JOURNÉES ÉQUATIONS AUX DÉRIVÉES PARTIELLES

FRANK MERLE

Recent progress on the blow-up problem of Zakharov equations

Journées Équations aux dérivées partielles (1995), p. 1-7
<http://www.numdam.org/item?id=JEDP_1995____A20_0>

Recent Progress on the Blow-up Problem for Zakharov Equations

By Frank Merle

Université de Cergy-Pontoise
Mathématiques, 8 Avenue du Parc, Le Campus,
95033 Cergy-Pontoise Cedex France

In this paper, we present recent progress for the blow-up problem for Zakharov equations.

More precisely, we consider Zakharov equations

$$\text{(I)} \qquad \begin{aligned} &i\partial u/\partial t = -\Delta u + nu \\ &\partial n/\partial t = -\nabla . v \\ &c_0^{-2}\partial v/\partial t = -\nabla n - \nabla |u|^2 \end{aligned}$$

with initial data $(u(-1),n(-1),v(-1)) = (u_0,n_0,v_0)$
where $u: R^2 \to C$, $n: R^2 \to R$, $v: R^2 \to R^2$,

and related equations which are

the cubic nonlinear Schrödinger equation

$$\text{(II)} \qquad i\partial u/\partial t = -\Delta u - |u|^2 u$$

with initial data $u(-1) = u_0$
where $u: R^2 \to C$,

and the Elliptic equation associated with (II)

$$\text{(III)} \qquad u = \Delta u + |u|^2 u$$

where $u: R^2 \to C$.

1) The local Cauchy theory for equations (I),(II).

We are interested to find a space H for equation (I) or (II) such that there is a unique solution of the equation on $[0,T)$ and we have the following $T=+\infty$ or $T<+\infty$ and $|u(t)|_H \to +\infty$ as t goes to T.

i)Case of the nonlinear Schrödinger equation (II)
The case of the cubic nonlinear Schrödinger equation is now well-understood. A local (in time) Cauchy theory can be done in various natural space H^1, H^s, L^2 (see [GV],[K],[CaW],[Bo1]). Moreover, one can show that the blow-up time does not depend on the Cauchy space and in fact

we have at the blow-up a concentration phenomenon in L^2.

In [MT] (see also [W2], [GlM2]), it is proved the following. Let u(t) a blow-up solution (and T its blow-up time), there is then x(t) such that for all R>0, $\liminf_{t \to T} |u(t)|^2_{L^2(|x-x(t)|\leq R)} \geq a>0$ where a is an universal constant ($a=|Q|^2_{L^2}$ where Q will be defined in subsection 3).

In addition, we have the following conserved quantities for all t,

$|u(t)|_{L^2} = |u_0|_{L^2}$,
$E(u(t)) = E(u_0)$ where $E(u) = 1/2\int|\nabla u|^2 - 1/4\int|u|^4$.

ii)Case of Zakharov equations (I).
A local (in time) Cauchy theory can not be done up to now in the energy space $H_1=\{(u,n,v)\in H^1\times L^2\times L^2\}$ for a general initial data. The result is proved for the space $H_2=\{(u,n,v)\in H^2\times H^1\times H^1\}$ (see for exemple[OT2],[KePVg],[Bo2] and the references therein).

Moreover, one can show that we have at the blow-up time again the same concentration phenomenon in L^2. Indeed, let (u(t),n(t),v(t)) a blow-up solution (and T its blow-up time), there is then x(t) such that for all R>0, $\liminf_{t \to T} |u(t)|^2_{L^2(|x-x(t)|\leq R)} \geq |Q|^2_{L^2}$ where Q will be defined in subsection 3.

