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Joel Smoller
Blake Temple

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SHOCK-WAVE EXPLOSIONS IN GENERAL RELATIVITY

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Joel Smoller[1] *Blake Temple*[2]

Abstract

In earlier work the authors constructed a class of spherically symmetric, fluid dynamical shock-waves that solve the Einstein equations of general relativity. These shock-waves extend the celebrated Oppenheimer-Snyder result to the case of non-zero pressure. In general our shock-waves are determined by a system of ordinary differential equations (ODE's) that describe the matching of a Friedmann-Robertson-Walker metric, (a cosmological model for the expanding universe), to an Oppenheimer-Tolman metric, (a model for the interior of a star), across a shock interface. A global exact solution of these ODE's was found for isothermal equations of state, and in this exact solution, the *Big Bang* begins with a shock-wave explosion instead of the usual singularity of cosmology. In this talk we discuss new work of the authors in which we derive an alternate version of the general ODE's, and we use these to demonstrate that our theory generates a large class of physically meaningful outgoing (Lax admissible) shock-waves that model blast waves in a general relativistic setting. We also obtain formulas for the physical quantities that evolve according to the equations. The resulting formulas are important for the numerical simulation of these solutions.

1 Introduction

In [7] we constructed a class of physically interesting shock-wave solutions of the Einstein equations of General Relativity, and in [8] we applied these

[1]Department of Mathematics, University of Michigan, Ann Arbor, MI 48109; Supported in part by NSF Applied Mathematics Grant Number DMS-92-06631, in part by ONR, US NAVY grant number N00014-94-1-0691 and by the Institute of Theoretical Dynamics, UC-Davis.

[2]Department of Mathematics, University of California, Davis, Davis CA 95616; Supported in part by NSF Applied Mathematics Grant Number DMS-92-06631, in part by ONR, US NAVY grant number N00014-94-1-0691, a Guggenheim Fellowship, and by the Institute of Theoretical Dynamics, UC-Davis.

results to construct an exact solution of these equations that models an explosion into a static, singular, isothermal sphere. This solution provides a general relativistic model that parallels a Newtonian model for stellar formation, and in this model the expanding universe begins with a shock-wave explosion into a pre-existing universe of static material, instead of beginning with the usual *Big Bang* singularity in which the entire universe "bursts from a single point." One remarkable feature of this exact solution is that the Einstein geometries that we are presupposing for the solution constrains the possible equations of state to satisfy the physically required conditions that the temperature, pressure and sound speed be larger when the density is larger. Said differently, the *geometry* is forcing the *fluids* to behave correctly! We take this as indicating that our setting is an extremely natural one physically. In this report we review some of our previous results for motivation, and describe our results in [9], (work in progress), in which we set out a general theory of our shock-waves in the case of a model fluid in which the pressure is assumed to be a function of the density alone. In this we identify several new dimensionless parameters that arise naturally.

The paper [7] concluded with the derivation of a set of (rather complicated) ordinary differential equations (ODE's) that describe the matching of a Friedmann-Robertson-Walker (FRW) type metric, to an Oppenheimer-Tolman (OT) type metric, such that the interface between the two metrics defines a spherically symmetric, fluid dynamical shock-wave. In our work in progress, we derive an alternate, simpler version of these ODE's, and we use these ODE's to compute simplified formulas for the physical quantities, (the pressure, the density, and the sound, shock and characteristic speeds), that are determined by the equations. We make the assumption throughout that the equations of state are of the simple form $p = p(\rho)$, so that the sound speed is given by $\frac{dp}{d\rho}$, and we consider only shock-waves that lie outside of the Schwarzschild radius, ($A > 0$; c.f., (2.13) below). We show that in this setting, the general theory can be worked out almost completely. In particular, we obtain conditions under which the Lax shock conditions hold, conditions under which the pressure is greater behind the shock, and conditions under which all "speeds" are bounded by the speed of light, $c = 1$; c.f., [4]. These are not mute points because in the example of [8], the shock becomes supersonic relative to the fluid on both sides of the shock when the FRW sound speed exceeds (approximately) .68 the speed of light; and the shock speed exceeds the speed of light when the inner FRW sound speed exceeds (approximately) .86 the speed of light. We also demonstrate that the equation of state is unphysical for incoming shocks (implosions), so the theory at the level of a barotropic equation of state provides a realistic model only for out-going explosions. The ODE's and formulas we derive are applicable to the numerical simulation of these shock-wave solutions.

The FRW metric is a uniformly expanding (or contracting) solution of the Einstein gravitational field equations that is generally accepted as a cosmological model for the universe. The OT solution is a time-independent solution which models the interior of a star. Both metrics are spherically symmetric, and both are determined by a system of ODE's that close when an equation of state $p = p(\rho)$ for the fluid is specified. In our shock-wave solution, the FRW metric represents an exploding *inner* core, (of a star or the universe as a whole), and the boundary of this inner core is a shock surface that is driven by the expansion behind the shock into the *outer*, static, OT solution, which represents the outer layers of a star, or the outer regions of the universe.

