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FUNCTIONS OF THE LAPLACE–BELTRAMI OPERATOR

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Let M be a closed n-dimensional Riemannian manifold, $C^\infty(M;\Omega^{1/2})$ be the space of smooth half-densities on M, $L_2(M;\Omega^{1/2})$ be the Hilbert space of half-densities with the natural inner product $(u,v) = \int_M u(x)\,\bar v(x)\,dx$, and $H^s(M;\Omega^{1/2})$, $s \in \mathbf{R}^1$, be the corresponding Sobolev spaces. We shall always deal with operators acting in the spaces of half-densities. This is just a technical assumption, using the natural isomorphism

$$C^\infty(M;\Omega^{1/2}) \ni u \to g^{-1/2}u \in C^\infty(M)$$

one can easily reformulate all results for the operators acting in the space of functions.

The Laplace-Beltrami operator Δ on M is defined in any local coordinates by

$$\Delta u(x) \;=\; g^{-1/2}(x) \sum_{i,j} \partial_{x^i} \left(g(x)\, g^{ij}(x)\, \partial_{x^j} \left(g^{-1/2}(x) u(x)\right)\right) ,$$

where g^{ij} is the metric tensor and $g := |\det\{g^{ij}\}|^{-1/2}$ is the the canonical Riemannian density. Clearly, Δ is a symmetric negative operator in $L_2(M;\Omega^{1/2})$.

Let ν be a symmetric first order differential operator. Set

$$A_\nu \;=\; \sqrt{-\Delta+\nu}\,, \qquad A_0 \;=\; \sqrt{-\Delta}$$

(we retain the same notation for the self-adjoint extensions of the operators Δ and ν). It is well known that A_ν is a pseudodifferential operator (ψDO) of Hörmander's class $\Psi^1_{1,0}$ with principal symbol

$$|\xi|_x \;:=\; \Big(\sum_{i,j} g^{ij}(x)\,\xi_i\,\xi_j\Big)^{1/2}, \qquad (x,\xi) \in T^*M$$

(see definitions below). The main aim of the paper is to show that under some restrictions on $\omega \in C^\infty(\mathbf{R}^1)$ the operators $\omega(A_\nu)$ also belong to appropriate classes of ψDOs.

Throughout the paper we shall use some elementary notions and results from differential geometry which can be found, for example, in [KN].

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1. Hörmander's classes of pseudodifferential operators. Let $\langle\xi\rangle_x = \sqrt{|\xi|_x^2+1}$. We say that a function $a \in C^\infty(M_y \times T^*M_x)$ belongs to Hörmander's class $\mathrm{S}^m_{\rho,\delta}$ if in any local coordinates

$$|\partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma a(y,x,\xi)| \leqslant \mathrm{const}_{\alpha,\beta,\gamma}\ \langle\xi\rangle_x^{m-\rho|\alpha|+\delta|\beta|+\delta|\gamma|} \tag{1}$$

for all multiindices α, β and γ.

Let $A : C^\infty(M;\Omega^{1/2}) \to C^\infty(M;\Omega^{1/2})$ be a linear operator with Schwartz kernel $\mathcal{A}(x,y)$. The operator A is said to be a ψDO of the class $\Psi^m_{\rho,\delta}$ if

(1) $\mathcal{A}(x,y)$ is smooth outside the diagonal in $M\times M$;
(2) in each coordinate patch $\mathcal{U}\times\mathcal{U} \subset M\times M$ the kernel $\mathcal{A}(x,y)$ is represented modulo a smooth function by an oscillatory integral of the form

$$(2\pi)^{-n}\int e^{i(x-y)\cdot\xi} a(y,x,\xi)\,d\xi\,, \qquad a\in \mathrm{S}^m_{\rho,\delta}\,.$$

If $1-\rho \leqslant \delta < \rho$ then there exists a function $a_m \in C^\infty(T^*M)$ such that

$$a(x,x,\xi) \;=\; a_m(x,\xi) \pmod{\mathrm{S}^{m+\delta-\rho}_{\rho,\delta}}$$

for all the local amplitudes a. The function a_m is determined modulo $\mathrm{S}^{m+\delta-\rho}_{\rho,\delta}$ by the ψDO A and is called the *principal symbol* of A (see [H], [Sh], [T], [Tr] for more precise definitions).