In addition, we have the following conserved quantities for all t,

$|u(t)|_{L^2} = |u_0|_{L^2}$,
$H(u(t),n(t),v(t)) = H(u_0,n_0,v_0)$ where $H(u) = \int|\nabla u|^2 + n|u|^2 + n^2/2+|v|^2/2c_0$.

iii) Blow-up problem
We are now interrested in the case $T<+\infty$, that is the case of a blow-up solution (or equivalently a singular solution) for equation (I) or (II). Most of the results can be extend in dimension $N\geq 1$in the case of a critical power for the nonlinear Schrödinger equation. Part of the results for the Zakharov equation can be extend to the dimension 3 (only dimensions 2,3 are relevant).

2) Elementary relations between equations (I)-(II)-(III)

i) Limit as c_0 goes to infinity.
We can easily see that as c_0 goes to infinity, the wave part of equation (I) give formally

$$\nabla(n + |u|^2) = 0,$$

or equivallently

$$n + |u|^2 = 0.$$

Thus equation (I) transform in equation (II) as c_0 goes to infinity.
If the initial data are compatible, this result of convergence has been rigourously proved by several authors ([AA2], [OT1], [KePVe]) when the limit solution u(t) (of equation (II)) is regular. Near the blow-up time, we do not have convergence results and in some sense we can not expect some. For example, in [GlM2], there is the case of a blow-up solution of equation (II) with initial data u_0 such that for all finite c_0 and all n_0,v_0 the solution of (I) (u,n,v)(t) is globally defined in time. Therefore, in some sense at the singularity, equation (I) when c_0 is large, can

not be consider as a perturbation of equation (II).

ii) Periodic solutions of (I),(II)

By direct calculation, we can check that if w(x) is a solution of equation (III) then

- $u(t,x) = e^{it} w(x)$ is a periodic solution of equation (II)
- $(u(t,x),n(t,x),v(t,x)) = (e^{it} w(x),- |w(x)|^2,0)$ is also a periodic solution of equation (I).

iii)Conformally self-similar blowing-up solution

For this power in two dimension, the nonlinear Schrödinger equation has one more invariance : if u(t,x) is a solution of equation (II) then

$$1/t\bar{u}(1/t,x/t) \exp(i|x|^2/4t)$$

is also a solution of equation (II).

In particular, if w(x) is a real solution of the equation (III), then

$$1/t\, w(x/t) \exp(-i/t + i|x|^2/4t)$$

is also a solution of equation (II) which blow-up at T = 0. We then obtain explicit blow-up solutions of equation (II).

Unfortunatly, such invariance does not exist for the Zakharov equation. In particular, there is no direct way to obtain explicit blow-up solutions of Zakharov equations.

3) On minimal solutions of (III)

In this section, we recall briefly some results on the elliptic equation (III). From [BeL],[St] it is now classical that equation (III) have infinitly many solutions in H^1 (up to the invariance of the equation).

Let us defined the unique positive radially symmetric solution of equation (III) (see [Kw] for uniqueness). We have in fact that the solution w=0 is isolated in the set of solution in L^2. More precisely,

i)Assume that w(x) is a nonzero solution of equation (III) then $|w|_{L^2} \geq |Q|_{L^2}$.

ii) Moreover, we have the following carracterisation of the minimal solution (or ground state) of equation (III). Assume that w is a nonzero solution of equation (III) and $|w|_{L^2} = |Q|_{L^2}$ then up to the invariance of the equation w = Q (that is there exist x',ω,θ such that $w(x) = e^{i\theta}\omega Q(\omega(x-x'))$).

4) Equation (II)

The problem of singularity for equation (II) has been studied in the last 20 years, and we give here part of results obtained.

i) No blow-up for small data

In [W1], it has been proved that for $u \in H^1$, we have the following

$$1/4\int|u|^4 \leq 1/2\int|\nabla u|^2 \{\int|u|^2 /\int Q^2 \}.$$

It follows from this identity that if

$$|u_0|_{L^2} < |Q|_{L^2}$$

then there is non blow-up phenomon and the solution is globally defined in time.

ii)blow-up for large data

For this equation there are two way to obtain blow-up solutions.

- explicit blow-up solution.

