JOURNÉES ÉQUATIONS AUX DÉRIVÉES PARTIELLES

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Scattering matrix for asymptotically euclidean manifolds

Journées Équations aux dérivées partielles (1994), p. 1-14
<http://www.numdam.org/item?id=JEDP_1994____A17_0>

Scattering matrix for asymptotically Euclidean manifolds (joint work with Richard Melrose).

Maciej Zworski

1 Introduction and statement of the result.

The purpose of this exposé is to present the result of [8] and to indicate the methods used there.

We consider scattering in a setting generalizing the Euclidean one and introduced by Melrose in [4]. The main purpose there was to obtain a systematic framework for the study of scattering theory without relying on the symmetries of the Euclidean situation. Roughly speaking, the sphere at infinity was replaced by an arbitrary Riemannian manifold, which constituted in some sense a 'smooth' deformation of infinity. In the future, one can envision allowing also 'singular' infinities such as arise in the N-body problem or in scattering by non-compact obstacles.

In the Euclidean case the absolute scattering matrix acts on functions on the sphere at infinity and is essentially the pull-back by the antipodal map. From the microlocal point of view it is a Fourier Integral Operator associated to the geodesic flow on the sphere at time π. We show that in the general situation, the scattering matrix has the same property with the geodesic flow now on the boundary at infinity, proving a statement conjectured in [4] – see Fig.1

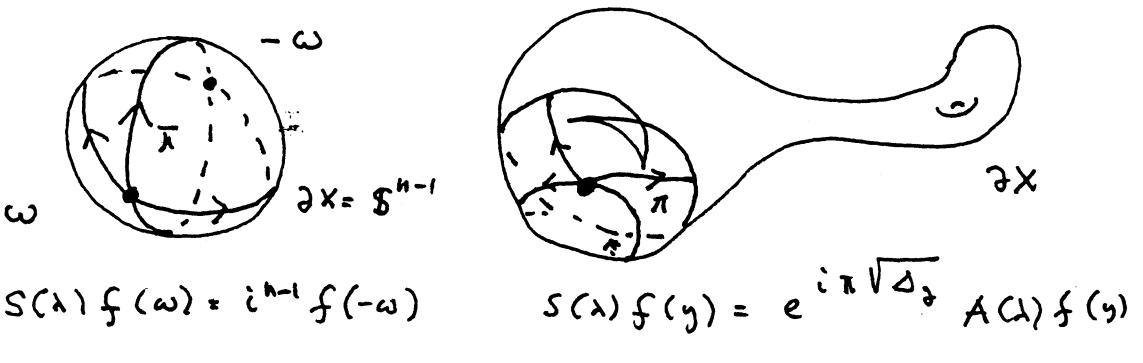

Figure 1: Geometric structure of the scattering matrix

To make this precise, let X be a compact $\mathcal{C}^\infty$ manifold with boundary. If x is a boundary defining function, that is

$$x|_{\partial X} = 0, \quad dx|_{\partial X} \neq 0, \quad x|_{X^\circ} > 0,$$

then X° admits a complete metric which near the boundary takes the form

$$g = \frac{dx^2}{x^4} + \frac{h}{x^2} \tag{1.1}$$

where h is a smooth symmetric 2-cotensor with $h|_{\partial X}$ positive definite, that is, defining a non-degenerate metric on ∂X. Following [4] we call metrics of the form (1.1) *scattering metrics.* We denote by Δ the Laplacian corresponding to g and by Δ_∂ the Laplacian on

∂X corresponding to $h|_{\partial X}$. We have the following basic fact analogous to that in classical scattering theory (see Sect.2 below for an indication of proof, and [4], Sect.15 for a detailed presentation): if $f \in \mathcal{C}^\infty(\partial X)$ is chosen then for each $\lambda \neq 0$, $\lambda \in \mathbb{R}$, there exists a *unique* function $u \in \mathcal{C}^\infty(X^o)$ satisfying

$$(\Delta - \lambda^2)u = 0, \quad u = e^{-\frac{i\lambda}{x}} x^{\frac{n-1}{2}} f' + e^{\frac{i\lambda}{x}} x^{\frac{n-1}{2}} f'', \tag{1.2}$$

where $f', f'' \in \mathcal{C}^\infty(X)$ and $f'|_{\partial X} = f$. The *absolute* scattering matrix, $S(\lambda)$ is defined as the map

$$S(\lambda) : \mathcal{C}^\infty(\partial X) \ni f \longmapsto f''|_{\partial X} \in \mathcal{C}^\infty(\partial X). \tag{1.3}$$

