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Symmetry of the Ginzburg-Landau minimizer in a disc

SYMMETRY OF THE GINZBURG-LANDAU MINIMIZER IN A DISC

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ABSTRACT:
The Ginzburg-Landau energy minimization problem for a vector field on a two dimensional disc is analyzed. This is the simplest nontrivial example of a *vector field* minimization problem and the goal is to show that the energy minimizer has the full geometric symmetry of the problem. The standard methods that are useful for similar problems involving real valued *functions* cannot be applied to this situation. Our main result is that the minimizer in the class of symmetric fields is stable, i.e., the eigenvalues of the second variation operator are all nonnegative.

I. INTRODUCTION

There are many energy minimization problems having a geometric symmetry and for which one can show that the energy minimizer has the same symmetry as the problem itself. Typically this is done by using a rearrangement inequality of some sort. However, and this is the important point, rearrangement inequalities work (if they work at all) only when the variable is a function and not something more complicated like a vector field.

There are several important problems in which the variable is one or more vector or tensor fields and for which the minimizer is believed to be symmetric. Examples include the full multi-field Ginzburg-Landau problem for a superconductor in a magnetic field, the 't Hooft-Polyakov monopole and the Skyrme model (see [LE2] for a review). They are all unresolved. In this paper we analyze the simplest possible nontrivial example of a vector field energy minimization problem—the Ginzburg-Landau problem for a complex scalar

field in a disc. It has exercised many authors (see, e.g., [JT], [BBH] and references therein) but no one has been able to show that the obvious symmetric vector field minimizes the energy (except in the weak coupling regime where convexity holds). In fact, it has been shown only very recently, [MP], [LL], that the symmetric solution is stable under perturbations. In [LL] we used a mixture of rearrangement inequalities on different components of the vector field and, while our methods are highly specialized to this problem, we believe that it is one of the few examples in which light can be shed on the symmetry of an energy minimizing vector field.

For the Ginzburg-Landau problem in the unit disc $\mathbf{D} = \mathbf{B}^2$ in $\mathbf{R}^2$, the variable is a real vector field $\psi(x) = (f(x), g(x))$. It is customary to introduce the complex valued function $\phi(x) = f(x) + ig(x)$. The energy functional is

$$\mathcal{E}(\psi) = \int_{\mathbf{D}} \{(\nabla f(x))^2 + (\nabla g(x))^2 + J(f(x)^2 + g(x)^2)\} \mathrm{d}x = \int_{\mathbf{D}} \{|\nabla \phi|^2 + J(|\phi|^2)\}. \quad (1.1)$$

Usually, $J : \mathbf{R}^+ \to \mathbf{R}^+$ is taken to be the function $J(t) = \lambda(1 - t)^2$ with $\lambda > 0$. For our purposes we can generalize this to J satisfying certain conditions, which we assume henceforth:

 (i) $J(0) = \lambda > 0$, $J(1) = 0$, $J(t) \geq 0$ if $t > 1$,
 (ii) $J(t)$ is monotone decreasing and convex on the interval $[0, 1]$,
(iii) J is twice differentiable on $[0, 1]$.

The gradients of ψ are assumed to be square integrable and the condition on ψ on the boundary of $\mathbf{D}$ is

$$\psi(x) = x = (x_1, x_2) = (\cos\theta, \sin\theta). \quad (1.2)$$

We denote the class of $H^1(\mathbf{D})$ functions satisfying (1.2) by $\mathcal{C}$. *The problem is to minimize $\mathcal{E}(\psi)$ subject to $\psi \in \mathcal{C}$.*

For this problem it is a standard fact that a minimizer exists and satisfies the Euler-Lagrange pair of equations

$$-\Delta\psi + J'(\psi^2)\psi = 0 \quad (1.3)$$

with $\psi^2 = f^2 + g^2$. The obvious conjecture about a minimizer ψ is that it is a "hedgehog", i.e., for some nonnegative function f defined on $[0, 1]$ with $f(1) = 1$

$$\psi(x) = f(r)(\cos\theta, \sin\theta) \quad (1.4)$$

where $r := \sqrt{x_1^2 + x_2^2}$. There is always a function ψ_0 that minimizes the energy in the class of vector fields of type (1.4), and it satisfies (1.3). The problem is to show that this ψ_0 is a global minimizer. In terms of $f(r)$, (1.3) reads

$$-f'' - \frac{1}{r}f' + \frac{1}{r^2}f + J'(f^2)f = 0 \quad (1.5)$$

with $f(0) = 0$ and $f(1) = 1$. The solution to this problem is unique [HH]. It is not hard to see that f is monotone increasing, but this fact is not needed.

