JOURNÉES ÉQUATIONS AUX DÉRIVÉES PARTIELLES

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Journées Équations aux dérivées partielles (1996), p. 1-24
<http://www.numdam.org/item?id=JEDP_1996____A13_0>

Relative determinants of elliptic operators and scattering theory

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0 Introduction

The zeta function regularization is one possible way to introduce a regularized determinant for a positive elliptic operator on a compact manifold. Regularized determinants play an important role in various fields of mathematics and physics.

We use the same method to regularize the ratio of two determinants of elliptic operators if the individual determinants can not be defined by ζ–function regularization. This situation occurs if the underlying manifold is noncompact. A typical example is the Schrödinger operator $\Delta + V$, $V \in C_0^\infty(\mathbb{R}^n)$. Since $\Delta + V$ has continuous spectrum, the ζ–function of $\Delta + V$ is not defined and therefore, the determinant of $\Delta + V$ can not be defined by zeta function regularization. However, the ratio of the determinants of $\Delta + V$ and Δ can be regularized by the same approach. We denote this regularized ratio of the determinants by $\det(\Delta + V, \Delta)$ and call it *relative determinant.* As the example may indicate, scattering theory plays an essential role in the study of relative determinants. This becomes even more transparent in the case of surfaces with hyperbolic ends. Then the relative determinant can be defined in terms of the resonances and the eigenvalues of the Laplacian by a formula which is similar to the one used in the compact case.

At the end of §1, we have listed a number of cases where determinants of elliptic operators on compact manifolds naturally arise. We think that the relative determinants can be used in much the same way in the corresponding noncompact setting.

1 Determinants of elliptic operators on compact manifolds

Let M be an n–dimensional compact C^∞ manifold without boundary and let E be a complex vector bundle over M. We pick a Riemannian metric on M and a Hermitian fibre metric in E. Let $L^2(E)$ be the associated Hilbert space of measurable square integrable sections of E. Let

$$A\colon C^\infty(E) \to C^\infty(E)$$

be an elliptic pseudo–differential operator of order $m > 0$, acting on the smooth sections of E. Suppose that A is symmetric and nonnegative with respect to the inner product in $C^\infty(E)$ induced by the metrics of M and E. Then A has a unique extension to a self–adjoint operator in $L^2(E)$ which we also denote by A. Since M is closed, A has pure point spectrum consisting of a sequence of eigenvalues

$$0 \le \lambda_0 \le \lambda_1 \le \cdots \longrightarrow \infty \tag{1.1}$$

of finite multiplicity. Each eigenvalue in this sequence is repeated according to its multiplicity. Using the eigenvalue sequence one can form the zeta function

$$\zeta_A(s) = \sum_{\lambda_j > 0} \lambda_j^{-s} \tag{1.2}$$

which is absolutely convergent in the half–plane $\mathrm{Re}(s) > n/m$. By the work of Seeley [S], one knows that $\zeta_A(s)$ has a meromorphic extension to the entire complex plane and $s = 0$ is not a pole of $\zeta_A(s)$. The zeta function can also be expressed in terms of the trace of the heat operator by

$$\zeta_A(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} (\mathrm{Tr}(e^{-tA}) - \dim \ker\mathrm{A})\, dt, \quad \mathrm{Re}(s) > n/m, \tag{1.3}$$

and the heat expansion

$$\mathrm{Tr}(e^{-tA}) \sim t^{-n/m} \sum_{j=0}^{\infty} a_j t^{j/m} \tag{1.4}$$

as $t \to 0$ can be employed to obtain the analytic continuation of $\zeta_A(s)$.

Since $\zeta_A(s)$ is holomorphic at $s = 0$, a regularized determinant of A can be defined by

$$\det A = \exp\left(-\frac{d}{ds}\zeta_A(s)|_{s=0}\right). \tag{1.5}$$

The ζ–function regularization was first introduced by Ray and Singer [RS1] to define a regularized determinant for the Laplacian on differential forms with coefficients in a flat vector bundle. Hawking [H] has used the same method to regularize path integrals that arise in the theory of gravitation. Note that purely formally

$$\frac{d}{ds}\zeta_A(s)|_{s=0} = -\sum_{\lambda_j>0} \log \lambda_j,$$

so that det A, as defined by (1.5), may indeed be regarded as a regularization of $\prod_{\lambda_j>0} \lambda_j$.

The definition of the determinant can be extended to the case when A has a positive definite symbol, is invertible, but is not necessarily self–adjoint and positive. If only the symbol is positive definite, then all but a finite number of eigenvalues, say $\lambda_1, \ldots, \lambda_k$, lie in a positive cone about the positive axis and hence, the complex power A^{-s} is defined [S] and can be used to introduce the zeta function $\zeta_A(s) = \mathrm{Tr}(A^{-s})$, $\mathrm{Re}(s) >> 0$, in this case too.

Furthermore, for elliptic operators which are self–adjoint but not bounded from below, the determinant can be defined by introducing a phase factor. For example, suppose that M has a spin structure and let S be the spinor bundle of M. Then the twisted Dirac operator

$$D_E \colon C^\infty(S \otimes E) \to C^\infty(S \otimes E)$$

is a first order elliptic self–adjoint operator with spectrum

$$\cdots \le \lambda_{-i} \le \lambda_{-i+1} \le \cdots \le \lambda_{-1} \le 0 \le \lambda_1 \le \cdots \le \lambda_i \le \lambda_{i+1} \le \cdots$$

Taking into account the signs of the eigenvalues, Atiyah, Patodi and Singer introduced in [APS] the η–function of D

$$\eta_D(s) = \sum_{\lambda_j \neq 0} \frac{\text{sign}\lambda_j}{|\lambda_j|^s} \tag{1.6}$$

which is absolutely convergent for $\text{Re}(s) > n$, and admits a meromorphic extension to $\mathbb{C}$. Moreover, $s = 0$ is not a pole of $\eta_D(s)$ [APS]. Then formally, $\eta_D(0)$ is the number of positive eigenvalues minus the number of negative eigenvalues of D. Using $\eta_D(0)$, one can define a regularized determinant for D by

$$\det \text{D} = \det|D| \cdot \exp\left(\frac{\pi i}{2}(\eta_D(0) - \zeta_{|D|}(0))\right) \tag{1.7}$$

[Si].

