varepsilon} \quad \text{and} \quad g_\varepsilon(z) = \frac{f(z + i\varepsilon) - f(z)}{i\varepsilon}.$$

Then $f_\varepsilon \to \partial_z f$ and $g_\varepsilon \to \partial_z f$ in the topology of $M_O^\mu(X)$.

2.15 Proposition. By $\mathcal{G}$ denote the space of all finite sums of functions of the form $f = f(t, x) = t^m f_1(t, x)$, where $f_1 \in C_0^\infty([0, 1) \times (-1, 1)^n)$ and $\operatorname{Re} m \geq 0$. By $\mathcal{G}^{(0)}$ denote the subspace of all $f \in \mathcal{G}$ with $\operatorname{Re} m > 0$ for all m involved.
   (a) We can define a linear map $T : \mathcal{G} \to \mathbb{C}$ by

$$Tf = \int_{(-1,1)^n} f(0, x)\, dx.$$

It vanishes on all functions f for which there is an $F \in \mathcal{G}$ with $t\partial_t F = f$ or $\partial_{x_j} F = f$ for some $1 \leq j \leq n$. Any other linear mapping $\mathcal{G} \to \mathbb{C}$ with this property coincides with T up to a multiplicative constant.
   (b) We have a linear map $T : \mathcal{G}^{(0)} \to \mathbb{C}$ by

$$T_0 f = \int_0^1 \int_{(-1,1)^n} f(s, x)/s\, dx\, ds.$$

It vanishes on all f for which there is an $F \in \mathcal{G}^{(0)}$ with $t\partial_t F = f$ or $\partial_{x_j} F = f$ for some $1 \leq j \leq n$. Any other linear mapping with this property coincides with T_0 up to a multiplicative constant.

2.16 Theorem. Any continuous trace on $C(X^\wedge, g)^+/C_{M+G}(X^\wedge, g)^+$ coincides with a multiple of "res".

Any continuous trace on the subalgebra $C(X^\wedge, g)_0^+/C_{M+G}(X^\wedge, g)_0^+$ coincides with a multiple of "res^0".

The only continuity assumption we need here is that the convergence of a sequence of Mellin symbols $h_j \to h$ in $C^\infty(\overline{\mathbb{R}}_+, M_P^\mu(X))$ entails the convergence of the traces of the operators $\omega_1[op_M^\gamma h_j]\omega_2$ to the trace of $\omega_1[op_M^\gamma h]\omega_2$.

Proof. Let "tr" be such a trace, and let $A \in C^\mu(X^\wedge, g)$ be an operator supported inside a single coordinate neighborhood U for X, where U is diffeomorphic to $(-1, 1)^n$. Let $\omega_1, \omega_2, \omega_3$ be cut-off functions with $\omega_1\omega_2 = \omega_1, \omega_2\omega_3 = \omega_2$. Since the trace by assumption vanishes on the C_{M+G}-ideals, let $A = \omega_1 t^m[op_M^\gamma h]\omega_2$ with $h \in C^\infty(\overline{\mathbb{R}}_+, M_O^\mu(X))$, $\operatorname{Re} m \geq 0$. By X_j and $D_j, j = 1, \ldots, n$, denote the operators with the symbols $g_1(t)(x, \xi, z) = \omega_3(t)x_j$ and $g_2(t)(x, \xi, z) = \omega_3(t)\xi_j$, respectively, on U.

The commutator $[X_j, A]$ has the Mellin symbol $it^m\omega_1(t)\partial_{\xi_j}h$, while $[D_j, A]$ has the Mellin symbol $-it^m\omega_1(t)\partial_{x_j}h$. Since "tr" vanishes on commutators, it vanishes on all operators whose Mellin symbols are derivatives with respect to some x_j or $\xi_j, 1 \leq j \leq n$. Similarly we have the operator $\omega_3(t)t\partial_t$ with the Mellin symbol $\omega_3(t)z$; the commutator $[\omega_3(t)t\partial_t, A] = [t\partial_t, A]$ has the Mellin symbol $t\partial_t\{t^m h(t)\}$. Hence "tr" also vanishes on Mellin symbols that are totally characteristic derivatives with respect to t.

Finally we let T_ε be the operator with the Mellin symbol $\omega_3(t)t^{i\varepsilon}, \varepsilon > 0$. It is a basic property of Mellin operators, see e.g. [10, Lemma 3.1.10], that the commutator $[T_\varepsilon, A]$ then has the Mellin symbol $t^{m+i\varepsilon}h(t)(x,\xi,z)\omega(t) - t^{m+i\varepsilon}h(t)(x,\xi,z-i\varepsilon)$. Since we may choose $m = -i\varepsilon$, "tr" vanishes on all symbols of the form $h(t)(x,\xi,z) - h(t,x,\xi,-i\varepsilon)$. The continuity in connection with Lemma 2.14 implies that it will also vanish on all Mellin symbols that are derivatives with respect to z. On $C(X^\wedge, g)_0^+$ we may work with multiplications by t^ε.