2. Classes of functions ω. Let $0<\rho\leqslant 1$. We denote by $\mathrm{S}^m_\rho(\mathbf{R}^1)$ the class of functions $\omega\in C^\infty(\mathbf{R}^1)$ such that

$$|\,\omega^{(k)}(s)\,| \;\leqslant\; \mathrm{const}_k\,(1+|s|)^{m-k\rho}\,, \qquad \forall\, k=0,1,\ldots,$$

where $\omega^{(k)}$ stands for the kth derivative $\partial_s^k\omega$. It is easy to see that for any function $\omega\in\mathrm{S}^m_\rho(\mathbf{R}^1)$ we have $t^k\hat\omega(t)\in C^{N_k}(\mathbf{R}^1)$, where $N_k\to+\infty$ as $k\to+\infty$ (in other words, $t^k\hat\omega(t)$ gets smoother and smoother as $k\to+\infty$). Therefore the Fourier transform $\hat\omega(t)$ of a function $\omega\in\mathrm{S}^m_\rho(\mathbf{R}^1)$ coincides with a smooth rapidly decreasing function outside any neighbourhood of the origin $t=0$.

3. The cases $\rho>1/2$ and $\rho<1/2$. It has been already proved [T, Ch. XII.3] that $\omega(A_\nu)$ is a ψDO of the class $\Psi^m_{\rho,1-\rho}$ with principal symbol $\omega(|\xi|_x)$ if $\omega\in\mathrm{S}^m_\rho(\mathbf{R}^1)$ with $\rho>1/2$. In [T, Ch. XII.3] the author conjectured that for $0<\rho\leqslant 1/2$ the operators $\omega(A_\nu)$ can also be included in some classes of ψDOs. In particular, this would imply that all such operators are pseudolocal, that is, sing supp $(\omega(A_\nu)u)\subset$ sing supp u for all $u\in L_2(M;\Omega^{1/2})$. Note that the latter may not be true if we only assume the function ω to be bounded with all its derivatives (this corresponds to $\rho=0$). For example, e^{iA_ν} is a Fourier integral operator which is not pseudolocal.

We have $\omega(|\xi|_x)\in\mathrm{S}^m_{\rho,1-\rho}$ for all $\omega\in\mathrm{S}^m_\rho(\mathbf{R}^1)$, $0<\rho\leqslant 1$. However, if $\rho<1/2$ then $1-\rho=\delta>\rho$. The condition $\delta<\rho$ plays the crucial role in the standard (coordinate) theory of ψDOs; if it is not fulfilled then almost all classical results fail. Moreover, in this case the principal symbol is not invariantly defined (it may depend on the choice of local oscillatory integrals). We overcome this difficulty by dealing with symbols from some special subclasses of $\mathrm{S}^m_{\rho,1-\rho}$ and introducing ψDOs in an invariant (coordinate free) way.

4. Classes of symbols. We say that a function $a \in C^\infty(T^*M)$ belongs to the class $\mathrm{S}^m_\rho(\mathbf{g})$ if a admits an asymptotic expansion of the form

$$a(x,\xi) \sim \sum_{k=0}^{\infty} c_k(x,\xi)\,\omega_k(|\xi|_x)\,, \qquad |\xi|_x \to \infty\,,$$

where $c_k \in \mathrm{S}^{l_k}_{1,0}$, $\omega_k \in \mathrm{S}^{m_k}_\rho(\mathbf{R}^1)$, $l_0+m_0 = m$, $l_k+m_k \geqslant l_{k+1}+m_{k+1}$, and $l_k+m_k \to -\infty$ as $k \to \infty$. Here and further on the sign $\sim$ means that the asymptotic expansion is uniform with respect to all the parameters involved and can be differentiated infinitely many times.

Obviously, if $a \in \mathrm{S}^m_\rho(\mathbf{g})$ and $b \in \mathrm{S}^{m'}_\rho(\mathbf{g})$ with $m' \leqslant m$ then $ab \in \mathrm{S}^{m+m'}_\rho(\mathbf{g})$ and $a+b \in \mathrm{S}^m_\rho(\mathbf{g})$. We also have $\mathrm{S}^m_\rho(\mathbf{g}) \subset \mathrm{S}^m_{\rho,1-\rho}$, so in any local coordinates the functions from $\mathrm{S}^m_\rho(\mathbf{g})$ satisfy (1) (as these functions are independent of y, all the y-derivatives in (1) vanish).