Although we cannot prove the full hedgehog conjecture, we are able to verify that the hedgehog is stable, that is to say that all the eigenvalues of the self-adjoint second variation operator H, defined by the quadratic form,

$$\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2}\mathcal{E}(\psi_0 + \varepsilon v)\Big|_{\varepsilon=0} = (v, Hv), \tag{1.6}$$

are nonnegative. Specifically H is given by

$$Hv = -\Delta v + J'(\psi_0^2)v + 2J''(\psi_0^2)\langle\psi_0, v\rangle\psi_0 \tag{1.7}$$

for vector fields v that vanish on $\partial\mathbf{D}$. Here $\langle a, b\rangle$ is the inner product on $\mathbf{R}^2$.

Independently, Mironescu [MP] proved that the eigenvalues of H are not only non-negative but strictly positive in fact. His beautiful analysis is completely different from ours and proceeds by an ingenious comparison of the Euler–Lagrange equation (1.5) with the second variation operator (1.7).

II. STATEMENTS OF THEOREMS AND LEMMAS

The following three theorems will be proved in the next section in the order 2, 3, 1. Theorem 1 is our main result. Theorem 3 will require three lemmas which we list here. Lemmas 1 and 2 on rearrangements are well known.

The proof of Theorem 2 uses some simple facts about convexity. This theorem holds for the analogous Ginzburg-Landau problem in the ball $\mathbf{B}^n \subset \mathbf{R}^n$ for any n, not just for $n = 2$. Theorem 1 is a Corollary of Theorems 2 and 3.

THEOREM 1 (Weak stability of the symmetric minimizer). *The eigenvalues of H in (1.8, 1.9) (with Dirichlet boundary conditions) are all nonnegative. The complex eigenfunctions of H can all be chosen to have the following form*

$$v(r, \theta) = e^{im\theta}\begin{pmatrix} a(r)e^{i\theta} + b(r)e^{-i\theta} \\ -ia(r)e^{i\theta} + ib(r)e^{-i\theta} \end{pmatrix}, \tag{2.1}$$

for suitable real functions $a = a_m$ and $b = b_m$ and with $m = 0, \pm1, \pm2, \ldots$. Clearly, $\bar{v}$, the complex conjugate, is also an eigenvector with the same eigenvalue as v. The lowest eigenvalue of H belongs either to $m = 0$ or to $m = \pm1$.

THEOREM 2 (Partial convexity of the energy functional $\mathcal{E}(\psi)$). *Suppose $\psi = (\psi_1, \ldots, \psi_n)$ is a real vector field in $H^1(\mathbf{B}^n)$ that satisfies $\psi(x) = x$ on the boundary of $\mathbf{B}^n$. Suppose that $|\psi(x)|^2 = \sum_{i=1}^{n}\psi_i(x)^2 \leq 1$ for all x and suppose that each component ψ_i satisfies*

$$\int_{\mathbf{S}^{n-1}} \psi_i(r\omega)\mathrm{d}\omega = 0 \tag{2.2}$$

for all $r \leq 1$. Define the vector field $\widetilde{\psi}(x)$ by

$$\widetilde{\psi}(r\omega) = h(r)\omega, \qquad \omega \in \mathbf{S}^{n-1} \tag{2.3}$$

where h is the spherical average of $|\psi|^2$, i.e.,

$$h(r) = \left[\frac{1}{|\mathbf{S}^{n-1}|} \int_{\mathbf{S}^{n-1}} \left(\sum_i \psi_i(r\omega)^2 \right) d\omega \right]^{1/2} \tag{2.4}$$

and $|\mathbf{S}^{n-1}| = \int_{\mathbf{S}^{n-1}} d\omega$. Then (with $\mathcal{E}(\psi)$ given by the obvious generalization to $\mathbf{B}_n$ of (1.3)),

$$\mathcal{E}(\widetilde{\psi}) \leq \mathcal{E}(\psi). \tag{2.5}$$

If we assume that $h(r) > 0$ for all $r > 0$ then equality occurs in (2.5) only if $\psi = \widetilde{\psi}$.

THEOREM 3 (Rearrangements of special vector fields). *Suppose that ψ is a vector field in $\mathcal{C}$ and suppose that there exists some fixed vector $\omega_0 \in \mathbf{S}^1$ and some function $z : [-1, 1] \to \mathbf{R}$ such that*

$$\psi(t\omega_0) = z(t)\omega_0 \tag{2.6}$$

for all $t \in [-1, 1]$. Then there is a vector field $\widehat{\psi} \in \mathcal{C}$ satisfying (2.6) and, additionally,

(i) $\widehat{\psi}(x) = -\widehat{\psi}(-x)$ for all $x \in \mathbf{D}$, $\qquad$ *(2.7)*