We observe that the regularized determinant is not multiplicative, i.e., in general,

$$\det(AB) \neq \det(A)\det(B).$$

The ratio

$$F(A,B) = \frac{\det(AB)}{\det(A)\det(B)}$$

defined on pairs of elliptic pseudo-differential operators which are close to self–adjoint ones has been studied by Kontsevich and Vishik in [KV] and by Okikiolu in [O].

Regularized determinants occur in various fields of mathematics and physics. We mention the following typical cases:

a) *Regularization of path integrals:* [H], [GMS]

b) *Quillen metric on the determinant line bundle:* [BGS1], [BGS2], [BGS3], [BF1], [BF2], [BL], [Fr]

c) *The arithmetic Riemann-Roch-Grothendieck theorem:* [Fa], [GS]

d) *Study of extremal metrics:* [OPS1], [BCY], [Do]

e) *Spectral geometry:* [OPS2]

f) Analytic torsion:[BZ], [BFK], [C], [Lo], [Mu3], [Mu4], [RS1], [RS2]

2 Relative determinants

If we are dealing with elliptic operators on noncompact manifolds, then the spectrum of the corresponding self–adjoint extensions may contain a continuous part. A typical example is the Schrödinger operator in $\mathbb{R}^n$. The presence of the continuous spectrum prevents us from using the ζ-function regularization to define a regularized determinant for such operators. In many cases, however, it is not the determinant of a single operator, but rather the ratio of two determinants which has to be regularized. For example, we may consider fermions coupled to a gauge field A in $\mathbb{R}^n$. Let $\not{D}(A)$ be the Dirac operator coupled to A and let $\not{D}$ be the free Dirac operator. Then the normalization of the measure of the corresponding path integral implies that the ratio

$$\frac{\det \not{D}(A)}{\det \not{D}}$$

and not $\det \not{D}(A)$ has to be regularized. This is what we call a relative determinant.

First, let A and B be two elliptic positive self–adjoint operators of order > 0 acting on the sections of a Hermitian vector bundle over a closed Riemannian manifold M. Then we define the relative determinant of (A, B) simply by

$$\det(A, B) = \frac{\det(A)}{\det(B)}. \tag{2.1}$$

We may also introduce the relative zeta function

$$\zeta(s; A, B) = \zeta_A(s) - \zeta_B(s), \tag{2.2}$$

and by (1.5), we have

$$\det(A, B) = \exp\left(-\frac{d}{ds}\zeta(s; A, B)|_{s=0}\right). \tag{2.3}$$

To see how this can be generalized to the case when the individual determinants are not defined, we employ (1.3) which gives

$$\zeta(s; A, B) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\,\mathrm{Tr}\left(e^{-tA} - e^{-tB}\right)\, dt. \tag{2.4}$$

Now, the right hand side can be defined under more general assumptions. The abstract setting is as follows:

Definition 2.1. Let $\mathcal{H}$ be a separable Hilbert space and let H, H_0 be two self–adjoint nonnegative linear operators in $\mathcal{H}$. The pair (H, H_0) is called admissible, if the following conditions hold:

1) $e^{-tH} - e^{-tH_0}$ is trace class for $t > 0$.

2) As $t \to 0$, there exists an asymptotic expansion of the form

$$\operatorname{Tr}\left(e^{-tH} - e^{-tH_0}\right) \sim \sum_{j=0}^{\infty}\sum_{k=0}^{k(j)} a_{jk} t^{\alpha_j} \log^k t \tag{2.5}$$

where $-\infty < \alpha_0 < \alpha_1 < \ldots$ and $\alpha_k \to \infty$. Moreover, if $\alpha_j = 0$ we assume that $a_{jk} = 0$ for $k > 0$.

3) As $t \to \infty$, there exists an asymptotic expansion of the form

$$\operatorname{Tr}\left(e^{-tH} - e^{-tH_0}\right) \sim \sum_{k=0}^{\infty} b_k t^{-\beta_k} \tag{2.6}$$

where $0 = \beta_0 < \beta_1 < \cdots$.

It follows from (2.5) that the integral

$$\int_0^1 t^{s-1} \operatorname{Tr}\left(e^{-tH} - e^{-tH_0}\right) dt$$

is absolutely convergent in the half–plane $\operatorname{Re}(s) > -\alpha_0$ and has a meromorphic continuation to the entire complex plane. The possible poles occur at $s = -\alpha_j$, $j \in \mathbb{N}$, and at the pole $s = -\alpha_j$, the Laurent expansion is given by

$$\sum_{k=0}^{k(j)} a_{jk} \frac{k!}{(s+\alpha_j)^{k+1}} + O(1).$$

Set

$$\zeta_1(s; H, H_0) = \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} \operatorname{Tr}\left(e^{-tH} - e^{-tH_0}\right) dt. \tag{2.7}$$

