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# ASYMPTOTICS OF THE FIRST NODAL LINE

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## INTRODUCTION

In this note, we announce the result that the first nodal line of a convex planar domain tends to a straight line as the eccentricity tends to infinity.

Let $\Omega$ denote a bounded convex domain in $\mathbf{R}^2$. Denote by $u$ a second Dirichlet eigenfunction. Then $u$ satisfies

$$\begin{aligned} \Delta u &= -\lambda_2 u \quad &&\text{in } \Omega \\ u &= 0 &&\text{on } \partial\Omega \end{aligned}$$

where $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$ represent the eigenvalues in increasing order. The first nodal line $\Lambda$ is the zero set of $u$.

$$\Lambda = \{z \in \Omega : u(z) = 0\}$$

In order to state a precise theorem, let us normalize the region to lie within an $N \times 1$ rectangle. Let $P_x$ and $P_y$ denote the orthogonal projection on the $x$ and $y$ axes, respectively. First rotate so that the projection $P_y\Omega$ is *smallest.* Then dilate and translate so that $P_y\Omega = (0,1)$ and $P_x\Omega = (0,N)$. (The choice of orientation of the $y$ axis is crucial for what follows, but the dilation and translation are merely for notational convenience.)

**Theorem 1.** *With the normalization above, there is an absolute constant $C$ such that*

$$\textit{length}\, P_x\Lambda \leq C/N$$

Furthermore, this estimate is sharp. The case of a long, thin, circular sector shows that $C \geq 1/2$.

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## Outline of the Proof

**Step 1 $O(1)$ estimate.** With the normalization of Theorem 1,

$$\Omega = \{(x,y) : f_1(x) < y < f_2(x),\, 0 < x < N\}$$

where $f_1$ is a convex function and $f_2$ is a concave function. Define the height function $h(x) = f_2(x) - f_1(x)$. Consider the ordinary differential operator $\mathcal{L}$ defined by

$$\mathcal{L} = -\frac{d^2}{dx^2} + \frac{\pi^2}{h(x)^2}$$

Recall the following theorem, which implies a weaker version of Theorem 1, namely, the same estimate without the factor $1/N$.

**Theorem 2** [J3]**.** *With the normalization of Theorem 1, let $\phi_2$ be the second eigenfunction for $\mathcal{L}$ with Dirichlet boundary conditions on $[0,N]$. Let $x_0$ be the unique zero of $\phi_2$ in $(0,N)$. There is an absolute constant $A$ such that*

$$P_x\Lambda \subset [x_0 - A, x_0 + A]$$

This theorem says in a very crude sense that $u$ resembles the function

$$\phi_2(x) \sin \ell_x(y)$$

where

$$\ell_x(y) = \frac{\pi(y - f_1(x))}{h(x)}.$$

The function $\ell_x(y)$ is chosen to be the linear function in $y$ that has the value 0 on the bottom, $(x, f_1(x))$, of $\Omega$ and $\pi$ on the top, $(x, f_2(x))$, of $\Omega$. Thus, $\sin \ell_x(y)$ is the lowest Dirichlet eigenfunction for $-(d/dy)^2$ on the interval $f_1(x) \le y \le f_2(x)$ of length $h(x)$. (The fact that we have rotated so that $h$ is as small as possible plays a crucial role.)

In addition to this estimate, we will need another consequence of [J3], expressed in terms of a parameter $L$ defined as follows.

**Definition.** *The length scale $L$ of $\Omega$ is the length of the rectangle $R$ contained in $\Omega$ with the lowest (first) Dirichlet eigenvalue.*

Up to order of magnitude, $L$ is the largest number such that $h(x) > 1 - 1/L^2$ on an interval of length $L$. When $\Omega$ is a rectangle, $R = \Omega$ and $L = N$. When $\Omega$ is a triangle of length $N$, then $L \approx N^{1/3}$. In general, $N^{1/3} \lesssim L \le N$. The example of a trapezoid shows that all intermediate sizes for $L$ are possible.