From the conformal invariance of the equation if w(x) is a real solution of the equation (III), then

$$1/t\, w(x/t) \exp(-i/t + i|x|^2/4t)$$

is also a solution of equation (II) which blow-up at T = 0.

In particular $S(t,x) = 1/t\, Q(x/t) \exp(-i/t + i|x|^2/4t)$

is a blow-up solution such that $|u_0|_{L^2} = |Q|_{L^2}$.

-Viriel identity.

From [SoSyZ], [Gla], we have the following property of the solution of equation (II).

Assume that $|x|u_0 \in L^2$ then for all time t, $|x|u_0 \in L^2$ and

$$d^2/dt^2 \{ \int |x|^2 |u(t,x)|^2\, dx \} = 16\, E(u_0).$$

From this viriel identity, we have that

if $E(u_0) < 0$ then the solution blow-up in finite time ($T < +\infty$).

iii)Minimal blow-up solutions

Since if $|u_0|_{L^2} < |Q|_{L^2}$ then there is non blow-up, and there is blow-up solution in the case where $|u_0|_{L^2} = |Q|_{L^2}$, one can ask is it possible to carracterize all minimal blow-up solutions in L^2 (that is solution which blows-up and such that $|u_0|_{L^2} = |Q|_{L^2}$).

In [M1] (see also [M4] for a another approach of the proof), the following is proved.

Assume that u(t) is a blow-up solution with minimal mass (and u(t) is an H^1 solution of equation (II)), that is $|u_0|_{L^2} = |Q|_{L^2}$. Then up to the invariance of the equation, we have

$$u(t,x) = S(t,x) = 1/t\, Q(x/t) \exp(-i/t + i|x|^2/4t)$$

(that is there exist x', x'',ω,θ such that $u(t,x) = e^{i\theta}\omega/t\, Q((x-x')\omega/t - x'') \exp(-i\omega^2/t + i|x-x'|^2/4t)$).

5) Equation (I)

Until recently, there were no results on existence of solutions which blow-up for Zakharov equations. Indeed the two ingredients; the conformal invariance and the viriel identity which give blow-up results for the limit equation as c_0 goes to infinity do not hold. We can note that there were numerical evidence of singular behavior of solution of equation (I) in [LPSSW] and [PSSW].

i) No blow-up for small data

One can show (see [AA1],[SS]) as for the Schrödinger equation, that if $|u_0|_{L^2} < |Q|_{L^2}$ then ther is non blow-up phenomon and the solution is globally defined in time.

ii)blow-up for large data

As for equation (II), we are able to construct explicit blow-up solution and give obstructions to regular behavior.

- explicit blow-up solutions.

We do not have anymore the conformal invariance to obtain explicit blow-up solutions. We use in fact a bifurcation argument at "infinity" (using the structure of the nonlinear Schrödinger equation) to obtain explicit blow-up solution.

In [GlM1], a family of blow-up solutions in the energy space of the form

$$u(t,x) = \omega/t\, P(\omega x/t) \exp(-\omega^2 i/t + i|x|^2/4t)$$

$$n(t,x) = \{\omega/t\}^2\, N(\omega x/t)$$

where $P(x) = P(|x|)$ and $N(x) = N(|x|)$
and

$$P = \Delta P + NP$$
$$(c_0\omega)^{-2}\{r^2N_{rr} + 6rN_r + 6N\} - \Delta N = \Delta P^2$$

is investigated.
More precisely, it is proved using this kind of construction, that there are blow-up solutions such that $|u_0|_{L^2} = |Q|_{L^2} + \varepsilon$, for *all* $\varepsilon>0$.

We can note that the solutions constructed are numerically stable (see [LPSSW]). The problem now is the following, we have construct blow-up solutions but we do not existence of many (or a large set) of singular solution. For this purpose, we use a different approach.

-viriel identity.