In [7], we described a general procedure for matching different metric solutions of the Einstein equations across an interface such that the metrics match Lipschitz continuously at the interface. In order for the interface to be a true fluid dynamical shock-wave, (as opposed to a surface layer), we must impose an additional constraint, called the *conservation constraint*, on the equations. In [7], we showed that for the matching of a FRW metric to an OT metric, the Lipschitz continuous matching at an interface can be achieved with any two arbitrary equations of state assigned to the FRW and OT solutions separately. However, in order to satisfy the conservation constraint, we must impose *one* additional constraint, and thus we lose the freedom to impose one of the two equations of state. Our approach is to choose the outer OT equation of state arbitrarily, and let the FRW equation of state be determined by the equations. Our conclusion is that, for any given fixed OT metric, our ODE's, (c.f. (3.9)-(3.10) below), describe the evolution of the shock position, together with the density, pressure and cosmological scale factor of the FRW solution. Any FRW metric that solves the ODE's, will match the given OT metric Lipschitz continuously across a true fluid dynamical shock-wave. (This matching can be improved to a $C^{1,1}$ matching of the metrics via a $C^{1,1}$ coordinate transformation; c.f. Theorem 2 below.)

It is well known that the FRW metric exhibits qualitatively different behavior depending on the sign of k, (a parameter in the metric, [12]), which determines the sign of the scalar curvature on the constant curvature surfaces at each fixed time. When $k > 0$, the shock-wave solutions described in [7] reduce to the well known model of Oppenheimer and Snyder (OS) when the OT solution is taken to be the empty space Schwarzschild metric. In this case the general ODE's derived in [7] reproduce the OS equation of state $p \equiv 0$ in the FRW metric, and thus our ODE's reproduce the OS results in this limit. Thus our shock- wave solutions provide a natural generalization of the OS model to the case of non-zero pressure. However, there is an important difference between the OS solution and our shock-wave solutions; namely, the OS interface is a time-reversible *contact discontinuity*, but the interfaces in our model describe true, *time-irreversible, fluid dynamical, shock-waves.* For a contact discontinuity, a smooth regularization of the solution at a fixed time will propogate as a nearby smooth solution for all times thereafter. In contrast, it is well known from the theory of hyperbolic conservation laws that, due to time-irreversibility, shock-wave solutions cannot be approximated globally by smooth, shock-free solutions of the hyperbolic equations, [4, 5].

2 Preliminaries

In this section we review the results in [7] and [8]. We consider the Einstein gravitational field equations

$$G = \kappa T, \tag{2.1}$$

where G denotes the Einstein curvature tensor for the spacetime metric g, T denotes the stress-energy tensor for a perfect fluid,

$$T = (\rho + p)u \otimes u + pg, \tag{2.2}$$

and $\kappa = 8\pi\mathcal{G}$. (We assume the speed of light $c = 1$.) Here u is the 4-velocity of the fluid, $\mathcal{G}$ is Newton's gravitational constant, and we assume a barotropic equation of state of the form $p = p(\rho)$, where p is the pressure and ρ is the density. In a given coordinate system, T takes the form

$$T_{ij} = pg_{ij} + (p + \rho)u_i u_j, \tag{2.3}$$

where i, j are assumed to run from 0-3, and we use the Einstein summation convention throughout. The Einstein tensor G is constructed from the

Riemann curvature tensor so as to satisfy $divG = 0$. Thus, on solutions of (2.1), $divT = 0$, and this is the relativistic version of the classical Euler equations for compressible fluid flow. The compressible Euler equations provide the setting for the mathematical theory of shock-waves, [4]. We now briefly recall the FRW and OT metrics, and the results of [7].

The FRW metric describes a spherically symmetric spacetime that is homogeneous and maximally symmetric at each fixed time. In coordinates, the FRW metric is given by, [12],

$$ds^2 = -dt^2 + R^2(t)\left\{\frac{1}{1-kr^2}dr^2 + r^2 d\Omega^2\right\}, \tag{2.4}$$

where $t \equiv x^0$, $r \equiv x^1$, $\theta \equiv x^2$, $\varphi \equiv x^3$, $R \equiv R(t)$ is the 'cosmological scale factor', and $d\Omega^2 = d\theta^2 + sin^2\theta d\varphi^2$ denotes the standard metric on the unit 2-sphere. The constant k can be normalized to be either $+1, -1$, or 0 by appropriately rescaling the radial variable, and each of the three cases is qualitatively different. (When $k > 0$, $1 - kr^2$ defines the boundaries of the physical universe, and in everything to follow presumes that this is not violated.) We assume that the fluid is perfect, (i.e., (2.3) holds), and that the fluid is co-moving with the metric. The fluid is said to be co-moving relative to a background metric g_{ij} if $u^i = 0$, $i = 1, 2, 3$, so that g diagonal and u having length one imply, [12],

$$u^0 = \sqrt{-g_{00}}. \tag{2.5}$$

Substituting (2.4) into the field equations, and making the assumption that the fluid is perfect and co-moving with the metric, yields the following constraints on the unknown functions $R(t), \rho(t)$ and $p(t)$, [12, 7]:

$$3\ddot{R} = -4\pi\mathcal{G}(\rho + 3p)R, \tag{2.6}$$

$$R\ddot{R} + 2\dot{R}^2 + 2k = 4\pi\mathcal{G}(\rho - p)R^2, \tag{2.7}$$

together with

$$\dot{p}R^3 = \frac{d}{dt}\{R^3(p+\rho)\}. \tag{2.8}$$

Equation (2.8) is equivalent to

$$p = -\rho - \frac{R\dot{\rho}}{3\dot{R}}. \tag{2.9}$$

Substituting (2.6) into (2.7) we get

$$\dot{R}^2 + k = \frac{8\pi\mathcal{G}}{3}\rho R^2. \tag{2.10}$$

Since ρ and p are assumed to be functions of t alone in (2.4), equations (2.9) and (2.10) give two equations for the two unknowns R and ρ under the assumption that the equation of state is of the form $p = p(\rho)$. It follows from (2.9)-(2.10), c.f., [7], that $(R(t), \rho(t))$ is a solution if and only if $(R(-t), \rho(-t))$ is a solution, and that

$$\dot{\rho}\dot{R} < 0. \tag{2.11}$$

Thus to every expanding solution there exists a corresponding contracting solution, and conversely.

The OT metric describes a time-independent, spherically symmetric solution that models the interior of a star. In coordinates the components of the metric are given by

$$d\bar{s}^2 = -B(\bar{r})d\bar{t}^2 + A(\bar{r})^{-1}d\bar{r}^2 + \bar{r}^2 d\Omega^2. \tag{2.12}$$

We write this metric in bar-coordinates so that it can be distinguished from the unbarred coordinates when the metrics are matched. Assuming the stress tensor is that of a perfect fluid which is co-moving with the metric, and substituting (2.12) into the field equations (2.1), yields, (c.f. [12]),

$$A(\bar{r}) = \left(1 - \frac{2\mathcal{G}M}{\bar{r}}\right), \tag{2.13}$$

where $M \equiv M(\bar{r})$, $\bar{\rho} \equiv \bar{\rho}(\bar{r})$ and $\bar{p} \equiv \bar{p}(\bar{r})$ satisfy the following system of ordinary differential equations in the unknown functions $(\bar{\rho}(\bar{r}), p(\bar{r}), M(\bar{r}))$:

$$\frac{dM}{d\bar{r}} = 4\pi\bar{r}^2\bar{\rho}, \tag{2.14}$$

$$-\bar{r}^2\frac{d\bar{p}}{d\bar{r}} = \mathcal{G}M\bar{\rho}\left\{1 + \frac{\bar{p}}{\bar{\rho}}\right\}\left\{1 + \frac{4\pi\bar{r}^3\bar{p}}{M}\right\}\left\{1 - \frac{2\mathcal{G}M}{\bar{r}}\right\}^{-1}. \tag{2.15}$$

Equation (2.15) is called the Oppenheimer-Volkov equation, and is referred to by Weinberg as *the fundamental equation of Newtonian astrophysics.* ([12], page 301).

In this paper we assume the case of a barotropic equation of state $\bar{p} = \bar{p}(\bar{\rho})$, in which case equations (2.14), (2.15) yield a system of two ODE's in the two unknowns $(\bar{\rho}, M)$. We always assume that

$$0 < \frac{\bar{p}}{\bar{\rho}} \equiv \bar{\mu} < 1,$$

and that the sound speed is less than the (normalized) speed of light; i.e.,

$$0 < \bar{\sigma} \equiv \frac{d\bar{p}}{d\bar{\rho}} \leq 1.$$

The total mass M inside radius $\bar{r}$ is then defined by

$$M(\bar{r}) = \int_0^{\bar{r}} 4\pi\xi^2 \bar{\rho}(\xi) d\xi. \tag{2.16}$$

The metric component $B \equiv B(\bar{r})$ is determined from $\bar{\rho}$ and M through the equation

$$\frac{B'(\bar{r})}{B} = -2\frac{\bar{p}'(\bar{r})}{\bar{p} + \bar{\rho}}. \tag{2.17}$$

We remark that for any given FRW and OT metrics, there are maximal domains of definitions for the variables. We assume that the FRW metric is defined on the maximal interval $t_- < t < t_+$ and $0 \leq r_- < r < r_+$, and the OT metric is defined on the maximal interval $0 < \bar{r}_- < \bar{r} < \bar{r}_+$. For example, if $k > 0$, then we must have $r < \frac{1}{\sqrt{k}}$, t must be restricted so that $\rho(t)$ and $R(t)$ are positive, and by (2.10), we must require $\frac{8\pi\mathcal{G}}{3}\rho(t)R(t)^2 - k \geq 0$.