Given local coordinates x, let us denote

$$\nabla_{x^k} = \partial_{x^k} + \sum_{i,j} \Gamma^i_{kj}(x)\,\xi_i\,\partial_{\xi_j}\,, \tag{2}$$

where Γ^i_{kj} are the Christoffel symbols of the Levi–Civita connection. The first order differential operator ∇_{x^k} is identified with a vector field on T^*M which is called the horizontal lift of the coordinate vector field ∂_{x^k}. By definition of the Levi–Civita connection, $\nabla_{x^k}(|\xi|_x) = 0$ for all k. Therefore for any function $a \in \mathrm{S}^m_\rho(\mathbf{g})$ we have

$$|\partial^\alpha_\xi \nabla_{x^{i_1}} \nabla_{x^{i_2}} \ldots \nabla_{x^{i_p}} a(x,\xi)| \leqslant \mathrm{const}_{\alpha,i_1,\ldots,i_p}\, \langle\xi\rangle_x^{m-\rho|\alpha|} \tag{3}$$

for all multiindices α and indices $i_1,\ldots,i_p$ (cf. (1)).

The differential operators ∇_{x^k} are not commuting; for example,

$$[\nabla_{x^k}, \nabla_{x^l}] = \sum_{i,j} R^i{}_{jkl}(x)\,\xi_i\,\partial_{\xi_j}\,,$$

where $R^i{}_{jkl}$ are the components of the curvature tensor. Let $\tilde{x}$ be the normal (exponential) coordinates with origin x and $\tilde{\nabla}_k$ be the operators (2) corresponding to these coordinates. The symmetrization of the tensor $\tilde{\nabla}_{i_1}\tilde{\nabla}_{i_2}\ldots\tilde{\nabla}_{i_p}a(x,\xi)$ with respect to the indices $i_1,\ldots,i_p$ is said to be the pth symmetric *horizontal differential* of the function a at the point (x,ξ). We shall denote the components of this tensor by $\nabla^\alpha_x a(x,\xi)$, where α are the multiindices of length p.

Remark. One can define the classes $\Psi^m_{\rho,0}(\Gamma)$ of symbols on T^*M assuming (3) instead of (1) (here Γ stands for the Levi–Civita connection). For the corresponding classes of ψDOs all the classical results remain valid under condition $\rho > 1/3$ (see [S]).

5. Definition of ψDOs. Let $\mathcal{V}$ be some sufficiently small neighbourhood of the diagonal in $M \times M$. For $(x,y) \in \mathcal{V}$ let $\gamma_{y,x}(t)$ the shortest geodesic joining x and y such that $\gamma_{y,x}(0) = x$ and $\gamma_{y,x}(1) = y$. This geodesic exists and is uniquely defined.

We define the global *phase function* $\varphi(x,\xi,y)$ by

$$\varphi(x,\xi,y) = -\langle\dot\gamma_{y,x}(0),\xi\rangle\,, \qquad (x,y)\in\mathcal{V}\,,\ \xi\in T_x^*M$$

(similar phase functions have been considered in [D]). Obviously, the phase function φ is linear in ξ. If y are the same coordinates as x then

$$\varphi(x,\xi,y) \sim (x-y)\cdot\xi - \frac{1}{2}\sum_{i,j,k}\Gamma_{ij}^k(x)\,(x^i-y^i)\,(x^j-y^j)\,\xi_k + O(|x-y|^3)\,, \qquad y\to x\,. \quad (4)$$

If x are arbitrary coordinates and y are the normal coordinates with origin x such that $\partial x^k/\partial y^j=\delta_j^k$ at the origin then

$$\varphi(x,\xi,y) = (x-y)\cdot\xi\,. \quad (5)$$

Remark. In the classical (coordinate) theory of ψDOs one deals with phase functions of the form (5) assuming, however, that the coordinates y are the same as x.

