Journées Équations aux dérivées partielles

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Journées Équations aux dérivées partielles (1996), p. 1-11
<http://www.numdam.org/item?id=JEDP_1996____A14_0>

Electrical Impedance Tomography in Non-Linear Media

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1. Introduction

This is a report on the paper [Su- U I] concerning an inverse boundary value problem for anisotropic quasilinear materials. We describe in this section the problem and the main results of [Su- U I]. In the remaining sections we outline the proof of the main results

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with $C^{2,\alpha}$ boundary, $0 < \alpha < 1$. Let $\gamma(x,t) = (\gamma_{ij}(x,t))_{n\times n} \in C^{1,\alpha}(\overline{\Omega} \times \mathbb{R})$ be a symmetric, positive definite matrix function satisfying

$$\gamma(x,t) \geq \epsilon_T I, \quad (x,t) \in \overline{\Omega} \times [-T,T], T > 0, \tag{1.1}$$

where $\epsilon_T > 0$ and I denotes the identity matrix.

It is well known (see e.g. [G-T]) that, given $f \in C^{2,\alpha}(\overline{\Omega})$, there exists a unique solution of the boundary value problem

$$\begin{cases} \nabla \cdot \gamma(x,u)\nabla u = 0 & \text{in } \Omega \\ u\Big|_{\partial\Omega} = f. \end{cases} \tag{1.2}$$

We define the Dirichlet to Neumann map (DN) $\Lambda_\gamma : C^{2,\alpha}(\partial\Omega) \to C^{1,\alpha}(\partial\Omega)$ as the map given by

$$\Lambda_\gamma : f \to \nu \cdot \gamma(x,f)\nabla u\Big|_{\partial\Omega}, \tag{1.3}$$

where u is the solution of (1.2) and ν denotes the unit outer normal of $\partial\Omega$.

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* Partly supported by NSF Grant DMS-9501401
** Partly supported by NSF Grant DMS-9322619 and ONR Grant N00014-93-1-0295.

Physically, $\gamma(x,u)$ represents the (anisotropic) conductivity of Ω and $\Lambda_\gamma(f)$ the current flux at the boundary induced by the voltage f.

We study the inverse boundary value problem associated to (1.2): how much information about the coefficient matrix γ can be obtained from knowledge of the DN map Λ_γ?

In the isotropic case, that is, $\gamma(x,t) = \alpha(x,t)I$ where I denotes the identity matrix and γ is a positive function having a uniform positive lower bound on $\overline{\Omega} \times [-T,T]$ for each $T > 0$, the above question is well-understood: the Dirichlet to Neumann map Λ_γ for $\gamma = \alpha I$ determines uniquely the scalar coefficient $\alpha(x,t)$ on $\overline{\Omega} \times \mathbb{R}$. This uniqueness result was proven in [S-U, I] ($n \geq 3$), in [N] ($n = 2$) for the linear case (i.e. $\gamma(x,t) = \gamma(x)$) and in [Su] for the quasilinear case. We refer the readers to the survey paper [U] for other related results.

The uniqueness, however, is false in the case where γ is a general matrix function: if $\Phi : \overline{\Omega} \to \overline{\Omega}$ is a smooth diffeomorphism which is the identity map on $\partial\Omega$, and if we define

$$(\Phi_*\gamma)(x,t) = \frac{(D\Phi(x))^T\gamma(x,t)(D\Phi(x))}{|D\Phi|} \circ \Phi^{-1}(x) \tag{1.4}$$

then it follows that (see Proposition (2.1))

$$\Lambda_{\Phi_*\gamma} = \Lambda_\gamma,$$

where $D\Phi$ denotes the Jacobian matrix of Φ and $|D\Phi| = \det(D\Phi)$.