In [7], we described a procedure for constructing a coordinate transformation

$$(\bar{t}, \bar{r}) \to (t, r), \tag{2.18}$$

such that the FRW metric (2.4) matches the OT metric (2.12) Lipschitz continuously across a shock surface Σ. This shock surface is given implicitly by the equation

$$M(\bar{r}) = \frac{4\pi}{3}\rho(t)\bar{r}^3. \tag{2.19}$$

Equation (2.19) defines the radial coordinate $\bar{r}$ of the OT metric as a function of the time coordinate t of the FRW metric along the shock surface Σ. Note that for a given FRW density $\rho(t)$, (2.19) determines $\bar{r} = \bar{r}(t)$, the shock position. However, we can also solve (2.19) for ρ to obtain ρ as a function of $\bar{r}$; namely,

$$\rho = \frac{3}{4\pi}\frac{M(\bar{r})}{\bar{r}^3}.$$

¿From here on we let ρ refer to either $\rho(t)$ or $\rho(\bar{r})$, as given by (2.19), so that (with slight abuse of notation),

$$\rho(t) = \rho(\bar{r})(t),$$

on the shock surface. With this notation, $\rho(\bar{r})$ is determined by the OT solution alone.

For (2.19) to be meaningful in a given problem we must assume that $\bar{r} = \bar{r}(t)$ is defined for $t \in (t_-, t_+)$, $\bar{r} \in (\bar{r}_-, \bar{r}_+)$, and $r = \frac{\bar{r}(t)}{R(t)} \in (r_-, r_+)$. Equation (2.10) applies when any equation of state $p = p(\rho)$ is assigned to the FRW metric, and any equation of state $\bar{p} = \bar{p}(\bar{\rho})$ is assigned to the OT metric. The transformation $\bar{r} = \bar{r}(t, r)$ is given by

$$\bar{r} = R(t)r, \tag{2.20}$$

in the mapping $(\bar{t}, \bar{r}) \to (t, r)$, but the transformation $\bar{t} = \bar{t}(t, r)$ is more complicated, and its existence is demonstrated in [7]. Condition (2.20) implies that the areas of the sphere's of symmetry agree in the barred and unbarred frames in a neighborhood of the shock. It is somewhat remarkable that, other than it's existence we do not require any detailed information about the $\bar{t}$ transformation for the general theory.

Our construction in Theorem 1 guarantees that the FRW metric matches the OT metric Lipschitz continuously across the shock (2.19), and thus the following general theorem, (which is proved in [7], Theorem 4), applies: (See also [1].)

Theorem 1 *Let Σ denote a smooth, 3-dimensional shock surface in spacetime with spacelike normal vector* $\mathbf{n}$. *Assume that the components g_{ij} of the gravitational metric g are smooth on either side of Σ, (continuous up to the* *boundary on either side separately), and Lipschitz continuous across Σ in some fixed coordinate system. Then the following statements are equivalent:*

(i) $[K] = 0$ at each point of Σ. (Here, $[f]$ denotes the jump in the quantity f across the surface Σ, and K denotes the extrinsic curvature, or second fundamental form, which is determined by g separately on each side of the shock surface Σ.)

(ii) The curvature tensors R^i_{jkl} and G_{ij}, viewed as second order operators on the metric components g_{ij}, produce no delta function sources on Σ.

(iii) For each point $P \in \Sigma$ there exists a $C^{1,1}$ coordinate transformation defined in a neighborhood of P, such that, in the new coordinates, (which can be taken to be the Gaussian normal coordinates for the surface), the metric components are $C^{1,1}$ functions of these coordinates. (By $C^{1,1}$ we mean that the first derivatives are Lipschitz continuous.)

(iv) For each $P \in \Sigma$, there exists a coordinate frame that is locally Lorentzian at P, and can be obtained from the original coordinates by a $C^{1,1}$ coordinate transformation. (A coordinate frame is locally Lorentzian at a point P if $g_{ij}(P) = diag(-1,1,1,1)$ and $g_{ij,k}(P) = 0$ for all $i,j,k = 0,...,3$.)

Moreover, if any one of these equivalencies hold, then the Rankine-Hugoniot jump conditions, $[G^{ij}]n_i = 0$, hold at each point on Σ. (This expresses the weak form of conservation of energy and momentum across Σ when $G = \kappa T$.)

In the case of spherical symmetry, the conservation condition $[G^{ij}]n_i = 0$, is implied by the single condition

$$[G^{ij}]n_i n_j = 0. \tag{2.21}$$

This condition *alone* implies the equivalencies in Theorem 1. The intuition for this is that the Einstein equations convert smoothness of the metric into constraints on conservation, so that for metrics that are only Lipschitz continuous across a shock, there is no constraint on conservation, and for metrics that are $C^{1,1}$, the above result implies that conservation is *guaranteed.* For spherical shock-waves, the matching condition $\bar{r} = Rr$ holds in a *neighborhood* of the shock, and this represents one degree of smoothness beyond Lipschitz continuity. It is interesting to us that the Einstein equations naturally admit solutions *weaker than shock-waves*, i.e., Lipschitz continuous metric solutions, and it is important for our theory that the shock position is determined at this weaker level. This separates off the the problem of finding the shock position from the issue of conservation.