In [M2], it is derived a perturbed viriel identity for the Zakharov equation. More precisly, for a regular solution with decay at infinity we have

$$d^2t/dt^2 \{1/4 \int |x|^2|u(t,x)|^2 \, dx + c_0^{-2} \int_0^t \int (x.v(t,x))n(t,x) \, dx \, dt \} = 2H(u_0,n_0,v_0) - c_0^{-2} \int |v(t,x)|^2 \, dx.$$

From this pertubed viriel identity, we have in [M2] that

if $H(u_0,n_0,v_0) < 0$ and the initial data are radially symmetric then the solution blow-up in finite time ($T<+\infty$) or in infinite time in H_1 (with a concentration of $u(t)$ in L^2 as t goes to infinity).

We suspect that in the case where $H(u_0,n_0,v_0) < 0$ then the solution away blows-up in finite time. This result give in particular the existence of a large class of singular solutions.

iii)Minimal blow-up solutions

Since if $|u_0|_{L^2} < |Q|_{L^2}$ then there is non blow-up, and there are blow-up solution such that in the case where $|u_0|_{L^2} = |Q|_{L^2} + \varepsilon$, for all $\varepsilon>0$. one can ask, as for the nonlinear Schrödinger equation about minimal blow-up solutions (that is solution which blows-up and such that $|u_0|_{L^2} = |Q|_{L^2}$).

In [GlM2], we in fact proved that there is no minimal blow-up solution:

if $|u_0|_{L^2} = |Q|_{L^2}$ then there is non blow-up.

Therefore, the situation is different from the one of the nonlinear Schrödinger equation.

iv)Instability and stability results of blow-up behavior

Let us first recall some results for the nonlinear Schrödinger equation. We have explicit blow-up solution such that the blow-up rate in H^1 is of the type $1/(T-t)$. In particular, the one which the minimal blow-up solution has this rate of blow-up. From a physical point of view, we can expect that this rate is stable. It is not the case. Indeed,in [LPSS] for example it is observed numerically blow-up rate of the type $\mathrm{Log}|\,\mathrm{Log}|T-t|\,| \,/\, (T-t)^{1/2}$.

In [M3], it show for the Zakharov equation (with c_0 finite but eventually very large), that the blow-up rate is stonger than $1/(T-t)$. More precisly, let $(u,n,v)(t)$ a blow-up solution and T its blow-up time, we have for t near

$$|u(t)|_{H^1} \geq c/(T-t).$$

This shows that in fact the blow-up rate of the type $\mathrm{Log}|\,\mathrm{Log}|T-t|\,| \,/\, (T-t)^{1/2}$ is unstable with respect to perturbations of the equation (with a term involving a wave equation). In contrary, the one with blow-up rate $1/(T-t)$ seem numerically stable. This in particular shows the physical interest of the minimal blow-up solution of the nonlinear Schrödinger equation : the solution of the form

$$S(t,x) = 1/t \, Q(x/t) \exp(-i/t + i|x|^2/4t).$$

References:

[AA1] H. Added, S. Added, Existence globale de solutions fortes pour les équations de la turbulence de Langmuir en dimension 2, C.R. Acad. Sci. Paris 200, (1984) 551-554.

[AA2] H. Added, S. Added, Equations of Langmuir turbulence and nonlinear Schrödinger equations : Smoothness and approximation, J. Funct. Anal. 79, (1988) 183-210.

[BeL] H. Berestycki, P.L. Lions, Nonlinear scalar field equations, I Existence of ground state; II Existence of infinitely many solutions, Arch. Rational Mech. Anal. 82, (1983) 313-375.

[Bo1] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations, Part I Schrödinger equations, G.A.F.A. 3, (1993) 107-178.

[Bo2] J. Bourgain, On the Cauchy and invariant measure problem for the periodic Zakharov system, Duke Math. J. 76, (1994) 175-202.

[CaW] T. Cazenave, F. Weissler, Some remarks on the nonlinear Schrödinger equation in the critical case, in nonlinear semigroups, partial equations, and attractors, T.L. Gill and Zachary (eds.) Lect. Notes in Math. 347 Springer (1989) 18-29.