We associate with a function $a\in \mathrm{S}_\rho^m(\mathbf{g})$ the oscillatory integral

$$\mathcal{A}(x,y) = (2\pi)^{-n}\,g^{-1/2}(x)\,g^{1/2}(y)\int e^{i\varphi(x,\xi,y)}a(x,\xi)\,d\xi\,, \qquad (x,y)\in\mathcal{V}\,. \quad (6)$$

Under change of coordinates $\mathcal{A}$ behaves as a half-density on $M\times M$ and, in view of (4), in any coordinate patch $\mathcal{A}$ coincides with the Schwartz kernel of a ψDO of the class $\Psi^m_{\rho,1-\rho}$.

We say that an operator A is a ψDO of the class $\Psi_\rho^m(\mathbf{g})$ if its Schwartz kernel is smooth outside the diagonal and is represented in a neighbourhood of the diagonal by the oscillatory integral (6) with some $a\in\mathrm{S}_\rho^m(\mathbf{g})$. The function a is said to be the (full) *symbol* of A and is denoted by σ_A.

The Schwartz kernel of a ψDO $A\in\Psi^m_{1,0}$ can be represented by the oscillatory integral (6) with $a\in\mathrm{S}^m_{1,0}$ (see [S]). Therefore $\Psi^m_{1,0}\subset\Psi_\rho^m(\mathbf{g})$.

6. Composition of ψDOs and adjoint operators. Let

$$\psi(x,\xi;y,z) = \langle\dot\gamma_{y,x},\xi\rangle - \langle\dot\gamma_{z,x},\xi\rangle - \langle\dot\gamma_{y,z},\Phi_{z,x}\xi\rangle\,,$$

where $\xi\in T_x^*M$ and $\Phi_{z,x}:T_x^*M\to T_z^*M$ is the parallel displacement along the geodesic $\gamma_{z,x}$. Let x be arbitrary coordinates and y and z be the normal coordinate systems with origin x such that $\partial x^k/\partial y^j=\partial x^k/\partial z^j=\delta_j^k$ at the origin. We define

$$P_{\beta,\gamma}(x,\xi) = g^{-1}(x)\,(\partial_y+\partial_z)^\beta\Big(g(z)\sum_{|\beta'|\leqslant|\beta|}\frac{1}{\beta'!}D_\xi^{\beta'}\,\partial_y^{\gamma+\beta'}e^{i\psi}\Big)\Big|_{y=z=x}\,.$$

The functions $P_{\beta,\gamma}\in C^\infty(T^*M)$ are polynomials in ξ; we denote their degrees by $d_{\beta,\gamma}$. The coefficients of $P_{\beta,\gamma}$ are components of some tensors, which are polynomials in the curvature tensor and its covariant differentials. One can prove (see [S]) that $P_{0,0}\equiv1$, $P_{\beta,0}\equiv0$ for all $\beta\neq0$, $P_{0,\gamma}\equiv0$ for all $\gamma\neq0$, and

$$d_{\beta,\gamma} \leqslant \min\{\,|\beta|,|\gamma|,(|\beta|+|\gamma|)/3\,\}\,, \qquad \forall\,\beta,\gamma\,. \quad (7)$$

Proposition 1 ([S]). *Let* $A \in \Psi^{m_1}_{1,0}$, $B \in \mathrm{S}^{m_2}_\rho(\mathbf{g})$ *or* $A \in \Psi^{m_1}_\rho(\mathbf{g})$, $B \in \mathrm{S}^{m_2}_{1,0}$. *Then* $AB \in \Psi^{m_1+m_2}_\rho(\mathbf{g})$ *and*

$$\sigma_{AB}(x,\xi) \sim \sum_{\alpha,\beta,\gamma} \frac{1}{\alpha!}\,\frac{1}{\beta!}\,\frac{1}{\gamma!}\, P_{\beta,\gamma}(x,\xi)\, D_\xi^{\alpha+\beta}\sigma_A(x,\xi)\, D_\xi^\gamma \nabla_x^\alpha \sigma_B(x,\xi)\,, \qquad |\xi|_x \to \infty\,. \tag{8}$$

Note that in view of (7) the terms in the right-hand side do form an asymptotic series.