We then have

Theorem *The absolute scattering matrix, $S(\lambda)$, is a Fourier Integral Operator of order* 0 *on ∂X, associated to the canonical diffeomorphism*

$$\exp(\pi H_{\sqrt{h}}) : T^*\partial X \setminus 0 \to T^*\partial X \setminus 0$$

given by the geodesic flow at distance π for the induced metric on ∂X, h. More precisely

$$S(\lambda) = \exp(i\pi\sqrt{\Delta_\partial})A(\lambda), \quad A(\lambda) \in \Psi^0(\partial X), \tag{1.4}$$

and A is elliptic.

The basic example is of course given by the Euclidean case: under the stereographic projection

$$\mathrm{SP} : \mathbb{R}^n \ni z \longmapsto \left((1+|z|^2)^{-\frac{1}{2}}, z(1+|z|^2)^{-\frac{1}{2}}\right) \in \mathbb{S}^n_+,$$

$$\mathbb{S}^n_+ = \{t = (t_0, \cdots, t_n) \in \mathbb{R}^{n+1} \; ; \; t_0 \geq 0, |t| = 1\},$$

$\mathbb{R}^m$ is identified with $(\mathbb{S}^n_+)^o$ where $X = \mathbb{S}^n_+$ is the compact manifold with boundary. The Euclidean metric becomes

$$|dz|^2 = dr^2 + r^2|d\omega|^2 = \frac{dx^2}{x^4} + \frac{|d\omega|^2}{x^2}, \quad |z| = r = \frac{1}{x}, \quad \omega = \frac{z}{|z|},$$

where $|d\omega|^2$ is the standard metric on the sphere $\mathbb{S}^{n-1} = \partial\mathbb{S}^n_+$. This is precisely a scattering metric of the form (1.1). To compute the absolute scattering matrix for the Euclidean space we follow the argument of Appendix to [3]. Thus, let us consider $u(x,\omega) = \exp(i\lambda\omega \cdot \theta / x)$, $\theta \in \mathbb{S}^{n-1}$. Then $(\Delta - \lambda^2)u = 0$ and in the sense of distributions in ω and as $x \to 0$

$$e^{i\lambda\frac{\omega\cdot\theta}{x}} \sim (2\pi)^{\frac{n-1}{2}} \lambda^{-\frac{1}{2}} x^{\frac{n-1}{2}} \left[e^{\frac{i\lambda}{x}} e^{-\frac{i\pi}{4}(n-1)} \delta_\theta(\omega) + e^{-\frac{i\lambda}{x}} e^{\frac{i\pi}{4}(n-1)} \delta_{-\theta}(\omega)\right] \tag{1.5}$$

which follows from the stationary phase method applied in the variable ω after integration against a function in $\mathcal{C}^\infty(\mathbb{S}^{n-1})$. Applying somewhat formally the definitions (1.2) and (1.3) we conclude that

$$S(\lambda) : \mathcal{C}^\infty(\mathbb{S}^{n-1}) \ni f \longmapsto i^{n-1} j^* f \in \mathcal{C}^\infty(\mathbb{S}^{n-1}), \quad j : \mathbb{S}^{n-1} \ni \omega \mapsto -\omega \in \mathbb{S}^{n-1}.$$

As in the Euclidean case an addition of a short-range perturbation does *not* change the geometric structure of the scattering matrix. In the generalized setting this can be stated as follows: if $V \in x^2\mathcal{C}^\infty(X)$ then the theorem above remains true for the scattering matrix for the operator $\Delta + V$. Clearly, any metric perturbation, $\tilde{g}$, which preserves the scattering metric structure (1.1) is also allowed and that corresponds to $g - \tilde{g} = \mathcal{O}(x^2)$, which is another short-range condition.