Then $\zeta_1(s; H, H_0)$ is a meromorphic function of $s \in \mathbb{C}$. By our assumption about the asymptotic expansion (2.5), $\zeta_1(s; H, H_0)$ has no pole at $s = 0$. Next consider the integral

$$\int_1^\infty t^{s-1} \operatorname{Tr}\left(e^{-tH} - e^{-tH_0}\right) \, dt.$$

By (2.6), this integral is absolutely convergent in the half–plane $\operatorname{Re}(s) < \beta_0$ and it has a meromorphic extension to $\mathbb{C}$. Set

$$\zeta_2(s; H, H_0) = \frac{1}{\Gamma(s)} \int_1^\infty t^{s-1} \operatorname{Tr}\left(e^{-tH} - e^{-tH_0}\right) \, dt.$$

Then $\zeta_2(s; H, H_0)$ is a meromorphic function of $s \in \mathbb{C}$ with possible poles at $s = \beta_k,\ k \in \mathbb{N}^+$. All poles are simple and we have

$$\operatorname{Res}_{s=\beta_k} \zeta_2\left(s; H, H_0\right) = \frac{b_k}{\Gamma(\beta_k)}, \quad k > 0.$$

Furthermore, $\zeta_2\left(s; H, H_0\right)$ is holomorphic at $s = 0$.

Definition 2.2. Let (H, H_0) be admissible. Then the *relative zeta function* of (H, H_0) is defined by

$$\zeta\left(s; H, H_0\right) = \zeta_1\left(s; H, H_0\right) + \zeta_2\left(s; H, H_0\right).$$

Summarizing we have proved

Proposition 2.3. *Let (H, H_0) be a pair of admissible operators in $\mathcal{H}$. Then the relative zeta function $\zeta(s; H, H_0)$ is a meromorphic function of $s \in \mathbb{C}$ with poles contained in the set $\{-\alpha_j \mid j \in \mathbb{N}\} \cup \{\beta_k \mid k \in \mathbb{N}^+\}$. Moreover, $s = 0$ is not a pole of $\zeta(s; H, H_0)$.*

Remarks. 1) For special cases, relative zeta functions have been studied by L. Guillopé [Gu1], [Gu2], and V. Bruneau [Br].
2) Suppose that the essential spectrum of H_0 has a positive lower bound. Since $e^{-tH} - e^{-tH_0}$ is trace class for $t > 0$, the essential spectrum of H has the same lower bound. Using the spectral theorem, it follows that as $t \to \infty$,

$$\operatorname{Tr}\left(e^{-tH} - e^{-tH_0}\right) = h + O(e^{-ct})$$

for some $h \in \mathbb{Z}$ and $c > 0$. Then the relative zeta function can be defined in the same way as in the compact case by Mellin transform

$$\zeta(s; H, H_0) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \left(\mathrm{Tr}(e^{-tH} - e^{-tH_0}) - h\right) dt, \quad \mathrm{Re}(s) > -\alpha_0.$$

3)Let H_0 and H_1 be two nonnegative self–adjoint operators in $\mathcal{H}$ such that

i) $\exp(-tH_i)$, $i = 0, 1$, is trace class for $t > 0$,

ii) As $t \to 0$, there exists an asymptotic expansion of the form

$$\mathrm{Tr}(e^{-tH_i}) \sim t^{-n/m_i} \sum_{j \geq 0} a_{ij} t^{j/m_i}, \; i = 0, 1.$$

Then H_i, $i = 0, 1$, has pure point spectrum and

$$\mathrm{Tr}(e^{-tH_i}) = \dim\ker(H_i) + O(e^{-ct})$$

as $t \to \infty$. Hence, (H, H_0) is an admissible pair in the sense of Definition 2.1. Moreover, the zeta functions

$$\zeta_{H_i}(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \left(\mathrm{Tr}(e^{-tH_i}) - \dim\ker(H_i)\right) dt, \quad \mathrm{Re}(s) > n/m_i,$$

have meromorphic extensions to $\mathbb{C}$ and

$$\zeta(s; H_1, H_0) = \zeta_{H_1}(s) - \zeta_{H_0}(s). \tag{2.8}$$

Thus, the relative zeta function is the generalization of the difference of two zeta functions to the case when the individual zeta functions are not defined.

In a similar way we may define relative eta functions. Let D, D_0 be two self–adjoint operators in $\mathcal{H}$ and assume that
i) $De^{-tD^2} - D_0 e^{-tD_0^2}$ is trace class for $t > 0$,
ii) As $t \to 0$ and as $t \to \infty$, there exist asymptotic expansions of

$$\mathrm{Tr}\left(De^{-tD^2} - D_0 e^{-tD_0^2}\right)$$

similar to (2.5) and (2.6), respectively.
Then we put

$$\eta_1(s;D,D_0)=\frac{1}{\Gamma\left(\frac{s+1}{2}\right)}\int_0^1 t^{(s-1)/2}\operatorname{Tr}\left(De^{-tD^2}-D_0e^{-tD_0^2}\right)\,dt,\ \operatorname{Re}(s)>-\alpha_0$$

$$\eta_2(s;D,D_0)=\frac{1}{\Gamma\left(\frac{s+1}{2}\right)}\int_1^\infty t^{(s-1)/2}\operatorname{Tr}\left(De^{-tD^2}-D_0e^{-tD_0^2}\right)\,dt,\ \operatorname{Re}(s)<-\beta_0$$

Both integrals admit meromorphic extensions to $\mathbb{C}$. Put

$$\eta(s;D,D_0)=\eta_1(s;D,D_0)+\eta_2(s;D,D_0). \tag{2.9}$$

This is the *relative eta function* associated to (D,D_0). We assume that the asymptotic expansions are such that $s=0$ is not a pole of $\eta(s;D,D_0)$. Then

$$\eta(D,D_0)=\eta(0;D,D_0) \tag{2.10}$$

is the *relative eta invariant* of (D,D_0).
We can now introduce relative determinants.