Theorem 2 ([S]). *If* $A \in \Psi^m_\rho(\mathbf{g})$ *then* $A^* \in \Psi^m_\rho(\mathbf{g})$ *and*

$$\sigma_{A^*}(x,\xi) \;\sim\; \sum_\alpha \frac{1}{\alpha!}\, D_\xi^\alpha \nabla_x^\alpha\, \overline{\sigma_A(x,\xi)}\,, \qquad |\xi|_x \to \infty\,.$$

By Theorem 2, if A is a ψDO with symbol $\omega(|\xi|_x)$ then $A - A^*$ is an operator with infinitely smooth kernel.

7. The operator $\omega(A_\nu)$**.** Proposition 1 implies the following

Lemma 3 ([S]). *Let* $B \in \Psi^{m_1}_{1,0}$ *and* A *be a* ψDO *with symbol* $\omega(|\xi|_x)$, $\omega \in \mathrm{S}^m_\rho(\mathbf{R}^1)$. *Then*

$$\sigma_{BA}(x,\xi) - \sigma_B(x,\xi)\,\omega(|\xi|_x) \;\sim\; \sum_{j=1}^\infty b_j(x,\xi)\,\omega^{(j)}(|\xi|_x)\,, \qquad |\xi|_x \to \infty\,, \tag{9}$$

$$\sigma_{AB}(x,\xi) - \sigma_B(x,\xi)\,\omega(|\xi|_x) \;\sim\; \sum_{j=1}^\infty \tilde b_j(x,\xi)\,\omega^{(j)}(|\xi|_x)\,, \qquad |\xi|_x \to \infty\,, \tag{10}$$

where $b_j \in \mathrm{S}^{m_1-1}_{1,0}$ *and* $\tilde b_j \in \mathrm{S}^{m_1}_{1,0}$. *The functions* b_j *and* $\tilde b_j$ *depend only on the operator* B *and Riemannian metric* $\mathbf{g}$.

Remark. The asymptotic expansions (9) and (10) are easily obtained from (8); the only difficulty is to prove that $b_j \in \mathrm{S}^{m_1-1}_{1,0}$ (one would expect $b_j \in \mathrm{S}^{m_1}_{1,0}$).

Theorem 4 ([S]). *If* $\omega \in \mathrm{S}^m_\rho(\mathbf{R}^1)$ *then* $\omega(A_\nu) \in \Psi^m_\rho(\mathbf{g})$ *and*

$$\sigma_{\omega(A_\nu)}(x,\xi) \;\sim\; \omega(|\xi|_x) + \sum_{j=1}^\infty c_{\nu,j}(x,\xi)\,\omega^{(j)}(|\xi|_x)\,, \qquad |\xi|_x \to \infty\,,$$

where $c_{\nu,j} \in \mathrm{S}^0_{1,0}$. *The functions* $c_{\nu,j}$ *are determined recursively from the system of equations*

$$\sigma_{A_\nu^k}(x,\xi) \;=\; |\xi|_x^k + \sum_{j=1}^k \frac{k!}{(k-j)!}\, |\xi|_x^{k-j}\, c_{\nu,j}(x,\xi)\,.$$

In particular,

$$c_{\nu,1}(x,\xi) \;=\; \frac{1}{12}\,|\xi|_x^{-1}\Big(\sum_{j,k} R_{jk}(x)\,\hat\xi^j\,\hat\xi^k - 2\,S(x)\Big)$$
$$+\;\frac{1}{2}\,|\xi|_x^{-1}\,\sigma_\nu(x,\xi) \;+\; \frac{i}{4}\,|\xi|_x^{-2}\sum_j \hat\xi^j\,\nabla_{x^j}\sigma_\nu(x,\xi) \;-\; \frac{1}{8}\,|\xi|_x^{-3}\,\sigma_\nu^2(x,\xi) \qquad (\mathrm{mod}\ \mathrm{S}^{-2}_{1,0})$$

$$c_{\nu,2}(x,\xi) \;=\; -\frac{1}{12}\sum_{j,k} R_{jk}(x)\,\hat\xi^j\,\hat\xi^k$$
$$-\;\frac{i}{4}\,|\xi|_x^{-1}\sum_j \hat\xi^j\,\nabla_{x^j}\sigma_\nu(x,\xi)\;+\;\frac{1}{8}\,|\xi|_x^{-2}\,\sigma_\nu^2(x,\xi)\qquad (\mathrm{mod}\ \mathrm{S}_{1,0}^{-1}),$$

where $\hat\xi^j := \partial_{\xi_j}|\xi|_x$, R_{jk} *are the components of the Ricci tensor,* $S(x)$ *is the scalar curvature, and* σ_ν *is the symbol of the differential operator* ν *as defined in section 5.*