We conclude this section with a brief discussion of complete metrics on compact $\mathcal{C}^\infty$ manifolds with boundary, indicating an alternative, 'non-euclidean' origin of scattering metrics (1.1). Thus let X and h be as before. Then X admits three types of 'marginally complete' metrics:

$$\left(\frac{dx}{x}\right)^2 + h \quad \text{cylindrical end metric} \tag{1.6}$$

$$\left(\frac{dx}{x}\right)^2 + x^2 h \quad \text{finite volume asymptotically hyperbolic metric} \tag{1.7}$$

$$\left(\frac{dx}{x}\right)^2 + x^{-2} h \quad \text{infinite volume asymptotically hyperbolic metric} \tag{1.8}$$

For (1.6) the metric on the boundary, $h|_{\partial X}$, does *not* depend on the choice of the defining function while for (1.7) and (1.8) only the conformal class on h is determined on the boundary. This geometric fact is reflected in the structure of the scattering matrix – see [5] and [1] for (1.6) and [9] for a special case of (1.8). For the former, the structure of the scattering matrix depends on the spectrum of Δ_∂ and for the latter, the scattering matrix is a pseudodifferential operator acting on denisities constructed using the conformal structure on ∂X.

Scattering metrics (1.1) arise by multiplying (1.6) by x^{-2} and they are *not* very rigid under changes of the boundary defining function. In fact, for (1.1) to be preserved, the change of the boudary defining function has to be of the form $\tilde{x} = x + \mathcal{O}(x^2)$. Hence once we demand that the metric is of the form (1.1), the boundary metric $h|_{\partial X}$ is uniquely determined. Analytically, this is reflected in the theorem above, where $h|_{\partial X}$ plays a crucial rôle in describing the scattering matrix, and, as opposed to the situation for (1.8), we now have propagation on the boundary. Heuristically, it can be explained as follows: in the asymptotically hyperbolic case all geodesics immediately go away from the boundary at infinity and the scattering matrix is localized (pseudodifferential) while in the asymptotically euclidean one, the geodesics can 'creep' along the boundary causing propagation – see Fig.2.

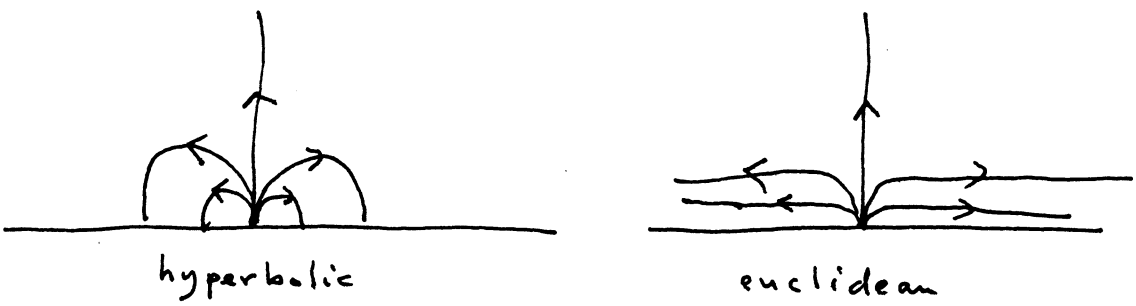

Figure 2: Geodesics near the boundary at infinity

It is not clear at the moment under what conditions one can expect meromorphic continuation of $S(\lambda)$ and what would be its analytic structure. It seems natural however to demand that the manifold is analytic in which case any relation with the complexified flow would be very interesting.

Finally, we should note that the methods of [8] have many similarities with more classical work in Euclidean scattering. Perhaps the closest is the Agmon-Hörmander approach through the analysis on the energy shell $|\xi|^2 = \lambda^2$ – see [2], Chapters 14 and 30 and references given there.