Definition 2.4. Let (H,H_0) be an admissible pair of operators in $\mathcal{H}$. Then the relative determinant of (H,H_0) is defined by

$$\det(H,H_0)=\exp\left(-\frac{d}{ds}\zeta(s;H,H_0)\mid_{s=0}\right). \tag{2.11}$$

Remarks. 1) The relative determinant can be defined for larger classes of operators. For example, let D,D_0 be two self-adjoint operators in $\mathcal{H}$ such that (D^2,D_0^2) is admissible and the above conditions i) and ii) are satisfied. It is easy to see that $(|D|,|D_0|)$ is admissible. Then, in analogy with (1.7), we define the relative determinant of (D,D_0) by

$$\det(D,D_0)=\det(|D|,|D_0|)\cdot\exp\left(\frac{\pi i}{2}(\eta(D,D_0)-\zeta(0;|D|,|D_0|))\right). \tag{2.12}$$

2) For $z\in\mathbb{C}$, $\operatorname{Re}(z)>0$, $\det(H+z,H_0+z)$ can be defined as above, and this function of z has an analytic continuation to a holomorphic function on $\mathbb{C}-(-\infty,0]$.

Some of the elementary properties of the relative determinant are described by the following

Proposition 2.5. *Suppose that* H_0, H_1 *and* H_2 *are self-adjoint nonnegative operators in* $\mathcal{H}$ *such that* (H_2, H_1) *and* (H_1, H_0) *are admissible. Then we have*

1) *If* $\det(H_i)$, $i = 0, 1$, *exists, then* $\det(H_1, H_0) = \det(H_1) \cdot (\det(H_0))^{-1}$.

2) $\det(H_0, H_1) = (\det(H_1, H_0))^{-1}$

3) $\det(H_2, H_0) = \det(H_2, H_1) \cdot \det(H_1, H_0)$

Next we consider the variation of relative determinants. Let H_u, $u \in (-\epsilon, \epsilon)$, be a differentialble 1–parameter family of nonnegative self–adjoint operators in $\mathcal{H}$. Let $\dot{H}_u = \frac{dH_u}{du}$. Assume that
i) For each $u \in (-\epsilon, \epsilon)$, (H_u, H_0) is admissible,
ii) H_u is invertible for each $u \in (-\epsilon, \epsilon)$,
iii) $\dot{H}_u e^{-tH_u}$ is trace class for $t > 0, u \in (-\epsilon, \epsilon)$, and as $t \to 0$, there exists an asymptotic expansion

$$\operatorname{Tr}\left(\dot{H}_u H_u^{-1} e^{-tH_u}\right) \sim \sum_{j=0}^{\infty} \sum_{k=0}^{k(j)} C_{jk}(u) t^{\gamma_j} \log^k t + C_1(u) + C_2(u) \log$$

where the exponents γ_j are such that $-\infty < \gamma_1 < \gamma_2 < \cdots$, $\gamma_j \neq 0$, $j \in \mathbb{N}$, and $\gamma_j \to \infty$.

Theorem 2.6. *Let the assumptions be as above. Then* $\det(H_u, H_0)$ *is differentiable in* $u \in (-\epsilon, \epsilon)$ *and*

$$\frac{d}{du} \log \det(H_u, H_0) = C_1(u) + \Gamma'(1) C_2(u).$$

Integrating this equality, we get

$$\det(H_1, H_0) = \exp\left\{ \int_0^1 \left(C_1(u) + \Gamma'(1) C_2(u)\right) du \right\}.$$

This formula for the relative determinant can be used whenever H_0 and H_1 can be connected by a differentiable family of self–adjoint operators satisfying the above conditions.

3 The large time asymptotic expansion and the scattering matrix

If we wish to study determinants for a particular class of operators, say elliptic operators on certain noncompact manifolds, then we have to verify conditions 1)–3) of Definition 2.1. For 1) and 2) one may use essentially local investigations of the heat kernels. Condition 3), however, needs some global arguments.

First of all, by Definition 2.1, 1), it follows that the Krein spectral shift function $\xi(\lambda) = \xi(\lambda; H, H_0)$ exists [BY]. More precisely, we have

Proposition 3.1. *Let H, H_0 be two nonnegative self-adjoint operators in $\mathcal{H}$ and assume that $e^{-tH} - e^{-tH_0}$ is trace class for $t > 0$. Then there exists a unique real valued locally integrable function $\xi(\lambda) = \xi(\lambda; H, H_0)$ on $\mathbb{R}$ such that for each $t > 0$, $e^{-t\lambda}\xi(\lambda) \in L^1(\mathbb{R})$ and*

1) $\operatorname{Tr}\left(e^{-tH} - e^{-tH_0}\right) = -t\int_0^\infty e^{-t\lambda}\xi(\lambda)\,d\lambda.$

2) *For every $\varphi \in C_0^\infty(\mathbb{R})$, $\varphi(H) - \varphi(H_0)$ is trace class and*

$$\operatorname{Tr}\left(\varphi(H) - \varphi(H_0)\right) = \int_{\mathbb{R}} \varphi'(\lambda)\xi(\lambda)\,d\lambda$$

3) $\xi(\lambda) = 0$ *for* $\lambda < 0$.