The main results of [Su -U I] concern with the converse statement. We have

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with $C^{3,\alpha}$ boundary, $0 < \alpha < 1$. Let γ_1 and γ_2 be quasilinear coefficient matrices in $C^{2,\alpha}(\overline{\Omega} \times \mathbb{R})$ such that $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$. Then there exists a $C^{3,\alpha}$ diffeomorphism $\Phi : \overline{\Omega} \to \overline{\Omega}$ with $\Phi\big|_{\partial\Omega}$ = identity, such that $\gamma_2 = \Phi_*\gamma_1$.*

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded simply connected domain with real-analytic boundary. Let γ_1 and γ_2 be real-analytic quasilinear coefficient matrices such that $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$. Assume that either γ_1 or γ_2 extends to a real-analytic quasilinear coefficient matrix on $\mathbb{R}^n$. Then there exists a real-analytic diffeomorphism $\Phi : \overline{\Omega} \to \overline{\Omega}$ with $\Phi\big|_{\partial\Omega}$ = identity, such that $\gamma_2 = \Phi_*\gamma_1$.*

Theorems 1.1 and 1.2 generalize all known results for the linear case ([S-U III]). In this case and $n = 2$, with a slightly different regularity assumption, Theorem 1.1 follows using a reduction theorem of Sylvester [S] and the uniqueness theorem of Nachman [N] for the isotropic case.

In the linear case and $n \geq 3$, Theorem 1.2 is a consequence of the work of Lee and Uhlmann [L-U], in which they discussed the same problem on real-analytic Riemannian manifolds. The assumption that one of the coefficient matrices can be extended to $\mathbb{R}^n$ can be replaced by a convexity assumption on the Riemannian metrics associated to the coefficient matrices. Thus Theorem 1.2 can also be stated under this assumption, which we omit here.

2. Invariance under the group of diffeomorphisms

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega$ in the $C^{m,\alpha}$ class, where $m \in \mathbb{Z}^+$, $\alpha \in [0,1)$. We denote by $\mathbb{G}_{m,\alpha}$ the group of diffeomorphisms given by

$$\mathbb{G}_{m,,\alpha} = \{\text{all } C^{m,\alpha} \text{ diffeomorphism } \Phi : \overline{\Omega} \to \overline{\Omega} \text{ with } \Phi\Big|_{\partial\Omega} = \text{identity}\}.$$

In the case that $\partial\Omega$ is in the real-analytic class, C^ω, we define

$$\mathbb{G}_\omega = \{\text{all } C^\omega \text{ diffeomorphisms } \Phi : \overline{\Omega} \to \overline{\Omega} \text{ with } \Phi\Big|_{\partial\Omega} = \text{identity}\}.$$

Let Φ be a diffeomorphism in one of the groups given above. As indicated in the introduction, the transformation $\Phi_* : \gamma \to \Phi_*\gamma$ preserves the Dirichlet to Neumann map in both linear and quasilinear cases. We give a proof below in the quasilinear case.

Proposition 2.1. *Let $\gamma(x,t)$ be a positive definite symmetric matrix in the $C^{1,\alpha}(\overline{\Omega})$ class, $0 < \alpha < 1$, satisfying (1.1) and $\Phi \in \mathbb{G}_{2,\alpha}$. Then*

$$\Lambda_{\Phi_*\gamma} = \Lambda_\gamma. \tag{2.1}$$

Proof. Let $\psi \in C^\infty(\overline{\Omega})$ be a test function. We write the equation (1.2) in the weak form:

$$\int_\Omega \nabla\psi \cdot \gamma(x,u)\nabla u dx = \int_{\partial\Omega} g\Lambda_\gamma(f)dS \tag{2.2}$$

where $g = \psi\Big|_{\partial\Omega}$. Let us define

$$\widetilde{u} = u \circ \Phi^{-1}, \quad \widetilde{\psi} = \psi \circ \Phi^{-1} \tag{2.3}$$

and make the change of variables $x \to \Phi(x)$ in (2.2). It is easy to verify that

$$\int_{\Omega} \nabla\widetilde{\psi} \cdot \Phi_{*}\gamma(x,\widetilde{u})(\nabla\widetilde{u})dx = \int_{\Omega} \nabla\psi \cdot \gamma(x,u)(\nabla u)dx. \tag{2.4}$$

By choosing in (2.4) $\psi \in C_0^{\infty}(\Omega)$, we have that $\widetilde{u}$ is the unique solution to

$$\begin{cases} \nabla \cdot \Phi_{*}\gamma(x,\widetilde{u})\nabla\widetilde{u} = 0 & \text{in } \Omega \\ \widetilde{u}\Big|_{\partial\Omega} = f \end{cases} \tag{2.5}$$