2 Microlocal approach to scattering theory.

We will now outline the microlocal approach to scattering theory viewed as a degenerate elliptic boundary value problem. Most of the material comes directly from [4] and for a general introduction to Melrose's appoach we refer to [6].

Let X be as in Sect.1 and let $\mathcal{V}_{\mathrm{b}}(X)$ denote the Lie algebra of $\mathcal{C}^\infty$ vector fields tangent to ∂X. A scattering metric (1.1) is an example of a metric on the structure bundle of the Lie algebra

$$\mathcal{V}_{\mathrm{sc}}(X) = x \cdot \mathcal{V}_{\mathrm{b}}(X).$$

This means that $\mathcal{V}_{\mathrm{sc}}(X)$ consists of smooth sections of the structure bundle ${}^{\mathrm{sc}}TX$. Roughly speaking, if we think of X as $[0,1)_x \times (\partial X)_y$ and the sections of TX as spanned by ∂_x and ∂_y, then the sections of ${}^{\mathrm{sc}}TX$ are spanned by $x^2\partial_x$ and $x\partial_y$. The Laplace operator for a scattering metric is precisely an elliptic polynomial in these vector fields. To study symbols of operators we need the corresponding 'cotangent' bundle, ${}^{\mathrm{sc}}T^*X$ sections of which are spanned by by dual forms $x^{-2}dx$ and $x^{-1}dy$. The usual cotangent bundle, T^*X embeds naturally in ${}^{\mathrm{sc}}T^*X$: $(x, y; \xi, \eta) \mapsto (x, y; x^2\xi, x\eta)$. We note that ${}^{\mathrm{sc}}T^*X = x^{-1}\,{}^{\mathrm{b}}T^*X$ where ${}^{\mathrm{b}}T^*X$ corresponds to $\mathcal{V}_{\mathrm{b}}(X)$ (see [2], Sect.18.3 for a slightly different definition; there ${}^{\mathrm{b}}T^*X$ is denoted by $\tilde{T}^*X$).

The enveloping algebra of $\mathcal{V}_{\mathrm{sc}}(X)$ consists of *scattering* differential operators, $\mathrm{Diff}^*_{\mathrm{sc}}(X)$, and the corresponding pseudodifferential operators can be defined using a systematic approach [6] (see Appendix B of [4]). However, a naive method is also possible and we will present it for $X = \mathbb{S}^n_+$ – the coordinate invariance and local identification of X with $\mathbb{S}^n_+$ gives then a general definition, see Sect.4 of [4].

We say that

$$A \in \Psi^{m,k}_{\mathrm{sc}}(\mathbb{S}^n_+, {}^{\mathrm{sc}}\Omega^{\frac{1}{2}})$$

if and only if $A' = \mathrm{SP}^* \circ A \circ (\mathrm{SP}^{-1})^*$ is of the form

$$A'(u(z)|dz|^{\frac{1}{2}}) = (2\pi)^{-n} \int e^{i(z-z')\cdot\zeta} a'((z+z')/2, \zeta) u(z') dz' d\zeta |dz|^{\frac{1}{2}},$$

with $a'(z,\zeta) = \mathrm{SP}_2^* a$, $a \in \rho_\sigma^{-m}\rho_\partial^k \mathcal{C}^\infty(\mathbb{S}^n_+, \mathbb{S}^n_+)$. Here, $\mathrm{SP}_2 = \mathrm{SP} \times \mathrm{SP} : \mathbb{R}^n_z \times \mathbb{R}^n_z \to \mathbb{S}^n_+ \times \mathbb{S}^n_+$, $\rho_\sigma \in \mathcal{C}^\infty(\mathbb{S}^n_+ \times \mathbb{S}^n_+)$ is the defining function of $\mathbb{S}^n_+ \times \partial\mathbb{S}^n_+$ and ρ_∂ is the defining function of $\partial\mathbb{S}^n_+ \times \mathbb{S}^n_+$.

We think of a as a *joint symbol* of A meaning that it measures both the behaviour at the boundary and at the fiber infinity of the cotangent bundle. To define it for any X we introduce a *compactified* scattering cotangent bundle ${}^{\mathrm{sc}}\bar{T}^*X$ – see Fig.3.