Thus, the behaviour of $\operatorname{Tr}(e^{-tH} - e^{-tH_0})$ as $t \to \infty$ is related to the behaviour of the spectral shift function $\xi(\lambda)$ as $\lambda \to +0$. In particular, we have

Lemma 3.2. *Suppose that there exist $\epsilon > 0$ and $0 \le \gamma_0 < \gamma_1 < \gamma_2 < \cdots \to +\infty$ such that for every $N \in \mathbf{N}$*

$$\xi(\lambda) = \sum_{j=0}^{N} C_j \lambda^{\gamma_j} + O\left(\lambda^{\gamma_{N+1}}\right), \ \lambda \in [0, \epsilon). \tag{3.1}$$

Then there is an asymptotic expansion

$$\int_0^\infty e^{-t\lambda}\xi(\lambda)\,d\lambda \sim t^{-1}\left(\frac{b_0}{t^{\gamma_0}} + \frac{b_0}{t^{\gamma_1}} + \cdots\right)$$

as $t \to \infty$.

One way to study the spectral shift function near $\lambda = 0$ is to relate it to the scattering matrix. Again, referring to condition 1) of Definition 2.1, it follows from the Birman–Krein invariance principle for wave operators [K] that the wave operators

$$W_{\pm} = s - \lim_{t \to \mp\infty} e^{itH} e^{-itH_0} P_{ac}(H_0)$$

exist and are complete. Then the scattering operator is defined by

$$S = W_-^* W_+.$$

Let $\{E_0(\lambda)\}_{\lambda \in \mathbb{R}^+}$ be the spectral family of H_0 and let $\sigma_{ac}(H_0)$ denote the absolutely continuous spectrum of H_0. Then S decomposes as

$$S = \int_{\sigma_{ac}(H_0)} S(\lambda) dE_0(\lambda).$$

The operator $S(\lambda)$, which acts in a Hilbert space $\mathcal{H}_\lambda$, is called the on-shell scattering matrix. Let I_λ denote the identity in $\mathcal{H}_\lambda$. It is known that $S(\lambda) - I_\lambda$ is trace class [BY], [Y] so that the Fredholm determinant $\det S(\lambda)$ of $S(\lambda)$ exists, and by a theorem of Birman and Krein [BK] we have the following relation between $\det S(\lambda)$ and the spectral shift function:

$$\det S(\lambda) = e^{-2\pi\xi(\lambda)}, \ a.e. \ \lambda \in \sigma_{ac}(H_0). \tag{3.2}$$

As a consequence we obtain

Theorem 3.3. *Let $\epsilon > 0$ and assume that $[0, \epsilon) \subset \sigma_{ac}(H_0)$. Let $D_\epsilon = \{z \in \mathbb{C} \mid |z| < \epsilon\}$ and let $\hat{D} \longrightarrow D_\epsilon$ be the ramified covering of order m given by $z \longmapsto z^m$. Suppose that*

1) *$\xi(\lambda)$ is continuous on $(0, \epsilon)$.*

2) *$\det S(\lambda)$, $\lambda \in (0, \epsilon)$, extends to a holomorphic function on $\hat{D}$.*

Then

$$\xi(\lambda) = \sum_{k=0}^{\infty} c_k \lambda^{k/m}, \ \lambda \in [0, \epsilon_1), \ 0 < \epsilon_1 < \epsilon.$$

This implies that condition 3) holds in this case. More precisely, we have

$$\operatorname{Tr}\left(e^{-tH} - e^{-tH_0}\right) = -\,\xi(0+) + \frac{1}{2\pi i}\int_0^{\epsilon_1} e^{-t\lambda^m} \operatorname{Tr}\left(S^*(\lambda^m)\frac{dS(\lambda^m)}{d\lambda}\right)\, d\lambda$$
$$+\, O(e^{-t\epsilon/2})$$

as $t \to \infty$, and the integral has an asymptotic expansion in negative powers of t. The coefficients of this expansion are given in terms of derivatives of $\det S(\lambda)$. These coefficients, on the other hand, appear as residues of the relative zeta function $\zeta(s; H, H_0)$. In particular, in this case $\det S(\lambda)$ is determined by the residues of $\zeta(s; H, H_0)$.

4 Examples

In this section we discuss a number of typical cases of admissible operators (H, H_0) which naturally arise in physics and geometry.

4.1 The Schrödinger operator in $\mathbb{R}^n$.

Let $V \in C_0^\infty(\mathbb{R}^n)$ and let H be the unique self–adjoint extension in $L^2(\mathbb{R}^n)$ of the Schrödinger operator $\Delta + V$ where $\Delta = -\sum_{i=1}^n \partial^2/\partial x_i^2$. Let $H_0 = \overline{\Delta}$. Then $e^{-tH} - e^{-tH_0}$ is trace class and as $t \to 0$, there exists an asymptotic expansion of the form

$$\operatorname{Tr}\left(e^{-tH} - e^{-tH_0}\right) \sim t^{-n/2}\sum_{j=1}^{\infty} a_j t^j \tag{4.1}$$

and the coefficients are given by

$$a_j = \int_{\mathbb{R}^n} P_j\left(V(x), DV(x), \ldots, D^\alpha V(x)\right)\, dx$$

where P_j is a universal polynomial in a finite number of partial derivatives of V [CV1], [Gu1], [Gu2]. In fact, (4.1) holds under less stringent assumptions on V (cf. [Ro]). We observe that $\operatorname{Tr}(e^{-tH} - e^{-tH_0})$ may also be regarded as regularized trace of e^{-tH}. Namely, let $K(x, y, t)$ be the kernel of e^{-tH} and let