Sketch of the proof. Let $0 < r < 1$ and $U_r(t) = \exp(itA_\nu^r)$. The operator $U_r(t)$ is the unique solution of the Cauchy problem

$$D_t U_r(t) - A_\nu^r\,U_r(t) = 0\,,\qquad U_r(0) = I\,. \tag{11}$$

Using Proposition 1 and (9), we construct an approximate solution of (11) as a ψDO of the class $\Psi_{1-r}^0(\mathbf{g})$ (in this construction the fact that $b_j \in \mathrm{S}_{1,0}^{m_1-1}$ is of central importance). Then, by well known *a priori* estimates, $U_r(t) \in \Psi_{1-r}^0(\mathbf{g})$. Finally, given $\omega \in \mathrm{S}_\rho^m(\mathbf{R}^1)$, we take $r \in (1-\rho, 1)$, set $\omega_r(s) = \omega(s^{1/r})$ and apply the inversion formula for the Fourier transform:

$$\omega(A_\nu) \;=\; (2\pi)^{-1}\int \hat\omega_r(t)\,e^{itA_\nu^r}\,dt\,.$$

Theorem 4 immediately implies

Corollary 5. *Let* $\omega \in \mathrm{S}_\rho^m(\mathbf{R}^1)$ *and* $Q = \omega(A_\nu) - \omega(A_0)$. *Then* $Q \in \Psi_\rho^{m-\rho}(\mathbf{g})$ *and*

$$\sigma_Q \;=\; \frac{1}{2}\,|\xi|_x^{-1}\,\omega'(|\xi|_x)\,\sigma_\nu \;+\; \frac{i}{4}\,|\xi|_x^{-1}\Big(|\xi|_x^{-1}\omega'(|\xi|_x) - \omega''(|\xi|_x)\Big)\sum_j \hat\xi^j\,\nabla_{x^j}\sigma_\nu$$
$$-\;\frac{1}{8}\,|\xi|_x^{-2}\Big(|\xi|_x^{-1}\omega'(|\xi|_x) - \omega''(|\xi|_x)\Big)\,\sigma_\nu^2\qquad (\mathrm{mod}\ \mathrm{S}_\rho^{m-3\rho}(\mathbf{g}))\,.$$

8. Operator series. The following lemma is a simple consequence of Lemma 3 and Theorem 4.

Lemma 6. *Let* A *be a* ψ*DO with symbol* $b(x,\xi)\,\omega(|\xi|_x)$, *where* $b \in \mathrm{S}_{1,0}^{m_1}$ *and* $\omega \in \mathrm{S}_\rho^m$. *Then*

$$A \;\sim\; B\,\omega(A_0) \;+\; \sum_{k=1}^{\infty} B_k\,\omega^{(k)}(A_0)\,,$$

where $B \in \Psi_{1,0}^{m_1}$ *is the* ψ*DO with symbol* $b(x,\xi)$ *and* $B_k \in \Psi_{1,0}^{m_1-1}$. *The operators* B_k *are determined by the function* b *and Riemannian metric* $\mathbf{g}$.

Here and further on (when we deal with operator series) the sign $\sim$ means that the difference between the left-hand side and the finite sum up to $k = N$ in the right-hand side becomes more and more smoothing operator as $N \to \infty$.

Obviously, Theorem 4 and Lemma 6 imply

Theorem 7. *$A \in \Psi_\rho^m(\mathbf{g})$ if and only if there exist ψDOs $C_k \in \Psi_{1,0}^{l_k}$ and functions $\omega_k \in \mathrm{S}_\rho^{m_k}$ such that*

$$A \sim \sum_{k=0}^{\infty} C_k\, \omega_k(A_0)\,,$$

$l_0 + m_0 = m$, $l_k + m_k \geqslant l_{k+1} + m_{k+1}$ and $l_k + m_k \to -\infty$ as $k \to \infty$.