Now, we write (2.5) in the weak sense. By using that $\Phi\Big|_{\partial\Omega}$ = identity we have

$$\int_{\Omega} \nabla\widetilde{\psi} \cdot \Phi_{*}\gamma(x,\widetilde{u})\nabla\widetilde{u}dx = \int_{\partial\Omega} g\Lambda_{\Phi_{*}\gamma}(f)dS.$$

Now comparing this formula with (2.2) and (2.4) we get

$$\int_{\partial\Omega} g\Lambda_{\gamma}(f)dS = \int_{\partial\Omega} g\Lambda_{\Phi_{*}\gamma}(f)dS, \quad \forall\, g \in C^{\infty}(\partial\Omega), f \in C^{2,\alpha}(\partial\Omega),$$

from which (2.1) follows. ■

3. First linearization and its consequences

In this section we shall linearize the quasilinear Dirichlet to Neumann map Λ_{γ} to obtain information about the coefficient matrix γ by using the linear results.

Let $\gamma(x,t)$ be a positive definite, symmetric matrix in the C^2 class satisfying (1.1) and $\partial\Omega$ in the $C^{2,\alpha}$ class. Fix $t \in \mathbb{R}$ and $f \in C^{2,\alpha}(\partial\Omega)$. Consider the function

$$s \to \Lambda_{\gamma}(t+sf). \tag{3.1}$$

By the definition of Λ_{γ}, (3.1) is a function from $\mathbb{R}$ to $C^{1,\alpha}(\partial\Omega)$.

It has been shown [Su] that the function (3.1) is twice differentiable in the weak sense. It turns out the first two derivatives of (2.1) at $s = 0$ yield important information about γ.

In this section we consider the first derivative. In section 4 we shall make use of the second derivative of (3.1) We shall use γ^t to denote the function of x obtained by freezing t in $\gamma(x,t)$.

Proposition 3.1. [Su]. *Let $\gamma(x,t)$ be a quasilinear coefficient matrix in $C^2(\overline{\Omega}\times\mathbb{R})$. Then for every $f\in C^{2,\alpha}(\partial\Omega)$ and $t\in\mathbb{R}$*

$$\lim_{s\to 0}\left\|\frac{1}{s}\Lambda_\gamma(t+sf)-\Lambda_{\gamma^t}(f)\right\|_{H^{\frac{1}{2}}(\partial\Omega)}=0.$$

Under the assumptions of Theorem 1.1., using Proposition 3.1. we have that

$$\Lambda_{\gamma_1^t}=\Lambda_{\gamma_2^t},\forall\, t\in\mathbb{R}. \tag{3.2}$$

Since Theorems 1.1 and 1.2 hold in the linear case, it follows that, there exists a diffeomorphism Φ^t, which is in $\mathbb{G}_{3,\alpha}$ when $n=2$ and is in $\mathbb{G}_\omega$ when $n\geq 3$, and the identity at the boundary such that

$$\gamma_2^t=\Phi_*^t\gamma_1^t. \tag{3.3}$$

It is proven in [Su- U I] that Φ^t is uniquely determined by γ_l^t, and thus by γ_l, $l=1,2$. We then obtain a function

$$\Phi(x,t)=\Phi^t(x):\overline{\Omega}\times\mathbb{R}\to\overline{\Omega}\times\mathbb{R}, \tag{3.4}$$

which is in $C^{3,\alpha}(\overline{\Omega})$ for each fixed t in dimension two and real analytic in dimension $n\geq 3$. It is also shown in [Su -U I] that Φ is also smooth in t. More precisely we have, in every dimension $n\geq 2$, that $\frac{\partial\Phi}{\partial t}\in C^{2,\alpha}(\overline{\Omega})$.

In order to prove Theorems 1.1 and 1.2, we must then show that Φ^t is independent of t. Without loss of generality, we shall only prove

$$\left.\frac{\partial\Phi}{\partial t}\right|_{t=0}=0\qquad\text{in }\overline{\Omega}. \tag{3.5}$$

It is easy to show, using the invariance (1.4) that we may assume that

$$\Phi(x,0)\equiv x,\text{that is,}\quad \Phi^0=\text{ identity}. \tag{3.6}$$

Let us fix a solution $u\in C^{3,\alpha}(\overline{\Omega})$ of

$$\nabla\cdot A\nabla u=0,\quad u|_{\partial\Omega}=f \tag{3.7}$$

where we denote $A=\gamma_1^0=\gamma_2^0$.