In light of (2.1), and (2.21), we conclude that conservation across the shock surface (2.19) is equivalent to the condition that the equation $[T^{ij}]n_i n_j = 0$ holds across Σ. In [8] we derived the following identity which is equivalent to $[T^{ij}]n_i n_j = 0$:

$$0 = (1-\theta)(\rho+\bar{p})(p+\bar{\rho})^2 + (1-\frac{1}{\theta})(\bar{\rho}+\bar{p})(\rho+p)^2 + (p-\bar{p})(\rho-\bar{\rho})^2, \quad (2.22)$$

where

$$\theta = \frac{A}{1-kr^2}. \quad (2.23)$$

This form of the constraint equation enabled us to construct the exact solution in [8]. The development to follow is likewise based on an analysis of (2.22). For completeness, we omit the derivation of (2.22) here. The condition

$$0 < \theta \leq 1 \quad (2.24)$$

turns out to be a very natural condition, and we show in ([9]) that it is equivalent to

$$\dot{R}^2 = \frac{8\pi\mathcal{G}}{3}\rho R^2 - k \geq 0. \quad (2.25)$$

In this paper we always assume (2.24) holds.

In the next section we summarize the results in [9] that represent the general theory of shock-waves that extend the Oppenheimer-Snyder model in the case $p = p(\rho)$. The analysis is based on a careful study of (2.22).

3 New Results

In this section we analyze (2.22) in detail. Solving for p in (2.22), we obtain the following formula for the FRW pressure p :

$$p_{\pm} = \frac{\frac{1}{2}\left\{-(\bar{\rho}+\rho)^2 + 2(\theta-1)\bar{\rho}\bar{p} + 2(\theta+\frac{1}{\theta})\rho\bar{\rho} + 2(\frac{1}{\theta}-1)\rho\bar{p} \pm SQ\right\}}{(1-\theta)\rho + (2-\theta-\frac{1}{\theta})\bar{p} + (1-\frac{1}{\theta})\bar{\rho}} \quad (3.1)$$

where

$$SQ = (6\bar{\rho}^2\rho^2 - 4\rho^3\bar{\rho} - 4\bar{\rho}^3\rho + \rho^4 + \bar{\rho}^4)^{1/2} = (\rho - \bar{\rho})^2. \tag{3.2}$$

Thus we conclude that every OT solution determines two possible FRW pressures p_- and p_+ through the conservation constraint. These implicitly determine the FRW equations of state $p = p(\rho)$. Algebraic simplifications lead to the following similar expressions for p_+ and p_- :

$$p_+ = \frac{\Theta\bar{p} - \rho}{1 - \Theta}, \tag{3.3}$$

$$p_- = \frac{\theta\bar{p} - \rho}{1 - \theta}, \tag{3.4}$$

where

$$\Theta \equiv \gamma\theta, \tag{3.5}$$

and

$$\gamma \equiv \frac{\rho + \bar{p}}{\bar{\rho} + \bar{p}}. \tag{3.6}$$

When $A > 0$ and $p = p(\rho)$, we prove in [9] that the case $\bar{\rho} > \rho$ leads to $\frac{dp}{d\rho} < 0$, a non-physical sound speed. Thus we will not consider the case $p = p_-$ further in these notes.

The following two theorems are consequences of (3.3), (proofs are omitted, and we assume throughout that results apply only to shocks at which $A > 0$ and $1 - kr^2 > 0$):

Theorem 2 *Assume that*

$$z \equiv \bar{\rho}/\rho < 1,$$

and

$$\bar{\mu} \equiv \frac{\bar{p}}{\bar{\rho}}.$$

Then $p_+ > 0$ if and only if $p - \bar{p} > 0$ if and only if $\theta_1 \leq \theta < 1$ at the shock, where

$$\theta_1 \equiv \theta_1(z, \bar{\mu}) \equiv \frac{1}{\gamma} = \frac{\bar{\rho} + \bar{p}}{\rho + \bar{p}} = \frac{1 + \bar{\mu}}{1 + \bar{\mu}z}z. \tag{3.7}$$

Theorem 3 *Assume that* $z < 1$. *Then for every choice of positive values for* $\bar{\rho}$, $\bar{p}$ *and* ρ, *the pressure* p_+ *monotonically takes on every value from* $[\bar{p}, +\infty)$, *and the pressure difference* $(p_+ - \bar{p})$ *monotonically takes on every value from* $[0, +\infty)$, *as* θ *ranges monotonically from* $[1, \theta_1)$.

Another direct consequence of (3.3), (3.4) is that if $A > 0$ and $\theta < 1$, then when $\rho > \bar{\rho}$, the only shock-waves with positive pressure must satisfy $p = p_+$ and

$$\Theta \equiv \gamma\theta > 1. \tag{3.8}$$

Using the formula (3.4) for p_+, we derive the following simplified set of equations for the dynamics of the shock surface and the FRW metric. The solutions describe an outgoing shock-wave exploding into a fixed outer OT solution. The time-reversal of the outgoing shocks are *rarefaction shocks* in the sense of Lax, [4], (details omitted):

Theorem 4 *Any FRW metric that matches a fixed OT metric Lipschitz continuously across the shock surface (2.19), (such that our other assumptions hold), and such that the Rankine Hugoniot jump conditions*

$$[T^{ij}]n_i = 0$$

also hold across the shock, must solve the ODE's

$$r\dot{R} = \pm\sqrt{1 - kr^2}\sqrt{1 - \theta} \tag{3.9}$$

$$R\dot{r} = \mp\frac{1}{1 - \Theta}\sqrt{1 - kr^2}\sqrt{1 - \theta}. \tag{3.10}$$