(ii) $\mathcal{E}(\widehat{\psi}) \leq \mathcal{E}(\psi)$ $\qquad$ *(2.8)*

(iii) $|\widehat{\psi}(x)|^2 \leq 1$ for all $x \in \mathbf{D}$. $\qquad$ *(2.9)*

Remark: The following might help to clarify the relation between Theorems 2 and 3. Write a minimizing $\psi \in \mathcal{C}$ in complex form as

$$\phi(r, \theta) = \sum_{k=-\infty}^{\infty} c_k(r)e^{ik\theta}, \tag{2.10}$$

with $c_k(1) = 0$ if $k \neq 1$ and $c_1(1) = 1$. If $c_0(r) \equiv 0$ then Theorem 2 applies and we learn that the hedgehog is the minimizer, i.e., $c_k(r) \equiv 0$ for $k \neq 1$.

Next suppose that we take a ϕ in the form (2.9) in which only at most two of the c_k's are not identically zero, say c_1 and c_m with $m \neq 1$. Then we claim that we can choose the two c's to be real functions without raising the energy. Having done this, Theorems 2 and 3 apply and we again learn that the energy minimizing choice in this restricted category has $c_m \equiv 0$ for $m \neq 1$. The proof of this assertion is the following. We write $c_j(r) = p_j(r) \exp[i\alpha_j(r)]$ with $p_j \geq 0$ and α_j real. Then

$$|\phi(r, \theta)|^2 = p_1(r)^2 + p_m(r)^2 + 2p_1(r)p_m(r)\cos[(m-1)\theta + \alpha_m(r) - \alpha_1(r)],$$

and we observe two things: If we replace α_1 and α_m by zero

then (i) the gradient term in $\mathcal{E}$ can only decrease; (ii) the J term does not change because by a trivial shift of θ, the θ integral does not depend on $\alpha_m(r) - \alpha_1(r)$. (The convexity of J plays no role here.)

The lemmas about symmetric decreasing rearrangements that we shall need are the following. The first was basically proved by Chiti [CG] and then by Crandall-Tartar [CT]. For some generalizations see [AL], 2.2 and 2.3.

Lemma 1. *Let f and g be nonnegative functions on $\mathbf{R}^n$ and let $J : \mathbf{R} \to \mathbf{R}^+$ be a convex function with $J(0) = 0$. Then*

$$\int_{\mathbf{R}^n} J(f^*(x) - g^*(x))\mathrm{d}x \le \int_{\mathbf{R}^n} J(f(x) - g(x))\mathrm{d}x \tag{2.11}$$

where f^* and g^* are the symmetric decreasing rearrangements of f and g.

Lemma 2 (Rearrangements and gradient norms). *For $u \in H_0^1[-a, a])$ define $u^* = |u|^*$. Then $u^* \in H_0^1([-a, a])$ and*

$$\int \left(\frac{\mathrm{d}u^*}{\mathrm{d}x}\right)^2 \le \int \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 . \tag{2.12}$$

Lemma 3 (Cutting argument). *Let $\psi = (f, g) \in \mathcal{C}$ and assume in addition that $g(x_1, 0) = 0$ for $x_1 \in [-1, 1]$. Then there exists $\widetilde{\psi} = (\widetilde{f}, \widetilde{g}) \in \mathcal{C}$ such that for all $x = (x_1, x_2)$ in $\mathbf{D}$*

(i) $\widetilde{g}(x_1, x_2) \ge x_2$ for $x_2 \ge 0$ and $\widetilde{g}(x_1, x_2) \le x_2$ for $x_2 \le 0$,
(ii) $|\widetilde{\psi}(x)| \le 1$ for all $x \in \mathbf{D}$ and hence $\widetilde{f}(x_1, x_2)^2 \le 1 - x_2^2$,
(iii) $\mathcal{E}(\widetilde{\psi}) \le \mathcal{E}(\psi)$.

III. PROOFS

3A. Proof of Theorem 2: Since $\sum\limits_{i=1}^{n} \psi_1(x)^2 \le 1$, and since $t \mapsto J(t)$ is convex we have, by Jensen's inequality, that

$$\frac{1}{|\mathbf{S}^{n-1}|} \int_{\mathbf{S}^{n-1}} J(|\psi(r\omega)|^2)\mathrm{d}\omega \ge J(h(r)^2) \tag{3A.1}$$

and hence

$$\int_{\mathbf{B}_n} J(|\psi(x)|^2)\mathrm{d}x \ge \int_{\mathbf{B}_n} J(h(r)^2)\mathrm{d}x = \int_{\mathbf{B}_n} J(|\widetilde{\psi}(x)|^2)\mathrm{d}x. \tag{3A.2}$$