$B_R \subset \mathbb{R}^n$ be the ball of radius R, centered at 0. Then

$$\operatorname{Tr}\left(e^{-tH} - e^{-tH_0}\right) = \lim_{R\to\infty}\left\{\int_{B_R} K(x,x,t)\,dx - \frac{\operatorname{Vol}(B_R)}{(4\pi t)^{n/2}}\right\}.$$

Let $S(\lambda) = S(\lambda; H, H_0)$ be the scattering matrix of (H, H_0). If n is odd, then $S(\lambda^2)$ extends to a meromorphic function on $\mathbb{C}$. For n even, we have to pass to the logarithmic covering of $\mathbb{C}$. We note that asymptotic expansions of $S(\lambda)$ near $\lambda = 0$ also exist under more general assumptions on the potential V (see, e.g., [JK], [J]). Let

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_N < \lambda_{N+1} = \cdots = \lambda_{N'} = 0$$

be the eigenvalues of H and let H_1 be the restriction of H to the orthogonal complement of the space spanned by the eigenfunctions of H. Then (H_1, H_0) is admissible and we put

$$\det(\Delta + V, \Delta) = \prod_{j=1}^{N} \lambda_j \cdot \det(H_1, H_0).$$

Since, for example, eigenvalues may converge to zero if the potential V is varied, this determinant will, in general, not depend continuously on V. But for an appropriate constant $m > 0$, $\det(\Delta + uV + m,\ \Delta + m)$ is always differentiable for $|u| < 1$ and we can use Theorem 2.6 to compute the variation.

4.2 Manifolds with cylindrical ends

Consider a compact C^∞ manifold M with boundary Y. Let

$$X = M \cup_Y (\mathbb{R}^+ \times Y)$$

where the bottom of the half-cylinder is glued to the boundary of M in the obvious way. We equip X with a Riemannian metric which is the product metric on $\mathbb{R}^+ \times Y$. Then X is a complete Riemannian manifold with a cylindrical end. Let Δ be the Laplacian of X. Then Δ is essentially self-adjoint and we put $H = \overline{\Delta}$. Let Δ_Y be the Laplacian of Y and let H_0 be the self-adjoint extension of

$$-\frac{\partial^2}{\partial u^2} + \Delta_Y : C_0^\infty(\mathbb{R}^+ \times Y) \to L^2(\mathbb{R}^+ \times Y)$$

obtained by introducing Dirichlet boundary conditions. We extend H_0 by zero on the orthogonal complement of $L^2(\mathbb{R}^+ \times Y)$ in $L^2(X)$. Then $e^{-tH} - e^{-tH_0}$ is trace class for $t > 0$ and as $t \to 0$ there is an asymptotic expansion

$$\operatorname{Tr}\left(e^{-tH} - e^{-tH_0}\right) \sim t^{-n/2} \sum_{j=0}^{\infty} a_j t^j.$$

Let $0 = \mu_0 < \mu_1 < \mu_2 <$ be the eigenvalues of Δ_Y. These are the thresholds of the continuous spectrum of H. The continuous spectrum of H consists of branches starting at each threshold with finite multiplicity equal to the dimension of the eigenspace, i.e.

$$\sigma_{ac}(H) = \bigcup_{i=0}^{\infty} [\mu_i, \infty).$$

Let $\mathcal{E}(\mu_i)$ be the eigenspace for the eigenvalue μ_i. If $\lambda \neq \mu_i$, $i \in \mathbb{N}$, the scattering matrix is a finite–rank operator

$$S(\lambda)\colon\ \oplus_{\mu_i<\lambda}\, \mathcal{E}(\mu_i) \to \oplus_{\mu_i<\lambda} \mathcal{E}(\mu_i)$$

In particular, if $\lambda < \mu_1$ then $S(\lambda)$ is a scalar and $S(\lambda^2)$ extends to a meromorphic function on $\{z \in \mathbb{C} \mid |z| < \mu_1\}$. Thus, by Theorem 3.3 and Lemma 3.2, (H, H_0) is admissible and $\det(H, H_0)$ is well defined.

Spinor Laplacians can be treated in the same way. For example, suppose that X is a spin manifold and let $F \to X$ be a Hermitian vector bundle over X with a Hermitian connection. We assume all structures are adepted to the product structure of $\mathbb{R}^+ \times Y$. Let

$$D\colon C^\infty(S \otimes F) \longrightarrow C^\infty(S \otimes F)$$

be the twisted Dirac operator. Then on $\mathbb{R}^+ \times Y$, D has the form

$$D = \gamma\left(\frac{\partial}{\partial u} + D_Y\right)$$

where γ denotes Clifford multiplication by the exterior normal vector field to Y and D_Y is a twisted Dirac operator on Y. Let $D_0 = \gamma(\partial/\partial u + D_Y)$ which we regard as operator in $L^2(\mathbb{R}^+ \times Y)$ with domain $C_0^\infty(\mathbb{R}^+ \times Y)$. Let $H = \overline{D}^2$ and let H_0 be the closure of D_0^2 with respect to Dirichlet boundary conditions. Then, as above, (H, H_0) is admissible and $\det(H, H_0)$ is well defined. We put $\det(D^2) = \det(H, H_0)$.