Conversely, any smooth solution of (3.9)-(3.10), (defined within the physical region of interest), will determine a solution of FRW type if we take

$$\rho = \frac{3}{4\pi}\frac{M}{\bar{r}^3},$$

and p *to be given by (3.3). This solution will match the OT metric Lipschitz continuously across the shock surface (2.19), (when we make the coordinate identification (2.18)), and the Rankine Hugoniot jump conditions will hold across the shock.*

Assuming that a fixed OT solution is given, we can use equations (3.9)-(3.10) to obtain a set of autonomous ODE's for the shock position $r(t)$ and the cosmological scale factor $R(t)$ whose solutions determine the FRW metrics that match the given OT metric Lipschitz continuously across the shock surface (2.19), such that conservation holds across the shock. The solution is determined by the coordinate mapping (2.18) so long as the solution stays within the physical domain of the variables. To see this, note that fixing the OT metric directly determines $M(\bar{r}), A(\bar{r}), \bar{\rho}(\bar{r})$ and $\bar{p}(\bar{r})$, and we can use the shock surface condition to determine $\rho = \frac{3}{4\pi}\frac{M(\bar{r})}{\bar{r}^3}$ as a known function of $\bar{r}$ as well. Since our coordinate identification sets $\bar{r} = Rr$, all of these functions can be taken as known functions of the shock position $r(t)$ and scale factor $R(t)$. Note that for the ODE's (3.9)-(3.10), we are free to choose two initial conditions, r and R. Moreover, the OT solution is determined by the choice of initial conditions M and $\bar{\rho}$ at given $\bar{r}$ for arbitrary equation of state $\bar{p} = \bar{p}(\bar{\rho})$. Thus we can determine local shock-wave solutions by arbitrarily assigning the OT equation of state, as well as $\bar{r}, \bar{\rho}, M$ and one of r or R, (because $Rr = \bar{r}$), thus allowing four initial conditions in all.

We always assume as given the OT equation of state and solution, and we then determined the FRW pressure, and ODE's for shock solution. One can also consider the "inverse" problem of assigning the FRW equation of state and solution, and of trying to determine the OT pressure and corresponding ODE's for the shock solution. For the pressure, one can solve (3.3) for $\bar{p}$. An easy calculation gives

$$\bar{p} = \frac{\theta\bar{\gamma}\rho - \bar{\rho}}{1 - \theta\bar{\gamma}}, \tag{3.11}$$

where

$$\bar{\gamma} = \frac{\bar{\rho} + p}{\rho + p}. \tag{3.12}$$

Note the remarkable symmetry between (3.11) and (3.3). However, this symmetry does not carry over to the corresponding shock equations. Indeed, when the FRW variables are known functions of t, we need to replace t with a known function of $\bar{r}$ and the unknown OT variables in order to derive a closed system of ODE's for $\bar{\rho}$ and M as functions of $\bar{r}$. For this, one must go to the shock surface equation $M = \frac{4\pi}{3}\rho\bar{r}^3$, and invert (the known) $\rho(t)$ in order to express t as a known function of the two variables M and $\bar{r}$. Moreover, in this case the conservation equation and the OT equations depend explicitly on $\bar{r}$ as well, and so fixing the FRW metric and solving for the OT metric leads to a considerably more complicated *non-autonomous* system of ODE's. The reason our approach is simpler and leads to an *autonomous* system of ODE's is because we can get $\bar{r}$ directly as a function of r and R from the identification $\bar{r} = Rr$, and we can solve for ρ as a known function of $\bar{r}$ from the shock surface equation. Thus the conservation equation, as well as the FRW equations, are autonomous. This justifies the approach we have taken.

The conditions for the shock speed to be less than the speed of light in a general solution of (3.9)-(3.10) are given by the following theorem, (see [9]):

Theorem 5 *The condition*

$$\theta > \frac{2\gamma - 1}{\gamma^2} \equiv \theta_-, \tag{3.13}$$

is equivalent to the condition that the shock speed s be less than the speed of light on solutions of (3.9)-(3.10) when $p \equiv p_+$.

Note that as $\gamma \to 1$, (which is equivalent to $\rho \to \bar{\rho}$, the weak shock limit), in (3.13),

$$\frac{2\gamma - 1}{\gamma^2} \to 1,$$

so $\theta \to 1$ and the shock speed tends to zero.