To estimate the kinetic energy we expand each component, ψ_j, into normalized spherical harmonics, Y_{lm}, with coefficients $c_j^{lm}(r)$.

$$\psi_j(r\omega) = \sum_{l=1}^{\infty} \sum_{m} c_j^{lm}(r) Y_{lm}(\omega). \tag{3A.3}$$

Here l denotes the irreducible representation of $SO(n)$, while m is a multi-index that labels the rows. The reason $l = 0$ is absent is that $\int_{\mathbf{S}^{n-1}} \psi_j(r\omega) d\omega = 0$ for every $0 < r \leq 1$. Note that $h(r)^2 = \sum\limits_{j=1}^{n} \sum\limits_{l=1}^{\infty} \sum\limits_{m} (c_j^{lm}(r))^2 / |\mathbf{S}^{n-1}|$. It is well known that

$$\int_{\mathbf{B}_n} |\nabla \psi_j|^2 = \sum_{l=1}^{\infty} \sum_m \int_0^1 \left\{ (dc_j^{lm}/dr)^2 + \frac{l(l+n-2)}{r^2} (c_j^{lm})^2 \right\} r^{n-1} dr. \qquad (3A.4)$$

Since $h\frac{dh}{dr} = \sum\limits_{j=1}^{n} \sum\limits_{l=1}^{\infty} \sum\limits_{m} \left(c_j^{lm} \frac{dc_j^{lm}}{dr} \right) \Big/ |\mathbf{S}^{n-1}|$ we get, by Schwarz's inequality, that

$$\left| \frac{dh}{dr} \right|^2 \leq \sum_{j=1}^{n} \sum_{l=1}^{\infty} \sum_m \left(\frac{dc_j^{lm}}{dr} \right)^2 \Big/ |\mathbf{S}^{n-1}|. \qquad (3A.5)$$

Obviously,

$$\sum_{l=1}^{\infty} \sum_m \int_0^1 l(l+n-2)(c_j^{lm})^2 r^{n-3} dr \geq \sum_{l=1}^{\infty} \sum_m \int_0^1 (n-1)(c_j^{lm})^2 r^{n-3} dr \qquad (3A.6)$$

with equality only if $c_j^{lm} \equiv 0$ for all $l \geq 2$. In that case we can write

$$\psi_j(x) = \sqrt{n} \sum_{k=1}^{n} d_j^k(r) \frac{x_k}{r} \qquad (3A.7)$$

and $h(r)^2 = \sum_{j,k=1}^{n} d_j^k(r)^2$. In general, by summing over j, we find that

$$\int_{\mathbf{B}_n} |\nabla \psi|^2 \geq |\mathbf{S}^{n-1}| \int_0^1 \{ (dh(r)/dr)^2 + \frac{n-1}{r^2} h(r)^2 \} r^{n-1} dr = \int_{\mathbf{B}_n} |\nabla \widetilde{\psi}|^2, \qquad (3A.8)$$

with equality only if (3A.7) is satisfied.

In short, (2.5) has been proved and we know that equality requires (3A.7). Our final task is to show that equality in (2.5) also requires $d_j^k(r) = h(r)\delta_{k,j}/\sqrt{n}$ when $h(r) > 0$ for all $r > 0$.

Inequality (3A.5) was obtained by using Schwarz's inequality. In order to have equality we must have that

$$\frac{d}{dr} d_j^k(r) = \lambda(r) d_j^k(r) \qquad (3A.9)$$

for some function $\lambda(r)$ not depending on j and k. By multiplying (3A.9) on both sides by $d_j^k(r)$ and summing over k and j we have $h(r)h'(r) = \lambda(r)h(r)^2$. Since $h(r) > 0$ for all $r > 0$ we have that

$$\lambda(r) = h'(r)/h(r). \qquad (3A.10)$$

This function is integrable away from the origin and hence (3A.9) yields

$$d_j^k(r) = \mu(r)d_j^k(1). \qquad (3A.11)$$

with $\mu(r) = \exp\left\{-\int_r^1 \lambda(s)\mathrm{d}s\right\}$. By assumption, however, $d_j^k(1) = n^{-1/2}\delta_{j,k}$, and this yields the desired conclusion with $\mu(r) = h(r)$. ∎

3B. Proof of Theorem 3: Without loss of generality, we can assume $\omega_0 = (1,0)$. Our hypothesis is that if $\psi(x) = (f(x), g(x))$ then $g(x_1, 0) = 0$ for $-1 \le x_1 \le 1$. By Lemma 3 (cutting argument) we can also assume two important facts about our f and g: (i). $f(x_1, x_2)^2 \le 1 - x_2^2$ and (ii). $g(x_1, x_2) \ge x_2$ if $x_2 > 0$ and $g(x_1, x_2) \le x_2$ if $x_2 \le 0$. We can also assume that $f^2 + g^2 \le 1$.