Remark. The trace $\mathrm{Tr}(e^{-tH}-e^{-tH_0})$ is closely related to the b–trace used by Melrose [Me] to define a regularized trace for e^{-tD^2}. The b–trace is defined as follows. For $R \geq 0$, let

$$X_R = M \cup_Y ([0,R] \times Y).$$

Then

$$b\text{-}\mathrm{Tr}(e^{-tD^2}) = \lim_{R\to\infty}\left\{\int_{X_R} tr\, e^{-tD^2}(x,x)\,dx - \frac{R}{\sqrt{4\pi t}}\mathrm{Tr}(e^{-tD_Y^2})\right\}.$$

By a similar formula, we may compute $\mathrm{Tr}(e^{-tH}-e^{-tH_0})$ in terms of the kernels which represent these operators. This gives

$$\mathrm{Tr}(e^{-tH}-e^{-tH_0}) = b\text{-}\mathrm{Tr}(e^{-tD^2}) + \frac{1}{4}\mathrm{Tr}(e^{-tD_Y^2})$$

Let M and B be connected Riemannian manifolds and assume that B is compact. Let $\pi\colon M \to B$ be a Riemannian submersion whose fibres $Z_b = \pi^{-1}(b)$ are $2k$–dimensional manifolds with cylindrical ends. We assume that Z_b is an oriented spin manifold. Such families can be constructed from fibrations of manifolds with boundary along the lines of [BC]. Let $E \to M$ be a Hermitian vector bundle which is adapted to the product structure and let $D_{\pm,b}$ be the twisted Dirac operator on the fibre Z_b. Let $\mathbb{R}^+ \times Y_b$ be the cylindrical end of Z_b. Then on $\mathbb{R}^+ \times Y_b$, $D_{\pm,b}$ takes the form

$$D_{\pm,b} = \gamma\left(\frac{\partial}{\partial u} + D_{Y_b}\right).$$

Let $D_\pm$ denote the corresponding family of Dirac operators. Assume that $0 \notin \mathrm{Spec}(D_{Y_b})$ for all $b \in B$. Then for each $b \in B$, D_b has discrete spectrum near zero and we may construct the determinant line bundle

$$\mathcal{L} = \det(\ker D_+)^* \otimes \det(\ker D_-)$$

in the same way as in [BF1]. Using the determinants $\det(D_{+,b}D_{-,b})$ and $\det(D_{-,b}D_{+,b})$ defined as above, we can equip $\mathcal{L}$ with the Quillen metric.

4.3 Complete surfaces of finite area

Let (X,g) be a complete surface of finite area such that the Gaussian curvature K satisfies $K \equiv -1$ in the complement of a compact set. Then X

admits a decomposition of the form

$$X = X_0 \cup Y_1 \cup \cdots \cup Y_m,$$

where X_0 is a compact surface with smooth boundary and

$$Y_i \cong [1, \infty) \times S^1, \ i = 1, \dots, m,$$

and the metric on Y_i equals

$$ds^2 = \frac{dy^2 + dx^2}{y^2}$$

where $(y, x) \in [1, \infty) \times S^1$. Each end Y_i is called a cusp of X.

Examples are the well known surfaces $\Gamma(N) \backslash H$ where H is the upper half-plane and $\Gamma(N) \subset SL(2, \mathbb{Z})$ is the principal convergence subgroup of level N.

Let Δ be the Laplacian of X. Then Δ is essentially self-adjoint in $L^2(X)$. Let $H = \overline{\Delta}$. The structure of the spectrum $\sigma(H)$ of H is well known (see, e.g., [Mu2]). Namely, $\sigma(H)$ is the union of an absolutely continuous part $\sigma(H)$, which equals $[1/4, \infty)$, and the point spectrum $\sigma_p(H)$. The point spectrum consists of a sequence of eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$ of finite multiplicity. The only possibly point of accumulation of the eigenvalue sequence is ∞ and, as proved by Colin de Verdiere [CV2], for a generic metric there exist only finitely many eigenvalues which are all contained in [0,1/4). The operator H_0 is obtained by restricting Δ to the zero Fourier coefficients on each cusp and imposing Dirichlet boundary conditions. It follows from Theorem 8.20 of [Mu1] that $\mathrm{Tr}(e^{-tH} - e^{-tH_0})$ has an asymptotic expansion of the form (2.5). Hence, (H, H_0) is admissible. Put

$$\det\Delta = \det(H, H_0).$$

In the present case, $\det \Delta$ can be expressed in a different way using the eigenvalues of Δ and the *resonances.* Here resonances are defined in terms of an analytic continuation of the resolvent of Δ to a meromorphic operator valued function with values in $\mathcal{L}\left(L^2_{comp}(X),\ L^2_{loc}(X)\right)$. A resonance is then, by definition, a pole of the analytic continuation of the resolvent which does not correspond to an eigenvalue in the sense that the pole is either not an eigenvalue or its multiplicity (defined in a proper sense) is bigger than the dimension of the eigenspace. Let $R(\Delta)$ be set of all poles of the analytic continuation of the resolvent. For each $\eta \in R(\Delta)$ one can define an algebraic multiplicity $m(\eta)$. In fact, there is a nonself–adjoint operator B – the generator of the Lax–Phillips semigroup – such that $R(\Delta)$ equals the set of generalized eigenvalues of B and $m(\eta)$ is then the algebraic multiplicity of this eigenvalue [Mu2].

Furthermore, the scattering matrix $S(\lambda) = S(\lambda; H, H_0)$ extends to a meromorphic function on the double covering of $\mathbb{C}$ defined by $\sqrt{\lambda}$, and resonances can also be defined as poles of the scattering matrix.