In the case of $X = \mathbb{S}^n_+$, ${}^{\mathrm{sc}}\bar{T}^*X$ is canonically isomorphic to $\mathbb{S}^n_+ \times \mathbb{S}^n_+$ and as we saw above, it is there that the symbol lives. In general we have by local identification of X and $\mathbb{S}^n_+$ (see [8])

Proposition 2.1 *For any compact manifold with boundary, X, the joint symbol of $A \in \Psi^{m,k}_{\mathrm{sc}}(X, {}^{\mathrm{sc}}\Omega^{\frac{1}{2}})$ is well defined as an element of*

$$\rho_\sigma^{-m}\rho_\partial^k\, \mathcal{C}^\infty({}^{\mathrm{sc}}\bar{T}^*X) \text{ modulo } \rho_\sigma^{-m+2}\rho_\partial^{k+2}\, \mathcal{C}^\infty({}^{\mathrm{sc}}\bar{T}^*X) \tag{2.1}$$

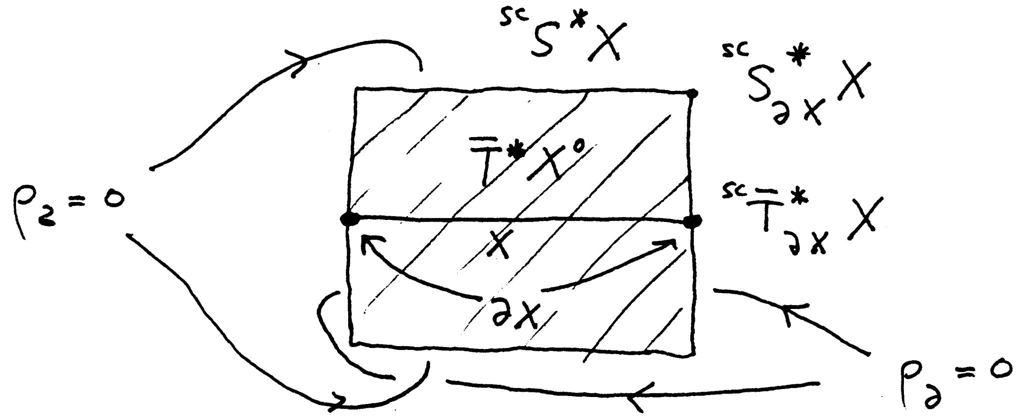

Figure 3: The compactified scattering cotangent bundle

We will denote the joint symbol of A by $j_{m,k}(A)$.

To study commutators we need the scattering Hamilton vector field – one way to obtain it is by extending the standard definition from the interior – see [8] and Sect.4 below for a more systematic approach. We have [4]

Proposition 2.2 *If* $a \in \rho_\sigma^{-m}\rho_\partial^k\, C^\infty({}^{sc}\bar{T}^*X)$ *then the Hamilton vector field extends from the interior to a vectorfield* ${}^{sc}H_a \in \rho_\sigma^{-m}\rho_\partial^k \mathcal{V}_{sc}({}^{sc}\bar{T}^*X)$.

If we define also a renormalized Hamilton vector field,

$$ {}^{sc}H_a^{m,k} = \rho_\sigma^{m-1}\rho_\partial^{k-1}\,{}^{sc}H_a \in \mathcal{V}_b({}^{sc}\bar{T}^*X), $$

then for any $b \in \rho_\sigma^{-m'}\rho_\partial^{k'}\, C^\infty({}^{sc}\bar{T}^*X)$

$$ \{a, b\} = \rho_\sigma^{-m+1}\rho_\partial^{k+1}{}^{sc}H_a^{m,k}b \in \rho_\sigma^{-m'-m+1}\rho_\partial^{k'+k+1}\, C^\infty({}^{sc}\bar{T}^*X), $$