The first step is to define $\widetilde{g}$ in the following way. For each, fixed x_2 we replace the function $x_1 \mapsto g(x_1, x_2)$ by its (one-dimensional) symmetric decreasing rearrangement $g^*(x_1, x_2)$ if $x_2 \ge 0$ and we replace it by $-|g(x_1, x_2)|^*$ if $x_2 \le 0$. By (ii) above, $\widetilde{g}$ satisfies the correct boundary condition, (i.e., $g(x_1, x_2) = x_2$ on $\partial\mathbf{D}$), and $\widetilde{g}$ also satisfies (ii) above.

The next step, the definition of $\widetilde{f}(x_1, x_2)$, is a bit more complicated. First, let $f^\#$ be the symmetric *increasing* rearrangement of $|f|$. [Another way to say this is that $1 - f^\# = (1 - |f|)^*$]. Once again, the rearrangement is done on each line $x_2 = $ constant. We note that $f(x_1, x_2)$ is continuous in x_1 for a.e. x_2 (because it is an $H^1(\mathbf{R})$ function for a.e. x_2) and has antisymmetric boundary values at $x_1 = \pm(1 - x_2^2)^{1/2}$. Therefore $f^\#$ is a continuous function of x_1 (indeed, it is an $H^1(\mathbf{R})$ function by Lemma 2) and $f^\#(0, x_2) = 0$. By (i) above, $f^\# \le (1 - x_2^2)^{1/2}$. Moreover, $f^\# = (1 - x_2^2)^{1/2}$ on $\partial\mathbf{D}$ since $x_1 \mapsto |f(x_1, x_2)|$ is continuous and $|f|$ satisfies the same boundary condition. Now define

$$\widetilde{f}(x_1, x_2) = \begin{cases} f^\#(x_1, x_2) & \text{if } x_1 > 0 \\ -f^\#(x_1, x_2) & \text{if } x_1 < 0 \end{cases} \qquad (3B.1)$$

which satisfies the correct conditions on $\partial\mathbf{D}$. We also note that $|\partial\widetilde{f}/\partial x_1| = |\partial f^\#/\partial x_1|$ and $|\partial\widetilde{f}/\partial x_2| = |\partial f^\#/\partial x_2|$.

Our task is to show that these rearrangements decrease both terms in the functional $\mathcal{E}$. We turn to the gradient norms first. By Lemma 2 we have that $\int_{\mathbf{D}}(\partial\widetilde{g}/\partial x_1)^2$ does not increase and the same is true for $\int_{\mathbf{D}}(\partial\widetilde{f}/\partial x_1)^2$. We next show that $\int_{\mathbf{D}}(\partial\widetilde{g}/\partial x_2)^2$ does not increase either. (The argument for $\widetilde{f}$ is essentially identical.) There are several ways to prove this, and one way is the following. An easy approximation argument shows that

$$\int_{\mathbf{D}}(\partial g/\partial x_2)^2 = \lim_{\delta \to 0} \delta^{-2}\int_{\mathbf{D}}[g(x_1, x_2 + \delta) - g(x_1, x_2)]^2 \mathrm{d}x_1\mathrm{d}x_2 \qquad (3B.2)$$

(Here, $g(x_1, x_2)$ has to be extended to be x_2 outside $\mathbf{D}$.) The result we want—that replacement of g by $\widetilde{g}$ does not increase the two sides of (3B.2)—follows from a trivial modification of Lemma 1.

To summarize, the vector field $\widetilde{\psi}$ is in $\mathcal{C}$ and its gradient norms are not bigger than those of ψ.

The next step is to prove that $|\widetilde{\psi}(x)|^2 \leq 1$ for all $x \in \mathbf{D}$. Let $K(t) = [\max(0, t)]^2$. Then

$$\int_{\mathbf{D}} K(g^2 - (1 - f^2)) = 0 \qquad (3B.3)$$

since $|\psi|^2 = f^2 + g^2 \leq 1$. By Lemma 1, however,

$$\int_{\mathbf{D}} K(g^2 - (1 - f^2)) \geq \int_{\mathbf{D}} K(\widetilde{g}^2 - (1 - \widetilde{f}^2)) \qquad (3B.4)$$

since $(g^2)^* = \widetilde{g}^2$ for each line, $x_2 = $ constant, and, similarly, $(1 - f^2)^* = (1 - \widetilde{f}^2)$ since $f^2 \leq 1$ and $\widetilde{f}^2 \leq 1$. If $\widetilde{g}^2 + \widetilde{f}^2 > 1$ on a set of positive measure the right side of (3B.4) would be positive, but this is precluded by (3B.3).