We now consider the problem of determining when the Lax shock conditions hold for the shocks determined by (3.9)-(3.10), in the case when the shock is an outgoing 2-shock, $\rho > \bar{\rho}$, $A > 0$, and the FRW metric is inside the OT metric. The Lax shock conditions express the requirement that the characteristics in the family of the shock impinge on the shock, and all other characteristics cross the shock. For an outgoing 2-shock, (the case that applies here), there are two Lax shock conditions which we refer to as FRW-Lax and OT-Lax conditions, respectively, [4]:

$$\sqrt{\sigma} > s \qquad \text{(FRW} - \text{Lax)} \tag{3.14}$$

and

$$s > \tilde{\lambda}_2^{OT}, \qquad \text{(OT} - \text{Lax)} \tag{3.15}$$

where

$$\sigma \equiv \frac{dp}{d\rho} = \frac{\dot{p}}{\dot{\rho}} = \frac{p'}{\rho'} \tag{3.16}$$

denotes the FRW sound speed, and $\tilde{\lambda}_2^{OT}$ denotes the characteristic speed of the outgoing OT sound wave as measured in the Local Minkowski coordinate frame fixed with the FRW fluid. Here we let dot denote $\frac{d}{dt}$ and prime denote $\frac{d}{d\bar{r}}$. In [9] we show that the OT fluid velocity as measured in the local Minkowski coordinate frame fixed with the FRW fluid turns out to be

$$\tilde{u} = -\sqrt{1-\theta}. \tag{3.17}$$

(Note that

$$|\tilde{u}| < 1 \tag{3.18}$$

when $\theta < 1$, as must be by covariance.) Conditions under which the shocks constructed here are Lax 2-shocks are summarized in the following results:

Theorem 6 *Assume that an OT solution $\bar{\rho} > 0$, $\bar{p}(\bar{r}) > 0$ and $M(\bar{r}) > 0$ of (2.14)-(2.15) is defined and smooth for all $\bar{r}$ in the interval*

$$\bar{r}_- < \bar{r} < \bar{r}_+ \leq \infty.$$

Assume also that the following additional conditions hold throughout the interval $(\bar{r}_-, \bar{r}_+)$:

$$0 < \bar{\mu} = \frac{\bar{p}(\bar{r})}{\bar{\rho}(\bar{r})} < 1, \tag{3.19}$$

$$0 < \bar{\sigma} = \frac{\bar{p}'(\bar{r})}{\bar{\rho}'(\bar{r})} < 1, \tag{3.20}$$

and

$$\rho = \frac{3}{4\pi}\frac{M(\bar{r})}{\bar{r}^3} > \bar{\rho}. \tag{3.21}$$

Then the solution $(r(t), R(t))$ of the shock equations $(3.9)_+$ and $(3.10)_-$ starting from initial data (r_0, R_0) satisfying

$$\bar{r}_- < \bar{r}_0 = r_0 R_0 < \bar{r}_+, \tag{3.22}$$

will exist, and will determine an outgoing shock-wave that satisfies,

$$0 < s < 1, \tag{3.23}$$

$$p > \bar{p}, \tag{3.24}$$

$$\rho > \bar{\rho}, \tag{3.25}$$

and

$$0 < \mu = \frac{p}{\rho} < 1, \tag{3.26}$$

together with the OT-Lax condition (3.15) throughout the maximal sub-interval of $(\bar{r}_-, \bar{r}_+)$ *containing* $\bar{r}_0$ *on which*

$$\theta_- < \theta < \theta_+, \tag{3.27}$$

where

$$\theta_- \equiv \theta_-(z, \bar{\mu}) = \frac{2\gamma - 1}{\gamma^2}, \tag{3.28}$$

and

$$\theta_+ \equiv \theta_+(z, \bar{\mu}, \bar{\sigma}) \equiv 1 - \left(\frac{1 - z}{1 + \bar{\mu}}\right)^2 \bar{\sigma}. \tag{3.29}$$

Moreover, the following inequalities hold for $z > 0.$*:*

$$\theta_1 < \theta_- < \theta_+, \tag{3.30}$$

and

$$\theta_- < 4z. \tag{3.31}$$

In particular, (3.27) is implied by the simpler, less sharp condition

$$4z < \theta < 1 - \frac{\bar{\sigma}}{(1 + \bar{\mu})^2}. \tag{3.32}$$

It remains only to discuss the FRW-Lax condition. A long calculation, (see [9]), leads to the following formula for the sound speed σ of the FRW solution as it evolves according to (3.9)-(3.10), (again see [9]):

$$(1-\theta\gamma)^2\sigma = \frac{1}{6}\frac{\theta(1-A)}{A(1+\bar{\mu})}\left\{\alpha(z,\bar{\mu}) + \beta(z,\bar{\mu})\frac{1-\theta}{\bar{\sigma}}\right\} + \frac{2}{3}\frac{\theta}{A} + \theta - \frac{5}{3}, \tag{3.33}$$

where

$$\alpha(z,\bar{\mu}) \equiv \frac{3 - 7z + 5\bar{\mu}z - 9\bar{\mu}z^2}{z}, \tag{3.34}$$

and

$$\beta(z,\bar{\mu}) \equiv \frac{(1+3\bar{\mu}z)(1+\bar{\mu}z)^2}{z(1-z)}. \tag{3.35}$$

Using this, it is straightforward to show that the FRW-Lax shock condition (3.14) is equivalent to

$$\frac{1}{6}\frac{\theta(1-A)}{A(1+\bar{\mu})}\left\{\alpha(z,\bar{\mu}) + \beta(z,\bar{\mu})\frac{(1-\theta)}{\bar{\sigma}}\right\} + \frac{2}{3}\frac{\theta}{A} + 2\theta > \frac{8}{3}. \tag{3.36}$$

Note now that $\theta_- \to 0$ and $\theta_+ \to 1 - \frac{\bar{\sigma}}{(1+\bar{\mu})^2} > 1 - \bar{\sigma}$ as $z \to 0$. Moreover, note that α, β and γ tend to $+\infty$ like $\frac{1}{z}$, as $z \to 0$. Using this it is straightforward to verify the following theorem which states, (roughly), that for outgoing shocks, the Lax characteristic conditions (3.14) and (3.15) hold, and the FRW sound speed is positive, and less than the speed of light, if the shock is sufficiently strong. This demonstrates that system (3.9)-(3.10) generates a large set of physically meaningful shock-wave solutions of the Einstein equations that model explosions.