Theorem 4 and Lemma 6 also imply

Corollary 8. *If $\omega \in \mathrm{S}_\rho^m$ then*

$$\omega(A_\nu) \sim \omega(A_0) + \sum_{j=1}^{\infty} P_{\nu,j}\, \omega^{(j)}(A_0)\,,$$

where $P_{\nu,j} \in \Psi_{1,0}^0$ are some ψDOs depending only on the operator ν and Riemannian metric $\mathbf{g}$. The symbols of $P_{\nu,j}$ coincide with $c_{\nu,j} - c_{0,j}$ modulo $\Psi_{1,0}^{-1}$ (here $c_{\nu,j}$ are the functions introduced in Theorem 4 and $c_{0,j}$ are those corresponding to $\nu = 0$). In particular,

$$\begin{aligned}
\sigma_{P_{\nu,1}}(x,\xi) &= \frac{1}{2}\,|\xi|_x^{-1}\,\sigma_\nu(x,\xi) \qquad (\mathrm{mod}\ \mathrm{S}_{1,0}^{-1})\,,\\
\sigma_{P_{\nu,2}}(x,\xi) &= -\frac{i}{4}\,|\xi|_x^{-1}\sum_j \hat{\xi}^j\,\nabla_{x^j}\sigma_\nu(x,\xi) + \frac{1}{8}\,|\xi|_x^{-2}\,\sigma_\nu^2(x,\xi) \qquad (\mathrm{mod}\ \mathrm{S}_{1,0}^{-1})\,.
\end{aligned}$$

9. Boundedness and composition of ψDOs of the class $\Psi_\rho^m(\mathbf{g})$. Clearly, if $\omega \in \mathrm{S}_\rho^m(\mathbf{R}^1)$ then $\omega(A_0) : H^s(M;\Omega^{1/2}) \to H^{s-m}(M;\Omega^{1/2})$ for all $s \in \mathbf{R}^1$. Therefore Theorem 7 implies

Theorem 9. *A ψDO $A \in \Psi_\rho^m(\mathbf{g})$ is bounded from $H^s(M;\Omega^{1/2})$ to $H^{s-m}(M;\Omega^{1/2})$ for all $s \in \mathbf{R}^1$.*

Let $\omega_1 \in \mathrm{S}_\rho^{m_1}(\mathbf{R}^1)$, $\omega_2 \in \mathrm{S}_\rho^{m_2}(\mathbf{R}^1)$, and A, B be the ψDOs with symbols $\omega_1(|\xi|_x)$, $\omega_2(|\xi|_x)$ respectively. Applying Lemma 6, Theorem 4 and Proposition 1, we see that $AB \in \Psi_\rho^{m_1+m_2}(\mathbf{g})$ and

$$\sigma_{AB}(x,\xi) \sim \sum_{j,k=0}^{\infty} a_{j,k}(x,\xi)\, \omega_1^{(j)}(|\xi|_x)\, \omega_2^{(k)}(|\xi|_x)\,, \qquad |\xi|_x \to \infty\,,$$

where the functions $a_{j,k} \in \mathrm{S}_{1,0}^0$ depend only on the Riemannian metric. Since $\omega'(|\xi|_x) = \sum_j \hat{\xi}_j\, \partial_{\xi_j}\omega(|\xi|_x)$, this asymptotic expansion can be rewritten in the form (8) with some functions $\tilde{P}_{\beta,\gamma}$ depending only on the Riemannian metric. By Proposition 1 the same result remains valid for arbitrary operators $A \in \Psi_\rho^{m_1}(\mathbf{g})$ and $B \in \Psi_\rho^{m_2}(\mathbf{g})$. Finally, if for all A and B (8) holds with some functions $\tilde{P}_{\beta,\gamma}$ independent of A and B then $\tilde{P}_{\beta,\gamma} = P_{\beta,\gamma}$. Thus, we have proved

Theorem 10. *If $A \in \Psi_\rho^{m_1}(\mathbf{g})$ and $B \in \Psi_\rho^{m_2}(\mathbf{g})$ then $AB \in \Psi_\rho^{m_1+m_2}(\mathbf{g})$ and the asymptotic expansion (8) holds.*

Remark. In the general case (7) does not directly imply that (8) is an asymptotic series; it seems that (8) contains terms of growing orders. However, all the bad terms disappear due to symmetries of the curvature tensor.

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