We now consider the J term in $\mathcal{E}$. We can define $L(t) = J(1 - t)$ for $0 \leq t \leq 1$. (The definition of $L(t)$ for $t < 0$ or $t > 1$ is not needed since $0 \leq t \leq 1$ in our application.) Then, by Lemma 1 and the same reasoning as for K above,

$$\int_{\mathbf{D}} L((1 - f^2) - g^2) \geq \int_{\mathbf{D}} L((1 - \widetilde{f}^2) - \widetilde{g}^2),$$

which is the same as $\int J(|\psi|^2) \geq \int J(|\widetilde{\psi}|^2)$.

Thus far we have constructed a $\widetilde{\psi}$ with $\mathcal{E}(\widetilde{\psi}) \leq \mathcal{E}(\psi)$, with $|\widetilde{\psi}(x)| \leq 1$ and with $\widetilde{f}(x_1, x_2) = -\widetilde{f}(-x_1, x_2)$ and $\widetilde{g}(x_1, x_2) = \widetilde{g}(-x_1, x_2)$. The final step is to use this $\widetilde{\psi}$ to construct a $\widehat{\psi}$ satisfying $\widehat{\psi}(x) = -\widehat{\psi}(-x)$ and $\mathcal{E}(\widehat{\psi}) \leq \mathcal{E}(\widetilde{\psi})$. Let $\mathbf{D}_+$ denote the upper hemidisc $\{(x_1, x_2) : x_2 \geq 0\} \cap \mathbf{D}$ and $\mathbf{D}_-$ the lower hemidisc. Let $(\widetilde{f}_{\pm}, \widetilde{g}_{\pm})$ denote $\widetilde{\psi}$ restricted to $\mathbf{D}_+$ and $\mathbf{D}_-$. Consider the following two vector fields.

$$\psi_1 = \begin{cases} (\widetilde{f}_+(x_1, x_2), \widetilde{g}_+(x_1, x_2)) & \text{in } \mathbf{D}_+ \\ (\widetilde{f}_+(x_1, -x_2), -\widetilde{g}_+(x_1, -x_2)) & \text{in } \mathbf{D}_- \end{cases}$$

$$\psi_2 = \begin{cases} (\widetilde{f}_-(x_1, -x_2), -\widetilde{g}_-(x_1, -x_2)) & \text{in } \mathbf{D}_+ \\ (\widetilde{f}_-(x_1, x_2), \widetilde{g}_-(x_1, x_2)) & \text{in } \mathbf{D}_- \end{cases}$$

Clearly $\psi_{1,2}(x) = -\psi_{1,2}(-x)$ and $|\psi_{1,2}(x)| \leq 1$. Also, ψ_1 and ψ_2 are in $\mathcal{C}$ because $\widetilde{g}(x_1, 0) = 0$. Moreover,

$$\mathcal{E}(\psi_1) + \mathcal{E}(\psi_2) = 2\mathcal{E}(\widetilde{\psi}).$$

Therefore, ψ_1 or ψ_2 is a vector field $\widehat{\psi}$ satisfying the conclusion of Theorem 3. ∎

3C. Proof of Theorem 1: The basic fact, which we shall prove later, is that the real eigenfunction of H can be chosen to have at least one of the following symmetry properties for all $x \in D$.

$$(a) \quad v(x) = -v(-x)$$
$$(b) \quad v(x) = Pv(P^{-1}x) \qquad (3C.1)$$

where $P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the reflection about the x_1-axis in $\mathbf{R}^2$. This is not to say that every eigenfunction has one of these properties, but we do assert that each *eigenvalue* of H has at least one eigenfunction of type (a) or (b). Since we are interested only in the eigenvalues of H, we may assume (a) or (b).

Now consider $\psi_\varepsilon := \psi_0 + \varepsilon v \in \mathcal{C}$, with v as in (3C.1). In case (a), $\psi_\varepsilon(x) = -\psi_\varepsilon(-x)$ for all ε and thus the hypothesis (2.2) of Theorem 2 is satisfied. In case the other condition, $|\psi_\varepsilon(x)| \leq 1$ is not satisfied we can replace $\psi_\varepsilon(x)$ by $\widehat{\psi}_\varepsilon(x) = \psi_\varepsilon(x)/|\psi_\varepsilon(x)|$ on the set where $|\psi_\varepsilon(x)| > 1$. Clearly (2.2) still holds and it is easy to check that $\mathcal{E}(\widehat{\psi}_\varepsilon) \leq \mathcal{E}(\psi_\varepsilon)$. By Theorem 2, there is a $\widetilde{\psi}_\varepsilon$ with $\mathcal{E}(\widetilde{\psi}_\varepsilon) \leq \mathcal{E}(\widehat{\psi}_\varepsilon)$ and $\widetilde{\psi}_\varepsilon$ is a hedgehog (1.4). Thus,

$$\mathcal{E}(\widehat{\psi}_\varepsilon) \geq \mathcal{E}(\widetilde{\psi}_\varepsilon) \geq \mathcal{E}(\psi_0) \qquad (3C.2)$$

since ψ_0 is the energy minimizer among all hedgehogs.