[GV] J. Ginibre, G. Velo, On a class of nonlinear Schrödinger equations I, II The Cauchy problem, general case, J. Funct. Anal. 32, (1979) 1-71.

[GlM1] L. Glangetas, F. Merle, Existence of self-similar blow-up solution for the Zakharov equation in dimension two, Commu. Math. Phys. 160, (1994) 173-215.

[GlM2] L. Glangetas, F. Merle, Concentration properties of blow-up solutions and instability results for the Zakharov equation in dimension two, Commu. Math. Phys. 160, (1994) 349-389.

[Gla] R.T. Glassey, On the blowing-up of solutions to the Cauchy problem for the nonlinear Schrödinger equation, J. Math. Phys. 18, (1977) 1794-1797.

[Ka] T. Kato, On nonlinear Schrödinger equations, Ann. Inst. Henri Poincaré, Physique Théorique 49, (1987) 113-129.

[KePVg] C. Kenig, G. Ponce, L. Vega, On the Zakharov and Zakharov-Schulman systems, J. Funct. Anal. 127, (1995) 204-234.

[Kw] M.A. Kwong, Uniqueness of positive solutions of $u = \Delta u + u^p$ in R^N, Arch. Rational Mech. Anal. 105, (1989) 243-266.

[LPSSW] M. Landman, G.C. Papanicolaou, C. Sulem, P.L. Sulem, X.P. Wang, Stability of isotropic self-similar dynamics for scalar collapse, Phys.Rev. A 46, (1992) 7869-7876.

[LPSS] M. Landman, G.C. Papanicolaou, C. Sulem, P.L. Sulem, Rate of the blow-up for solutions of the nonlinear Schrödinger equation in critical dimension, Phys. Rev. A 38, (1988) 3837-3843.

[M1] F. Merle, Determination of blow-up solutions with minimal mass for Schrödinger equation with critical power, Duke J. 69, (1993) 427-454.

[M2] F. Merle, Blow-up results of the viriel type for Zakharov equations, Commun. Math. Phys. (to appear).

[M3] F. Merle, Lower bounds for the blow-up rate of solutions of Zakharov equations in dimension two, preprint.

[M4] F. Merle, Asymptotics for L2 minimal blow-up solutions of critical nonlinear Schrödinger equation, Anal. I.H.P. Analyse non linéaire (to appear).

[MT] F. Merle, Y. Tsutsumi, L2 concentration of blow-up solutions for the nonlinear Schrödinger equation with the critical power nonlinearity, J. Diff. Equ. 84, (1990), 205-214.

[OT1] T. Ozawa, Y. Tsutsumi, The nonlinear Schrödinger limit and the initial layer of the Zakharov equations, preprint.

[OT2] T. Ozawa, Y. Tsutsumi, Existence of smoothing effect of solutions for the Zakharov equations, preprint.

[PSSW] G.C. Papanicolaou, C. Sulem, P.L. Sulem, X.P. Wang, Singular solutions of the Zakharov equations for the Langmuir turbulence, Phys. Fluids B3, (1991) 969-980.

[SoSyZ] V.V. Sobolev, V.S. Synach, V.E. Zakharov, Character of the singularity and stochastic phenomena in self-focussing, Zh. Eksp. Theor. Fiz., Pis'ma Red 14, (1974) 173-176.

[St] W.A. Strauss, Existence of solitary waves in higher dimensions, Commun. Math. Phys. 55, (1977) 149-162.

[SS] C. Sulem, P.L. Sulem, Quelques résultats de régularité pour les équations de la turbulence de Langmuir, C.R.Acad. Sci. Paris 289, (1979) 173-176.

[W1] M.I. Weinstein, Nonlinear Schrödinger equations and sharp interpolation estimates, Commun. Math. Phys. 87, (1983) 567-576.

[W2] M.I. Weinstein, On the stucture and formation of singularities in solutions to the nonlinear dispersive evolution equations, Commun. Partial Diff. Equ. 11, (1986) 545-565.