For every $t \in \mathbb{R}$ and $l = 1, 2$, we solve the boundary value problem (3.4) with γ^t replaced by γ_l^t. We obtain a solution $u_{(l)}^t$:

$$\begin{cases} \nabla \cdot \gamma_l^t \nabla u_{(l)}^t = 0 & \text{in } \Omega \\ u_{(l)}^t \Big|_{\partial\Omega} = f \end{cases} \qquad l = 1, 2. \tag{3.8}$$

It follows from the proof of Proposition (2.1) (see also (2.3)) that

$$u_{(1)}^t(x) = u_{(2)}^t(\Phi^t(x)), \qquad x \in \overline{\Omega}.$$

Differentiating this last formula in t and evaluating at $t = 0$ we obtain

$$\left(\frac{\partial u_{(1)}^t}{\partial t} - \frac{\partial u_{(2)}^t}{\partial t} \right) \Bigg|_{t=0} - X \cdot \nabla u = 0, \qquad x \in \overline{\Omega}, \tag{3.9}$$

where

$$X = \frac{\partial \Phi^t}{\partial t} \Bigg|_{t=0}. \tag{3.10}$$

It is easy to show that $X \cdot \nabla u = 0$ for every solution of (3.7) implies $X = 0$. so we are reduced to prove

$$\left(\frac{\partial u_{(1)}^t}{\partial t} - \frac{\partial u_{(2)}^t}{\partial t} \right) \Bigg|_{t=0} = 0. \tag{3.11}$$

¿From (3.8) we get

$$\nabla \cdot (\gamma_1(x,t) \nabla u_{(1)}^t) - \nabla \cdot (\gamma_2(x,t) \nabla u_{(2)}^t) = 0. \tag{3.12}$$

Differentiating (3.12) in t at $t = 0$ we conclude

$$\nabla \cdot \left[\left(\frac{\partial \gamma_1}{\partial t} - \frac{\partial \gamma_2}{\partial t} \right) \Bigg|_{t=0} \nabla u \right] + \nabla \cdot \left[A \nabla \left(\frac{\partial u_{(1)}^t}{\partial t} - \frac{\partial u_{(2)}^t}{\partial t} \right) \Bigg|_{t=0} \right] = 0. \tag{3.13}$$

We claim that to prove (3.11) it is enough to show that

$$\nabla \cdot \left[\left(\frac{\partial \gamma_1}{\partial t} - \frac{\partial \gamma_2}{\partial t} \right) \Bigg|_{t=0} \nabla u \right] = 0. \tag{3.14}$$

This is the case since we get from (3.13) and (3.14)

$$\nabla \cdot \left[A\nabla \left(\frac{\partial u^t_{(1)}}{\partial t} - \frac{\partial u^t_{(2)}}{\partial t} \right)\Big|_{t=0} \right] = 0.$$

The claim now follows since the operator $\nabla \cdot A\nabla : H^2(\Omega) \cap \overset{0}{H^1}(\Omega) \to L^2(\Omega)$ is an isomorphism and

$$\left(\frac{\partial u^t_{(1)}}{\partial t} - \frac{\partial u^t_{(2)}}{\partial t} \right)\Big|_{t=0}\Big|_{\partial\Omega} = 0.$$

4. Second linearization and products of solutions

In order to show (3.14) we now study the second derivative of (3.1). We introduce, for every $t \in \mathbb{R}$, the map $K_{\gamma,t} : C^{2,\alpha}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ which is defined implicitly as follows (see [Su]): for every pair $f_1, f_2) \in C^{2,\alpha}\partial\Omega) \times C^{2,\alpha}(\partial\Omega)$,

$$\int_{\partial\Omega} f_1 K_{A,t}(f_2)ds = \int_\Omega \nabla u_1 \frac{\partial A}{\partial t} \nabla u_2^2 dx \tag{4.1}$$

with $u_l, l = 1, 2$, as in (3.8). with f replaced by $f_l, l = 1, 2$. We have

Proposition 4.1. [Su]. *Let $\gamma(x,t)$ be a positive definite symmetric matrix in $C^2(\overline{\Omega} \times \mathbb{R})$, satisfying (1.1). Then for every $f \in C^{2,\alpha}(\partial\Omega)$ and $t \in \mathbb{R}$,*

$$\lim_{s\to 0} \left\| \frac{1}{s}\left[\frac{1}{s}\Lambda_A(t+sf) - \Lambda_{A^t}(f) \right] - K_{A,t}(f) \right\|_{H^{\frac{1}{2}}(\partial\Omega)} = 0.$$