Now the resonance set $R(\Delta)$ can be used as a discrete set of spectral parameters in the same way as the eigenvalues are used in the case of a compact Riemannian manifold. For example, there is an analogue of Weyl's formula [Mu2], [Pa], and also a trace formula [Mu2]. In particular, one can introduce the *resonance zeta function* which is defined by

$$\zeta_{Res}(s) = \sum_{\substack{\eta \in R(\Delta) \\ \eta \neq 1}} \frac{m(\eta)}{(1-\eta)^s}. \tag{4.2}$$

This series is absolutely convergent for $\mathrm{Re}(s) > 2$ and admits a meromorphic continuation to $\mathbb{C}$ which is holomorphic at $s = 0$. Using this zeta function, we define a second regularized determinant by

$$\det{}_{Res}\Delta = \exp\left(-\frac{d}{ds}\zeta_{Res}(s)\mid_{s=0}\right).$$

Then formally, one has

$$\det{}_{Res}\Delta = \prod_{\substack{\eta \in R(\Delta) \\ \eta \neq 1}} |1-\eta|^2.$$

Furthermore, the two determinants are closely related. Namely, we have

$$\det\Delta = \exp\left(\frac{\mathrm{Area}(X)}{8\pi} - \frac{3\pi\gamma}{2}m\right)\det{}_{Res}\Delta$$

where γ is Euler's constant [Mu2]. More generally, for $\mathrm{Re}(z) > 1$, we can define

$$\zeta_{Res}(s;z) = \sum_{\substack{\eta \in R(\Delta) \\ \eta \neq 1}} \frac{m(\eta)}{(z-\eta)^s}, \ \mathrm{Re}(s) > 2. \tag{4.3}$$

Then, as a function of s, $\zeta_{Res}(s;z)$ has a meromorphic continuation to $\mathbb{C}$ and $s=0$ is not a pole. Put

$$\det_{Res}(\Delta+z-1)=\exp\left(-\frac{\partial}{\partial s}\zeta_{Res}(s;z)|_{s=0}\right).$$

For a hyperbolic surface $\Gamma\backslash H$ of finite area, $\det_{Res}(\Delta+z-1)$ can be expressed in terms of the Selberg zeta function $Z_\Gamma(s)$ and the determinant of the scattering matrix $S(\lambda)$. First recall that

$$Z_\Gamma(s)=\prod_\gamma\prod_{k=0}^{\infty}\left(1-e^{-(s+k)\ell(\gamma)}\right),\ \mathrm{Re}(s)>1,$$

where γ runs through the primitive closed geodesics and $\ell(\gamma)$ is the length of γ. Furthermore, we write the spectral parameter λ as usually as $\lambda=z(1-z)$, $z\in\mathbb{C}$, and regard the scattering matrix as a function of z. Then

$$\det{}^2_{Res}(\Delta+z-1)=\det S(z)Z_\Gamma(z)^2Z_\infty(z)^2\Gamma(z+1/2)^{-2m}e^{c(2z-1)+d},\quad (4.4)$$

where c and d are certain constants that depend on Area$(\Gamma\backslash H)$ and m, and

$$Z_\infty(s)=\left((2\pi)^s\Gamma_2(s)^2/\Gamma(s)\right)^{\mathrm{Area}(\Gamma\backslash H)/2\pi}$$

with $\Gamma_\infty(s)$ being the double Gamma function (see [Mu2]). It follows from (4.4) that $\det_{Res}(\Delta+z-1)$ extends to an analytic function on $\mathbb{C}$ with the set of zeros equal to $R(\Delta)$. We claim that this holds in general.

If $\Gamma=SL(2,\mathbb{Z})$, then the scattering matrix is a function $S(z)$ which is given in terms of the Riemannian zeta function $\zeta(s)$ by

$$S(z)=\sqrt{\pi}\frac{\Gamma(z-1/2)}{\Gamma(z)}\frac{\zeta(2z-1)}{\zeta(2z)}.$$

[He]. Thus in this case, the resonances are precisely the numbers $\rho/2$ where ρ runs over the nontrivial zeros of the Riemann zeta function. Furthermore, $\det_{Res}(\Delta+z-1)$ can be factorized in the product of two determinants where the first one is defined in terms of the eigenvalues similar to (1.5) and the second one is defined in terms of the nontrivial zeros ρ of $\zeta(s)$. We describe the second determinant. For $\mathrm{Re}(z)>1$ consider the Dirichlet series

$$\zeta(s,z)=(2\pi)^s\sum_\rho(z-\rho)^{-s}\quad (4.5)$$

where $\arg(z-\rho) \in (-\pi/2, \pi/2)$. It was proved in [De] that (4.5) converges absolutely for $\mathrm{Re}(s) > 1$ and for fixed z, it has an analytic continuation to a holomorphic function of $s \in \mathbb{C}\backslash\{1\}$. Moreover, one has

$$2^{-1/2}(2\pi)^{-2}\pi^{-z/2}\Gamma(z/2)\zeta(z)z(z-1) = \exp\left(-\frac{\partial}{\partial s}\xi(s,z)|_{s=0}\right). \qquad (4.6)$$

The right hand side is then the determinant associated with the resonances. Note that (4.5) resembles (4.3). This analogy becomes even closer if we recall Colin de Verdiere's result that for a generic metric, the number of eigenvalues is finite [CV2].

In conclusion, one can say that to some extent, for a surface with hyperbolic ends, the resonances can be used as a substitute for the eigenvalues of the Laplacian on a compact surface. It would be very interesting to see if in other cases, the resonances play a similar role. For example, we think that everything that we described here for surfaces can be extended to the case of manifolds with ends of hyperbolic type as studied in [Mu1]. Furthermore, the Laplacian on forms can be treated in the same way. In particular, we can introduce the L^2 analytic torsion for hyperbolic manifolds of finite volume.

Another interesting problem is the investigation of the finer structure of the distribution of resonances in the case of surfaces with hyperbolic ends. This should be seen in connection with quantum chaos [Sa].

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