The heuristic principle behind $L$ is that $\phi_2$ resembles $\sin(2\pi(x - x_0)/L)$, the second eigenfunction of the interval $[x_0 - L/2, x_0 + L/2]$ and $u$ resembles $\sin(2\pi(x - x_0)/L) \sin \pi y$, the second eigenfunction of the rectangle of length $L$ and width 1 with nodal line at $x = x_0$. This is true to within order of magnitude near the "central" portion of $\Omega$, with an exponential tail in the thin regions of $\Omega$. More precisely we have,

**Proposition.** *Let* $u_+(x,y) = \max\{u(x,y),0\}$ *and* $u_-(x,y) = \max\{-u(x,y),0\}$. *Then*

$$u_+(x_0 + A + 1, 1/2) \approx u_-(x_0 - A - 1, 1/2) \approx \max|u|/L$$
$$u_+(x_0 + L/20, 1/2) \approx u_-(x_0 - L/20, 1/2) \approx \max|u|$$

The proposition follows from the methods of [J3]. (See especially Proposition A of [J3].)

**Step 2.** Denote by

$$e(x,y) = (h(x)/2)^{-1/2} \sin \ell_x(y)$$

the first Dirichlet eigenfunction on $I_x = \{y : f_1(x) \le y \le f_2(x)$, normalized in $L^2(I_x)$. Denote

$$\psi(x) = \int_{f_1(x)}^{f_2(x)} u(x,y)e(x,y)dy$$

Then

$$u(x,y) = \psi(x)e(x,y) + v(x,y)$$

where $\psi(x)$ is the "best" coefficient possible and $v(x,y)$ should be a small error term. Because of Theorem 2, there exists a number $x_1$, such that $|x_0 - x_1| < A$ and $\psi(x_1) = 0$.

**Lemma 1.** $\psi'(x) \approx 1/L$ *on* $|x - x_0| \le L/20$. *In particular,* $\psi$ *is strictly increasing and* $x_1$ *is the only zero of* $\psi$ *on that interval.*

**Lemma 2.** $|v(x,y)| \lesssim S/L$ *where*

$$S = \max_{|x-x_1|\le L/20} (|f_1'(x)| + |f_2'(x)|)e^{-c|x-x_1|} + e^{-cL}$$

The number $S$ represents the slope of the boundary near $x_1$ plus the slope at a further distance decreased by an exponential factor. In the range $|x - x_0| \le L/20$, $|f_1'(x)| + |f_2'(x)| \le C/L^3$. The ideas of the proofs of Lemmas 1 and 2 will be presented in the next section. For now let us complete the outline of the proof of Theorem 1.

**Step 3.** If $u(x, 1/2) = 0$, then

$$\frac{1}{L}|x - x_1| \approx |\psi(x)| = |v(x,1/2)|/e(x,1/2) \lesssim S/L$$

Therefore,

$$|x - x_1| \lesssim S$$

Moreover,

$$S \lesssim 1/L^3 \lesssim 1/N$$

This is the end of the proof for points of the nodal line in the middle of $\Omega$. Near the boundary $\partial\Omega$, the denominator $e(x,y)$ is small, so additional ideas are needed. One uses maximum principle and Hopf type estimates of [J1,J2,J3] and extra estimates on the rate of vanishing of $v(x,y)$ at the boundary.

## Proofs of Lemmas 1 and 2

The idea of the proof of Lemma 1 is as follows. One calculates that

$$\mathcal{L}\psi - \lambda_2\psi = -\psi'' + \left(\frac{\pi^2}{h(x)^2} - \lambda_2\right)\psi = \sigma$$

where $\sigma$ is small and

$$\left|\lambda_2 - \frac{\pi^2}{h(x)^2}\right| \leq \frac{100}{L^2}$$

in the range $|x - x_0| \leq L/20$. Next one deduces from the proposition above that with the normalization $\max|u| = 1$,

$$\psi(x_0 + L/20) - \psi(x_0 - L/20) \approx 1$$

Then comparison with constant coefficient ordinary differential equations gives $\psi'(x) \approx 1/L$ for $|x - x_0| \leq L/20$.

The idea of the proof of Lemma 2 is to follow the Carleman method of differential inequalities. In that method, one considers a *harmonic* function, say $w$, in a region, say $\Omega$, which vanishes on a portion of the boundary. Then one considers the function

$$f(x) = \int_{f_1(x)}^{f_2(x)} w(x,y)^2\,dy$$

Using the equation $\Delta w = 0$, the zero boundary values, and integration by parts, one can find a differential inequality for $f$ of the form $f''(x) \geq a(x)f(x)$. This convexity property makes it possible to deduce rates of vanishing for $w$.

