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Shock Waves and General Relativity

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1. In 1939, J. R. Oppenheimer, and H. Snyder, (O.S.) produced the first mathematical model for gravitational collapse of stars [1]. This involved the matching of two metrics, (the Robertson-Walker metric, and the Schwarzschild metric), Lipschitz continuously across a surface of discontinuity for the fluid variables. In their paper too, O.S. deduced the theoretical existence of what we now call black holes. However, O.S. had to make the (unphysical) assumption that the pressure was always equal to zero. In this paper, we shall describe how the O.S. results can be extended to the case of non-zero pressure. In order to do this, we shall first clarify the role of "conservation" in stellar and cosmological models, and then we shall study shock waves in General Relativity. Finally we shall describe a possible physical application of our result to a dynamic models of cosmology. (See the papers [4,5] for complete details.)

2. Our first goal is to generalize the O.S. model for gravitational collapse to the case of non-zero pressure. This involves a watching of different metrics, Lipschitz continuously across a surface of discontinuity of the fluid variables; i.e., across a shock wave.

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In our shock-wave model, the inner metric is the Robertson-Walker (RW) metric:

$$d_s^2 = -dt^2 + R(t)^2\{\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\},$$

where $dr^2 = d\theta^2 + sin^2\theta d\phi^2$, is the standard metric on the 2-sphere. The RW metric is spherically symmetric, and is used to model a homogeneous, isotropic universe – no preferred point, no preferred direction. It expands or contracts in time according to $R(t)$, the cosmological scale factor; $R(t)$ also determines the red-shift factor for distant objects. The singularity, $(R(\bar{t}) = 0)$, in backwards time denotes the "big-bang". If we are g

iven an equation of state, $p = p(\rho)$ ($p =$ pressure, $\rho =$ density), $R(t)$ determined by the Einstein equations $G = \lambda T$ under the assumption $\rho = \rho(t)$.

The outer metric in our model is the Interior Schwarzschild (IS) metric:

$$ds^2 = -B(\bar{r})d\bar{t}^2 + \left(1 - \frac{2GM(\bar{r})}{\bar{r}}\right)^{-1} d\bar{r}^2 + \bar{r}^2 d\Omega^2,$$

where the following equations hold

$$B'/B = \frac{-2\bar{p}^1}{\bar{p}+\bar{p}}, -\bar{r}^2\bar{p}(\bar{r}) = GM(\bar{r})\bar{p}(\bar{r})\left(1+\frac{\bar{p}}{\bar{\rho}}\right)\left(1+\frac{4\pi\bar{r}^3\bar{p}}{M}\right)\left(1-\frac{2GM(\bar{r})}{\bar{r}}\right)^{-1}, M'(\bar{r}) = 4\pi\bar{r}^2\rho(\bar{r}).$$

This metric is a time independent, spherically symmetric solution of Einsteins equations $G = \lambda T$, and is used to model the interior of a star. The functions $B(\bar{r})$ and $M(\bar{r}) = \int_0^{\bar{r}} 4\pi s^2 \bar{\rho}(s)ds$ (the "total mass"), we determined by Einstein's equations for any equation of state $\bar{p} = \bar{p}(\bar{\rho})$.

Each of the above metrics are solutions of Einsteins equations $G = \lambda T$, where T is the stress-energy tensor of a "perfect fluid":

$$T = (p+\rho)u \otimes u + pg,$$

and $\lambda = 8\pi G, G =$ Newton's gravitational constant. (We assume that the speed of light $c = 1$.)

Now let's review the O.S. model. Here the outer metric is the (empty space) Schwarzschild metric

$$ds^2 = \left(1 - \frac{2GM}{\bar{r}}\right)d\bar{t}^2 + \left(1 - \frac{2GM}{\bar{r}}\right)^{-1} d\bar{r}^2 + \bar{r}^2 d\Omega^2$$

where M is a constant. The inner metric is the RW metric. The O.S. result is to take $p \equiv 0$; then there exists a coordinates transformation $(t, r) \longrightarrow (\bar{t}, \bar{r})$ such that under this identification of coordinates, the metrics agree and are Lipschitz continuous across a 3-dimensional interface, the surface of the star. The surface in (r,t) coordinates is $r = a$ (fixed in RW metric). In $(\bar{r}, \bar{t})$ coordinates, the surface collapses to a black hole; namely, $\bar{r}(\bar{t}) \longrightarrow \bar{r} = 2\mathrm{GM}$ (the Schwarzschild radius). The total mass inside the interface is fixed because mass does not cross the interface; thus the interface is a contact discontinuity. $R(t)$ vanishes at some finite time T, so "the fluid sphere of initial density $\rho(0)$ and zero pressure collapses from rest to a state of infinite density in the finite time T." This is the first example of gravitational collapse.

Our shock wave model is to take the inner metric to be the RW metric with $p \not\equiv 0, \rho \not\equiv 0$, and the outer metric to be the IS metric with given equation of state $\bar{p} = \bar{p}(\bar{\rho}), \bar{\rho} = \bar{\rho}\ (\bar{r})$. We define a coordinate mapping $(t, r) \longrightarrow (\bar{t}, \bar{r})$ such that the RW metric matches Lipschitz - continuously with the IS metric across a 3-dimensional shock surface. We prove

Theorem 1. There exists a shock wave solution for arbitrary equation of states $p = p(\rho)$ (RW) and $\bar{p} = \bar{p}(\bar{\rho})$ (IS).

The shock surface is given (implicitly) by

$$M(\bar{r}) = \frac{4\pi}{3}\rho(t)\bar{r}^3,$$

and represents a global conservation of mass principle.