Under the assumptions of Theorems 1.1 and 1.2, using Proposition 4.1 with $t = 0$, we obtain

$$K_{\gamma_{1,0}}(f) = K_{\gamma_{2,0}}(f), \qquad \forall\, f \in C^{3,\alpha}(\partial\Omega).$$

Thus, by (4.1) we have

$$\int_\Omega \nabla u_1 \left.\frac{\partial \gamma_1}{\partial t}\right|_{t=0} \nabla u_2^2 dx = \int_\Omega \nabla u_1 \left.\frac{\partial \gamma_2}{\partial t}\right|_{t=0} \nabla u_2^2 dx, \tag{4.2}$$

with u_1, u_2 solutions of (3.8) By writing

$$B = \left(\frac{\partial \gamma_1}{\partial t} - \frac{\partial \gamma_2}{\partial t} \right)\Big|_{t=0} \tag{4.3}$$

and replacing in (4.2) u_1 by u and u_2^2 by $(u_1+u_2)^2 - u_1^2 - u_2^2$, we obtain

$$\int_\Omega \nabla u \cdot B(x)\nabla(u_1 u_2)dx = 0 \tag{4.4}$$

with u, u_1 and u_2 solutions of (3.8).

To continue from (4.4), we need the following two lemmas.

Lemma 4.1. *Let $h(x) \in C^1(\overline{\Omega})$ be a vector-valued function. If*

$$\int_\Omega h(x)\nabla(u_1u_2)dx = 0$$

for arbitrary solutions u_1 and u_2 of (3.8), then $h(x)$ lies in the tangent space $T_x(\partial\Omega)$ for all $x \in \partial\Omega$.

Lemma 4.2. *Let $A(x)$ be a positive definite, symmetric matrix in $C^{2,\alpha}(\overline{\Omega})$. Define*

$$D_A = \mathrm{Span}_{L^2(\Omega)}\{uv; u, v \in C^{3,\alpha}(\overline{\Omega}), \nabla\cdot A\nabla u = \nabla\cdot A\nabla v = 0\}.$$

Then the following are valid:
(a) If $l \in C^\omega(\overline{\Omega})$ and $l \perp D_A$, then $l = 0$ in $\overline{\Omega}$
(b) If $n = 2$, then $D_A = L^2(\Omega)$.

Now we finish the proof of (3.14) concluding the proofs of Theorems 1.1 and 1.2.

By Lemma 4.1 we have that $\nu\cdot B(x)\nabla u \equiv 0$ in $\partial\Omega$. Integrating by parts in (4.4) we obtain

$$\int_\Omega [\nabla\cdot B(x)\nabla u]u_1u_2dx = 0. \tag{4.5}$$

We now apply Lemma 4.2 to (4.5). If $n \geq 3$, we have that γ_1 and γ_2 are real-analytic on $\overline{\Omega}\times\mathbb{R}$. Thus $B \in C^\omega(\overline{\Omega})$. Since the solutions u solves an elliptic equation with a real-analytic coefficient matrix, we have that u is analytic in Ω. If u is analytic on $\overline{\Omega}$, we can conclude from Lemma 4.2 that