In case (b), ψ_ε satisfies hypothesis (2.6) of Theorem 3, with $\omega_0 = (1,0)$. We first have to use Theorem 3 to obtain an intermediate $\widehat{\psi}$ that satisfies the hypothesis of Theorem 2. Again, (3C.2) holds. Since

$$\mathcal{E}(\psi_\varepsilon) = \mathcal{E}(\psi_0) + \varepsilon^2 \gamma \int_{\mathbf{D}} v^2 + o(\varepsilon^2),$$

where γ is the eigenvalue of H belonging to v, we see from (3C.2) that $\gamma \geq 0$.

There are two ways to derive the symmetry (3C.1) and we shall give both. The first is a fairly general argument and the second involves a detailed study of the eigenfunctions leading to (2.1).

General argument: Let P denote reflection about some axis through the origin. For any eigenfunction, w, its reflection, $(P^* w)(x) := P w(P^{-1} x)$, is also an eigenfunction with the same eigenvalue. If $v(x) = w(x) + (P^* w)(x)$ is not identically zero for *some* P then v is an eigenfunction satisfying (3C.1) (b). If v vanishes identically for *all* reflections P we claim that w must be of type (3C.1)(a). To see this recall that any rotation $\mathcal{R}$ is the product of two reflections and hence $\mathcal{R} w(\mathcal{R}^{-1} x) = w(x)$, i.e., w is rotationally symmetric. It is easy to see that w must then satisfy $w(x) = k(r)(-x_2, x_1)$ for some function k, and hence w satisfies (3C.1)(a).

Details of eigenfunctions: Let $\mathcal{R}_\alpha$ be the rotation through the angle α and let U_α be its representation given by

$$(U_\alpha v)(x) = \mathcal{R}_\alpha v(\mathcal{R}_{-\alpha} x). \qquad (3C.3)$$

U_α is a strongly continuous one-parameter subgroup of the unitary group of $L^2(\mathbf{D}; \mathbf{C}^2)$ and it commutes with H. Its infinitesimal generator is

$$L = i \frac{\partial}{\partial \theta} + i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \qquad (3C.4)$$

By standard arguments we can choose the eigenfunctions of H to be eigenfunctions of L. By solving $Lv = \nu v$ we find that $v(\theta)$ must be of the form (2.1). ν must be an integer since $v(0) = v(2\pi)$. Furthermore, a glance at the eigenvalue equation reveals that a and b can be taken to be real.

Next we verify that the lowest eigenvalue belongs to $m = 0$ or $m = \pm 1$. Suppose, on the contrary, that the lowest one belongs to $M > 1$ (m and $-m$ are the same by complex conjugation). Then define a comparison function by $\widehat{v} := e^{-i\theta}v$. Obviously the J-term is unchanged. The only term that changes in the gradient norm is the replacement of $I := \int_{\mathbf{D}} \left|\frac{\partial}{\partial\theta}v\right|^2 r^{-2}$ by $\widehat{I} := \int_{\mathbf{D}} \left|\frac{\partial}{\partial\theta}e^{-i\theta}v\right|^2 r^{-2}$. One easily computes that

$$I = 2 \int_{\mathbf{D}} \{(M+1)^2 a(r)^2 + (M-1)^2 b(r)^2\} r^{-2}$$

and hence $\widehat{I} < I$ if $M > 1$.

Now consider (2.1) with $m = 0$. This vector v satisfies $v(x) = -v(-x)$ (because changing x to $-x$ amounts to changing θ to $\theta + \pi$). Thus, all $m = 0$ eigenfunctions satisfy (3C.1)(a). Indeed all even-m eigenfunctions have this property.

When $m = 1$ we claim that (3C.1)(b) holds—thus completing our proof. We take the real part of (2.1), which is $v(r, \theta) = (a(r)\cos 2\theta + b(r), a(r)\sin 2\theta)$; this satisfies (3C.1)(b) with P being reflection about the x_1-axis. ∎

3D. Proof of Lemma 3: We shall construct $\widetilde{\psi}$ by a sequence of steps.