Now an important question arises; namely if two metrics match Lipschitz - continuously across a shock surface $\sum$, does the weak form of conservation of energy and momentum hold across $\sum$? That is, does the equation $[T^{ij}]n_i = 0$, hold? (Here n_i are the components of the normal vector $\bar{n}$, and the brackets denote the jump in T^{ij} across $\sum$.)

But since $G_{ij} = \lambda T_{ij}$, this is equivalent to $[G^{ij}]n_i = 0$. The answer is no, in general. But Israel, [1] has shown that if the extrinsic curvature, (2nd fundamental form), K is continuous across $\sum$, then conservation holds. (Here K maps the tangent space at $p \epsilon \sum$ into itself, and is given by $K(X) = \nabla_X \bar{n}$.) We prove that for spherically symmetric metrics conservation holds if the sing le condition

$$[T^{ij}]n_i n_j = 0$$

holds. Now for our problem, if $r = r(t) = R(t)r(t)$ denotes the shock surface, then

$$(*) \qquad [T^{ij}]n_i n_j = (\bar{p} + \rho)\dot{r}^2 - (\bar{p} + \bar{p})\frac{B}{A}\frac{(1 - kr^2)}{R^2} + (p - \bar{p})\frac{1 - kr^2}{R^2} = 0.$$

Notice that in the O.S. model, $\overline{p} = \overline{\rho} = 0$ so the above relation reduces to $\rho r^2 + p\frac{1-kr^2}{R^2} = 0$; thus if $\rho = 0$, then $p = 0 = r$, so there is no solution for non-zero pressure.

Our idea is to fix $\overline{\rho}, \overline{p}, M(\overline{r})$ on the outside IS solution, $\overline{p} = \overline{p}(\rho)$, and find the ode's for $R(t)$ and $\rho(t)$:

$$\dot{R}^2 = \frac{8\pi G}{3}\rho R^2 - k, \quad p\frac{d}{dt}(\rho R^3)/(-3R^2\dot{R}).$$

Note that if $p = p(\rho)$, this gives two ode's in ρ, R so we have a unique solution and we cannot guarantee that conservation (i.e. $(*)$) holds. We thus allow p to be free and we take $(*)$ as the extra constraint. Solving for the (unknown) p, we must be sure that p is "physical" ; e.g. is $p \geq 0$? That is, do these ode's determine "physically meaningful" shock waves? In general, this depends on the initial conditions, as well as on the outer IS equation of s

tate, $\overline{p} = \overline{p}(\overline{\rho})$.

3. If we consider the case $\overline{p} = \overline{\sigma}\overline{\rho}$, ($\overline{\sigma}$ = const.), then we can construct an explicit solution of an O.S. type shock wave with $p \not\equiv 0$, which satisfies conservation. For this, we choose $k = 0$ in the inner RW metric. We prove

Theorem 2. Under these assumptions, a RW equation of state of the form $p = \sigma\rho$ (σ = const.) is determined by the algebraic equation.

$$(**) \qquad 0 = 3\sigma + \frac{3\overline{\sigma}}{1+6\overline{\sigma}+\overline{\sigma}^2}(3\sigma+1)^2 - \frac{9}{1+\overline{\sigma}}(1+\sigma^2) - 1,$$

and we can obtain explicit formulas for the solution.

That is, given $\overline{\sigma}, \sigma$ is obtained from the above relation. In fact, near $\overline{\sigma} = 0$, we have

$$\sigma = \frac{7}{3}\overline{\sigma} + 0(\overline{\sigma}^2);$$

also if $\sigma = \frac{1}{3}$ (the extreme relativistic limit), $\overline{\sigma} \approx \frac{1}{10}$, and if $\sigma = 1$ (speed of light), $\overline{\sigma} \approx \frac{7}{24}$. We conclude too that the density, pressure and sound speed are larger behind the shock.

We still must see if the Lax shock conditions hold; c.f. [3, p249 ff], and also we must have the shock speed smaller than 1. We prove

Theorem 3. There exist numbers $\sigma_1, \sigma_2, 0 < \sigma_1 = .458 < .744 = \sigma_2 < 1$ such that

(i) The Lax characteristic condition holds iff $0 < \sigma < \sigma_1$, and

(ii) The shock speed is less-than the speed of light iff $0 < \sigma < \sigma_2$.

Thus our solution yields a physically meaningful shock-wave solution of the Einstein equations if $0 < \sigma < \sigma_1$. Moreover, if $\sigma_1 < \sigma < \sigma_2$, a new type of shock wave appears in which the sound speeds are supersonic on both sides of the shock. Note too that a fluid with sound speeds no larger than $\sqrt{\sigma_2} \approx \sqrt{.744}$, can drive shock waves with speeds all the way up to the speed of light.

Finally, we can give explicit formulas for the solution, and we can show that $\bar{r}(t)$ and $R(t)$ are equal to zero at the same time. Is this a different sort of "big bang"?

4. We conclude with describing a possible physical application; all of our remarks are highly speculative, at this point.

We have constructed exact, irreversible shock wave solutions of Einstein's equations based on the IS and RW metrics. One possible application is to a new cosmological model; namely, can there be a shock wave of cosmic dimensions at the edge of the universe? Could the $2.7°K$ microwave background radiation be related to the radiation from the shock-wave? Since our shock wave is an irreversible solution of Einstein's equations, we cannot reconstruct the past from the knowledge of the present, [3, p 2

61 ff]. Does this imply that the "big-bang" scenario is different from the usual model?

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