Theorem 7 *Fix the dimensionless constants $\bar{\mu}$, $\bar{\sigma}$, A, and θ, (assume $0 < A < \theta < 1$ if $k > 0$ or $0 < \theta \leq A$ if $k \leq 0$), such that (3.27) holds. Then, if $z = \bar{\rho}/\rho$ is sufficiently small, then $0 < \sigma < 1$, $p > \bar{p}$, and the FRW-Lax shock conditions (3.18) and (3.14) both hold.*

Since the FRW equation of state is determined by the OT equation of state through the ODE's (3.9)-(3.10), there remains the question of how the

FRW equation of state is restricted by these equations. Assume an outgoing shock with $\rho > \bar{\rho}$. We now show that given any values of $\bar{\sigma}_0, \bar{\mu}_0, \sigma_0, \mu_0$ and $\rho_0 > 0$ satisfying

$$0 < \bar{\sigma}_0, \bar{\mu}_0, \sigma_0, \mu_0 < 1, \tag{3.37}$$

there exists a range of values of z_0, near $z = 0$, such that, when $\bar{\sigma}_0, \bar{\mu}_0, \sigma_0, \mu_0, \rho_0$ and z_0 are taken as initial values for (3.9)-(3.10) and the outer OT solution, then the shock-wave solution so determined will be a Lax shock-wave which moves at less than the speed of light in some neighborhood of the initial point. This demonstrates (roughly) that, for sufficiently strong shocks, arbitrary equations of state can be approximated locally on each side of the shock in this theory. Note that σ_0, μ_0, ρ_0 and z_0 supply the requisite number of initial conditions, two for the OT equations (2.14)-(2.15) and two for the shock equations (3.9)-(3.10). In the next theorem we show that even after prescribing $\bar{\mu}_0$ and $\bar{\sigma}_0$ for the OT equation of state at a point, we still have enough freedom in the initial conditions to freely assign σ_0, μ_0, ρ_0, and z_0, at a point. To state the theorem, note first that for any values of $\bar{\sigma}_0, \bar{\mu}_0, \sigma_0$ and μ_0, satisfying (3.37), we can use

$$\mu = \frac{\gamma\theta z - 1}{1 - \gamma\theta} \tag{3.38}$$

to solve for θ and obtain

$$\theta = \frac{(1 + \bar{\mu}_0)(1 + \mu_0)}{(1 + \bar{\mu}_0 z)(z + \mu_0)} z. \tag{3.39}$$

It is readily shown that when (3.37) holds, equation (3.39) defines θ as a continuous function of z that takes on all values from 0 to 1 as z increases from 0 to 1.

Theorem 8 *For any choice of $\bar{\sigma}_0, \bar{\mu}_0, \sigma_0, \mu_0$ satisfying (3.37), and any choice of $z_0, \theta_0 \in (0, 1)$ such that*

$$\theta_0 = \frac{(1 + \bar{\mu}_0)(1 + \mu_0)}{(1 + \bar{\mu}_0 z)(z + \mu_0)} z_0, \tag{3.40}$$

and such that

$$\theta_0 < 1 - \bar{\sigma}, \tag{3.41}$$

there exists $A_0 \in (0,1)$ *such that, (c.f. (3.33)),*

$$\sigma_0 = (1-\gamma_0\theta_0)^{-2}\left\{\frac{1}{6}\frac{\theta_0}{A_0}\frac{(1-A_0)}{(1+\bar{\mu}_0)}\left(\alpha_0+\beta_0\frac{1-\theta_0}{\bar{\sigma}_0}\right)+\frac{2}{3}\frac{\theta_0}{A_0}+\theta_0-\frac{5}{3}\right\}. \tag{3.42}$$

Note that as $z_0 \to 0$, (strong shocks), (3.40) implies that θ_0 asymptotically looks like

$$\theta_0 \approx \frac{(1+\bar{\mu}_0)(1+\mu_0)}{\mu_0}z,$$

while θ_- and $\frac{2\gamma_0-1}{\gamma_0^2}$ asymptotically look like $2(1+\bar{\mu}_0)z_0$, and so $\theta_0 < \theta_+$, and both $\theta_- < \theta_0$, and $\frac{2\gamma_0-1}{\gamma_0^2} < \theta_0 < \theta_+$, for sufficiently small z_0. Thus we have shown (essentially) that *the equations of state can be freely assigned to second order at a point for sufficiently strong shocks.*

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