Step 1.
$$\psi \mapsto T_1\psi = \begin{cases} \psi & \text{if } |\psi| \leq 1 \\ \psi/|\psi| & \text{if } |\psi| \geq 1 \end{cases}$$

[with $|\psi| = (f^2 + g^2)^{1/2}$]. An easy exercise shows that $\|\nabla T_1\psi\|_2 \leq \|\nabla\psi\|_2$. Furthermore, $\int J(|\psi|^2) \geq \int J(|T_1\psi|^2)$ because $J(t) \geq 0$ when $t > 1$ while $J(1) = 0$. Therefore, without loss of generality we can assume that $|\psi(x)| \leq 1$ for all x.

Step 2. $\qquad \psi \to T_2\psi = (f, h) \quad$ with $h(x_1, x_2) = \begin{cases} \max\{x_2, g(x_1, x_2)\} & \text{if } x_2 \geq 0 \\ \min\{x_2, g(x_1, x_2)\} & \text{if } x_2 \leq 0. \end{cases}$

Obviously $|T_2\psi(x)| \geq |\psi(x)|$ for all x.

The condition $g(x_1, 0) = 0$ guarantees that $T_2\psi \in \mathcal{C}$.

Step 3. $\qquad\qquad\qquad\qquad \psi \mapsto T_3\psi = T_1 T_2\psi.$

If we write $T_1 T_2\psi = (\widehat{f}, \widehat{g}) = \widehat{\psi}$ we can easily verify the following for all x (using $|\psi| \leq 1$)

(a) $\qquad\qquad\qquad\qquad 1 \geq |T_3\psi(x)| \geq |\psi(x)|$

(b) $\qquad\qquad\qquad |\widehat{f}(x)| \leq |f(x)| \quad \text{and} \quad \text{sgn}\widehat{f}(x) = \text{sgn}f(x)$

(c) $\qquad\qquad\qquad |\widehat{g}(x)| \geq |g(x)| \quad \text{and} \quad \text{sgn}\widehat{g}(x) = \text{sgn}x_2$

(d) $\qquad\qquad \widehat{g}(x)^2 - g(x)^2 \geq \dfrac{1 - x_2^2}{1 + x_2^2}[x_2^2 - g(x)^2]_+ .$

(a) is obvious because T_2 does not decrease $|\psi|$ and T_1 only cuts off $|T_2\psi|$ at 1; but $|\psi| \leq 1$ everywhere. (b) is also obvious because T_2 leaves f invariant and T_1 can only decrease $|f|$. (c) follows from the facts that T_2 increases $|g|$, the map $t \mapsto t/(f^2 + t)$ is monotone increasing for $t > 0$, and $g^2 \leq g^2/(f^2 + g^2)$ since $f^2 + g^2 \leq 1$. Indeed, (d) gives a more quantitative estimate. To prove (d) we recall that $T_2\psi =: (f, h)$. If $f^2 + h^2 \leq 1$ and $x_2^2 \geq g^2$ then $|\widehat{g}| = |h| = |x_2|$ and (d) is certainly true. If $f^2 + h^2 > 1$ and $x_2^2 \geq g^2$ then

$$\widehat{g}^2 - g^2 = \frac{x_2^2}{f^2 + x_2^2} - g^2 \geq \frac{x_2^2}{1 - g^2 + x_2^2} - g^2$$
$$= \frac{1 - g^2}{1 - g^2 + x_2^2}[x_2^2 - g^2] \geq \frac{1 - x_2^2}{1 + x_2^2}[x_2^2 - g^2].$$

We claim that $\mathcal{E}(T_3\psi) \leq \mathcal{E}(\psi)$. As far as the gradient term is concerned T_2 replaces g by the harmonic function x_2 on the set where $|g| \leq |x_2|$. This certainly lowers the gradient term. The J term does not increase by property (a) above, since $J(t)$ is decreasing for $0 \leq t \leq 1$.

Now we iterate T_3 and denote $(f_n, g_n) = \psi_n := T_3^n\psi$. By (b) and (c) f_n and g_n are bounded monotone sequences and converge pointwise to limit functions $\widetilde{f}$ and $\widetilde{g}$. Since $\mathcal{E}(\psi)$ is weakly lower semicontinuous we have that $\mathcal{E}(\widetilde{\psi}) \leq \mathcal{E}(\psi)$, where $\widetilde{\psi} = (\widetilde{f}, \widetilde{g})$. It is clear that $\widetilde{\psi}$ satisfies the correct boundary conditions and hence is in $\mathcal{C}$.

The only thing left to check is that $\widetilde{g}(x)^2 \geq x_2^2$. If we define $a_n(x) = [x_2^2 - g_n(x)^2]_+$ property (d) can be rewritten as $a_{n+1}(x) \leq a_n(x)(2x_2^2/(1 + x_2^2))$, which shows that $a_n(x)$ converges to zero pointwise for all $x \in \mathbf{D}$. ∎

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