To prove Lemma 2, one considers

$$g(x) = \int_{f_1(x)}^{f_2(x)} v(x,y)^2\,dy,$$

and deduces a differential inequality of the form

$$g'' \geq 2\left(\frac{(2\pi)^2}{h(x)^2} - \lambda_2\right) g - \beta\sqrt{g} \geq g - \beta\sqrt{g}$$

The crucial point is that because we have subtracted the first eigenfunction in the $y$ direction ($v = u - \psi(x)e(x,y)$), the coefficient on $g$ involves $(2\pi)^2$ rather than $\pi^2$. It follows that

$$g(x) \approx \frac{\cosh(x - x_1)}{\cosh(L/2)} + \beta \text{ dependence}$$

The first term is exponentially small and the second term is controlled by $S$, proving Lemma 2.

To illustrate the mechanism of the lemmas explicitly, we carry out a sine series computation in a special case. Note that the size and sign of $(k\pi)^2 - \lambda_2$ for $k = 1$ versus $k \geq 2$ is at issue. We consider the special case in which $f_1(x) = 0$ and $f_2(x) = 1$ for $0 \le x \le N-1$. Then $N - 1 \le L \le N$, so $N$ and $L$ are comparable. By comparison with rectangles of length $N$ and $N - 1$ we find that

$$\pi^2 \left(1 + \frac{4}{N^2}\right) \le \lambda_2 \le \pi^2 \left(1 + \frac{4}{(N-1)^2}\right)$$

For $0 \le x \le N - 1$,

$$u(x, y) = \sum_{k=1}^{\infty} u_k(x) \sin(k\pi y)$$

where

$$u_k(x) = 2 \int_0^1 \sin(k\pi y) u(x, y) dy$$

Furthermore, the Fourier coefficient $u_k$ satisfies

$$u_k''(x) + (\lambda_2 - (k\pi)^2) u_k = 0.$$

The function $\psi(x) = u_1(x)/\sqrt{2}$ and $\lambda_2 - \pi^2 \approx 1/N^2 \approx 1/L^2$. Thus

$$u_1(x) = -c_1 \sin \sqrt{\lambda_2 - \pi^2} x.$$

The coefficient satisfies $c_1 > 0$ because $u$ is negative on the left half and positive on the right half of $\Omega$. Normalize so that $\max u = 1$. By the proposition, $u_\pm$ is large at $x_0 \pm L/20$, and hence $c_1$ is larger than a positive absolute constant. This yields Lemma 1, as well as the precise location of $x_1$ as a function of $\lambda_2$.

On the other hand, the remaining terms of the series are small. For all $k > 1$, $\lambda_2 - k^2\pi^2 < -1$. Therefore,

$$u_k(x) = c_k \sinh \sqrt{(k\pi)^2 - \lambda_2} x$$

The unit bound on $u$ implies

$$\sum_{k=1}^{\infty} u_k(x)^2 \le 2$$

In particular,

$$\sum_{k=2}^{\infty} c_k^2 \sinh^2 [\sqrt{(k\pi)^2 - \lambda_2} (N - 1)] \le 2$$

This implies that for $k \ge 2$,

$$|u_k(x)| \le Ce^{-kN} \quad \text{for} \quad |x - x_1| \le N/10$$

Hence $v(x, y)$ is exponentially small, which proves Lemma 2 in the special case.

## WHERE IS $\Lambda$?

Recall that the nodal line may be in the exact middle ($x_1 = N/2$), as in the case where $\Omega$ is a rectangle, or it may be very near the fat end of the region, as in the case of a circular sector with vertex at the origin: $x_1 \approx N - cN^{1/3}$. Theorems 1 and 2 give numerical schemes for approximating the location of the nodal line as follows.

Recall that $x_0$ was defined above as the zero of the eigenfunction $\phi_2$. Since the second eigenvalue for $\mathcal{L}$ on $[0, N]$ is the same as the first Dirichlet eigenvalue for the operator on the two intervals $[0, x_0]$ and $[x_0, N]$, Theorem 2 implies the following prescription.

**ODE Eigenvalue Scheme.** Choose $x_0$ to be the unique number such that the lowest Dirichlet eigenvalue for the operator $\mathcal{L}$ on the intervals $[0, x_0]$ and $[x_0, N]$ are equal. Then

$$P_x\Lambda \subset [x_0 - A, x_0 + A]$$

The min-max principle implies that any curve dividing the region $\Omega$ into two halves with equal eigenvalues must intersect the nodal line. Theorem 1 implies that $\Lambda$ is particularly close to a vertical straight line. This leads to the following prescription.