$$\nabla\cdot(B(x)\nabla u) = 0, \qquad x \in \overline{\Omega}. \tag{4.6}$$

We shall prove that (4.6) holds independent of whether u is analytic up to $\partial\Omega$ or not. This is due to the Runge approximation property of the equation (3.7) [L]. Using the assumptions of Theorem 1.2 we extend A analytically to a slightly larger domain $\widetilde{\Omega} \supset \overline{\Omega}$. For any solution $u \in C^{3,\alpha}(\overline{\Omega})$ and an open subset $\mathcal{O}$ with $\overline{\mathcal{O}} \subset \Omega$, we can find a sequence of solutions $\{u_m\} \subset C^\omega(\widetilde{\Omega})$, which solves (4.4) on $\widetilde{\Omega}$, and $u_m\Big|_{\mathcal{O}_1} \underset{m\to\infty}{\longrightarrow} u\Big|_{\mathcal{O}_1}$ in the L^2 sense, where $\overline{\mathcal{O}}_1 \subset \Omega$, $\overline{\mathcal{O}} \subset \mathcal{O}_1$. By the local regularity theorem of elliptic equations this convergence is valid in $H^2(\mathcal{O})$. Since (4.6) holds with $u = u_m$, letting $m \to \infty$ yields the desired result for u on $\mathcal{O}$. Thus (4.6) holds. If $n = 2$, Lemma 4.2 (b) implies that $\nabla\cdot(B(x)\nabla u) = 0$ for any solution $u \in C^{3,\alpha}(\overline{\Omega})$.

The proof of Lemma 4.1 follows an argument of Alessandrini [Al], which relies on the use of solutions with isolated singularities. It turns out that in our case, only solutions with Green's function type singularities are sufficient in the case $n \geq 3$, while in case $n = 2$, solutions with singularities of higher order must be used. There are additional difficulties since we are dealing with a vector function h. We refer the readers to [Su-U I] for details.

The proof of part (a) of Lemma 4.2 follows the proof of Theorem 1.3 in [Al] (which also follows the arguments of [K-V]). Namely, one constructs solutions u of (3.7) in a neighborhood of Ω with an isolated singularity of arbitrary given order at a point outside of Ω. We then plug this solution into the identity

$$\int_\Omega lu^2 dx = 0.$$

By letting the singularity of u approach to a point x in $\partial\Omega$, one can show that any derivative of h must vanish on x and thus by the analyticity of l, $l \equiv 0$ in $\overline{\Omega}$. We leave the details to the reader.

To prove the part (b) of Lemma 4.2, we first reduce the problem to the Schrödinger equation.

Using isothermal coordinates (see [A]), there is a conformal diffeomorphism $F : (\overline{\Omega}, g) \rightarrow (\overline{\Omega}', e)$, where g is the Riemannian metric determined by the linear coefficient matrix A with $g_{ij} = A_{ij}^{-1}$. One checks that F transforms the operator $\nabla \cdot A\nabla$ (on Ω) to an operator $\nabla \cdot A'\nabla$ (on Ω') with A' a scalar matrix function $\beta(x)I$. Therefore the proof of the part (b) is reduced to the case where $A = \beta I$, with $\beta(x) \in C^{2,\alpha}(\overline{\Omega})$. By approximating by smooth solutions, we see that the $C^{3,\alpha}$ smoothness can be replaced by H^2 smoothness. Thus we have reduced the problem to showing that

$$D_\beta = \text{Span}_{L^2}\{uv; u, v \in H^2(\Omega); \nabla \cdot \beta\nabla u = \nabla \cdot \beta\nabla v = 0\} = L^2(\Omega).$$

We make one more reduction by transforming the equation $\nabla \cdot \beta\nabla u = 0$ to the Schrödinger equation

$$\Delta v - qv = 0$$

with

$$u = \beta^{-\frac{1}{2}} v, q = \frac{\Delta\sqrt{\beta}}{\sqrt{\beta}} \in C^\alpha(\overline{\Omega}). \tag{4.7}$$

This allows us to reduce the proof to showing that

$$D_q = \mathrm{Span}_{L^2}\{v_1 v_2; v_i \in H^2(\Omega), \Delta v_i - q v_i = 0, i = 1,2\} = L^2(\Omega) \tag{4.8}$$

for potentials q of the form (4.7)

Statement (4.8) was proven by Novikov ([No].) In [Su-U I] it was shown that it is enough to use the Proposition below which is valid for any potential $q \in L^\infty(\Omega)$. This result uses some of the techniques of [Su -U II,III]

Proposition 4.2. *Let $q \in L^\infty(\Omega), n = 2$. Then D_q has a finite codimension in $L^2(\Omega)$.*

It is an interesting open question whether $D_q = L^2(\Omega)$ in the two dimensional case.

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