Now a construction similar to that in the proof of [5, Theorem 1.4] in connection with Lemma 2.15 completes the argument. $\triangleleft$

The Manifold Case and an Extension of the Wodzicki Residue

2.17 Definition. *By $C^\infty(\overline{\mathbb{R}}_+, \widetilde{L}^\mu(X;\mathbb{R}))$ we denote the set of all $p \in C^\infty(\overline{\mathbb{R}}_+, L^\mu(X;\mathbb{R}))$ for which there is a $q \in C^\infty(\overline{\mathbb{R}}_+, L^\mu(X;\mathbb{R}))$ with $p(t,\tau) = q(t,t\tau)$.*

We call these operator-valued symbols totally characteristic.

2.18 Mellin Quantization. *It was shown in Schrohe-Schulze [10, Section 2.4] that, for $h \in C^\infty(\overline{\mathbb{R}}_+, M_O^\mu(X))$, there is a $p \in C^\infty(\overline{\mathbb{R}}_+, \widetilde{L}^\mu(X;\mathbb{R}))$ with*

$$op\, p \equiv op_M^\gamma h \quad mod \ L^{-\infty}(X^\wedge) \qquad (2.1)$$

and vice versa. We have the asymptotic expansion

$$p(t)(x,\xi,\tau) \sim \sum_{k=0}^\infty \partial_{t'}^k D_\tau^k \left\{ h(t)(x,\xi,-iT(t,t')\tau)\,T(t,t')/t' \right\}|_{t'=t},$$

where $T(t,t') = \frac{t-t'}{\ln t - \ln t'}$. Note that $T(t,t) = t$. Applying this formula, one gets:

2.19 Proposition. *Let A be as in Definition 2.10, and let $op\, p \in L^\mu(X^\wedge)$ be the pseudodifferential operator associated with A modulo $L^{-\infty}(X^\wedge)$ according to Theorem 2.18. Then*

$$-res^0 A = \int_{X^\wedge} \int_{\{|\xi,\tau|=1\}} Tr\, p(t)_{-n-1}(x,\xi,\tau)\,\sigma_{\xi,\tau}\,dx \wedge dt = \ W\text{-res}\,op\,p,$$

where W-res for the moment denotes the Wodzicki residue.

This leads to the following theorem.

2.20 Theorem. *The dimension of the space of continuous traces on $C(\mathbb{B}, g)^+/C_{M+G}(\mathbb{B}, g)^+$ equals the number of conical points.*

On $C(\mathbb{B}, g)_0^+/C_{M+G}(\mathbb{B}, g)_0^+$ there is only one non-trivial continuous trace (up to multiples). It extends the Wodzicki residue.

Proof. In the interior, the only traces are the multiples of Wodzicki's residue. On the other hand, each trace on $C(\mathbb{B}, g)^+/C_{M+G}(\mathbb{B}, g)^+$ yields a trace on the cylinder algebra $C(X^\wedge, g)^+/C_{M+G}(X^\wedge, g)^+$. For operators supported by compact sets in $X \times (0,1)$ both have to agree. This shows that the only possible traces on $C(\mathbb{B}, g)^+/C_{M+G}(\mathbb{B}, g)^+$ are the multiples of "res". For simplicity we had been working with one conical point; the argument now shows that we may pick a constant for each of them.

In the case of $C(\mathbb{B}, g)_0^+ / C_{M+G}(\mathbb{B}, g)_0^+$ we know from Proposition 2.19 that the only possible choice on the cylinder, namely "res^0", extends Wodzicki's residue and therefore is a trace on the full algebra $C(\mathbb{B}, g)_0^+$. ◁

2.21 Remark. *On Schulze's original cone algebra there are infinitely many traces. In fact, for $k \in \mathbb{N}$ and $A \in C^\mu(\mathbb{B}, g)$ of the form $A = \omega_1 [op_M^\gamma h] \omega_2 + R$ with cut-off functions ω_1, ω_2 near zero and $R \in C_{M+G}(\mathbb{B}, g)$ let*

$$res^k A = \int_X \int_S \int_{-\infty}^{\infty} Tr\, h(0)_{-n-1-k}(x, \xi, i\tau) \tau^k \, d\tau \, \sigma_\xi \, dx.$$

Then res^k is a trace on $C(\mathbb{B}, g)$.

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