**PDE Eigenvalue Scheme.** Choose $x_2$ so that the least eigenvalues for the Dirichlet problem for the Laplace operator on the two regions

$$\Omega \cap \{(x, y) : x < x_2\} \quad \text{and} \quad \Omega \cap \{(x, y) : x > x_2\}$$

are equal. Then Theorem 1 implies

$$P_x\Lambda \subset [x_2 - C/N, x_2 + C/N]$$

The first scheme requires knowledge of the lowest eigenvalue of an ordinary differential equation, which is in standard numerical packages. The second scheme requires knowledge of the lowest eigenvalue on a convex domain, which is not quite as standard. Toby Driscoll [D] has recently developed a very effective program for computing both eigenfunctions and eigenvalues on polygons. Preliminary experiments with triangles with $3 \leq N \leq 150$ indicate that $A$ in the first scheme may be $1/100 + 1/N$. (This seems too good to be true, but perhaps $A = 1/10 + 1/N$ will work in general.) The bound $C/N$ in Scheme 2 seems to be $1/N$ as predicted by the case of a sector. We must confess, however, that the rigorous proofs of these bounds give ridiculous values like $C = 10^{20}$.

## CONJECTURES

The methods outlined here should also give information about the size of the first eigenfunction, improving by a factor of $\sqrt{L}$ the bounds given in [J3].

**Conjecture 1.** *With the normalizations on $\Omega$ of Theorem 1, let $u_1$ denote the first eigenfunction for $\Omega$ such that* $\max|u_1| = 1$. *Then there is an absolute constant $C$ and a suitable multiple of the first eigenfunction $\phi_1$ for the operator $\mathcal{L}$ on $[0, N]$ satisfies*

$$|u_1(x, y) - \phi_1(x) \sin \ell_x(y)| \leq C/L$$

Conjecture 1 is motivated by the elementary inequality

$$|\sin(x/L) - \sin(x/(L+1))| \le C/L \quad \text{on} \quad 0 \le x \le \pi L$$

The methods used to prove Theorem 1 also show the following.

**Corollary.** *Let $(x, y)$ be a point of $\Lambda$ satisfying $1/4 \le y \le 3/4$, that is, far from $\partial\Omega$. Let $\eta$ be a unit vector tangent to $\Lambda$ at $(x, y)$. Then*

$$|\eta \cdot e_1| \le CS \le C/L^3$$

**Conjecture 2.** *The corollary is valid up to the boundary.*

(Conjecture 2 implies Theorem 1.)

Finally let us speculate about the higher-dimensional case. We begin by explaining the significance of $L$ in another way. Let $e$ be a unit vector and define

$$\Omega(t, e) = \{x + se : x \in \Omega,\ 0 \le s < t\}$$

Thus $\Omega(t, e)$ is $\Omega$ stretched by $t$ in the direction $e$. Define

$$P(e) = -\frac{d}{dt}\lambda_1(\Omega(t, e))|_{t=0}$$

This is the first variation of the lowest eigenvalue. It is analogous to the projection body function in the theory of convex bodies. (See [J4].) In a convex domain normalized as above,

$$P(e_1) \approx \min_e P \approx 1/L^3 \quad \text{and}$$

$$P(e_2) \approx \max_e P \approx 1$$

Moreover, the direction $e_1$ is necessarily within $1/L^3$ of the values of $e$ for which the exact minimum is attained.

In $\mathbf{R}^n$, $n \ge 3$ one can define the same function $P$ on the unit sphere. In the spirit of quadratic forms, choose $v_1$ so that

$$P(v_1) = \min P$$

Choose $v_2$ perpendicular to $v_1$ such that

$$P(v_2) = \min_{v \perp v_1} P(v)$$

(One can continue inductively to form an orthonormal basis $v_1, v_2, \ldots, v_n$.)

**Conjecture 3.** *There is a dimensional constant $C$ such that if $v$ is a unit vector tangent to $\Lambda$, then*

$$|v \cdot v_1| \le CP(v_1)/P(v_2)$$

This conjecture is intended to give specific bounds on the way the nodal set tends to a plane as the eccentricity tends to infinity. It is an analogous conjecture concerning the shape of the second eigenfunction to conjectures in [J4] concerning the shape of the first eigenfunction. One could also formulate even more detailed and even more speculative conjectures relating all the numbers $P(v_k)$ to the location of $\Lambda$.

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