where we extend $\{\bullet, \bullet\}$ by continuity from the interior of ${}^{sc}T^*X$. As for the usual pseudodifferential operators this is significant since for $A \in \Psi^{m,k}(X, {}^{sc}\Omega^{\frac{1}{2}})$ and $B \in \Psi^{m',k'}(X, {}^{sc}\Omega^{\frac{1}{2}})$ we have $[A, B] \in \Psi^{m+m'+1,k+k'+1}(X, {}^{sc}\Omega^{\frac{1}{2}})$ and

$$ j_{m+m'+1,k+k'+1}([A, B]) = \frac{1}{i}\{j_{m,k}(A), j_{m',k'}(B)\}. $$

For $A \in \Psi^{m,k}(X, {}^{sc}\Omega^{\frac{1}{2}})$ we define the characterstic variety

$$ \Sigma_{sc}(A) = \{m \in \partial\,{}^{sc}\bar{T}^*X : j_{m,k}(A)(m) = 0\} $$

which immediately leads to the definition of the wave front set: for $u \in \mathcal{D}'(X)$

$$ WF_{sc}^{m,l}(u) = \bigcap\{\Sigma_{sc}(A) \ : \ A \in \Psi^{m,-l}(X, {}^{sc}\Omega^{\frac{1}{2}}) \text{ and } Au \in L^2(X, {}^{sc}\Omega)\}. $$

Clearly the index m measures the C^∞ regularity and the index l the decay at the boundary ∂X, that is, at infinity.

For operators with real joint symbols there exists an exact analogue of Hörmander's propagation of singularities theorem (Theorem 26.1.4 in [2]), now giving also information about decay at ∂X – see Proposition 7 of [4]. Here we recall only that for $P \in \Psi^{m,k}(X, {}^{sc}\Omega^{\frac{1}{2}})$, the wave front set is invariant under the flow of the renormalized Hamilton vector field

${}^{\mathrm{sc}}H^{m,k}_{j_{m,k}(P)}$. No information is of course given at radial points and expectedly these are the points of interest in scattering theory.

Thus let us consider $P = \Delta - \lambda^2 \in \Psi^{2,0}_{\mathrm{sc}}(X, {}^{\mathrm{sc}}\Omega^{\frac{1}{2}})$, $\lambda \in \mathbb{R}\backslash 0$. Then in using the coordinates $(x, y; \tau, \mu)$ in ${}^{\mathrm{sc}}T^*X$ near $\partial X = \{x = 0\}$ $(T^*X \ni (x, y; \xi, \eta) \mapsto (x, y; x^2\xi, x\eta) \in {}^{\mathrm{sc}}T^*X)$,

$$\Sigma_{\mathrm{sc}}(\Delta - \lambda^2) = \{(x, y; \tau, \mu) \ : \ \tau^2 + h(y, \mu) - \lambda^2, x = 0\} \subset {}^{\mathrm{sc}}T_{\partial X}X,$$

since $j_{2,0}(\Delta - \lambda^2)|_{{}^{\mathrm{sc}}T^*_{\partial X}X} = \tau^2 + h(y, \mu) - \lambda^2$. Then the scattering Hamilton vector field ${}^{\mathrm{sc}}H^{2,0}_{j_{2,0}(\Delta-\lambda^2)}$ on $\Sigma_{\mathrm{sc}}(\Delta - \lambda^2)$ is

$${}^{\mathrm{sc}}H^{2,0}_{j_{2,0}(\Delta-\lambda^2)}|_{\Sigma_{\mathrm{sc}}(\Delta-\lambda^2)} = 2\tau R_\mu - 2h(y, \mu)\partial_\tau + H_h, \tag{2.2}$$

where $R_\mu = \mu \cdot \partial_\mu$ is the radial vector field and H_h is the boundary Hamilton vector field in y and μ. The radial set has two components

$$R_\pm(\lambda) = \{\mu = 0, \tau = \pm|\lambda|\} \subset {}^{\mathrm{sc}}\bar{T}^*_{\partial X}X \tag{2.3}$$

and the subsets of the characteristic variety, $R_\pm(\lambda)$, $\Sigma_{\mathrm{sc}}(\Delta - \lambda^2) \setminus (R_+(\lambda) \cup R_-(\lambda))$, are closed under the scattering bicharacteristic flow. More precisely we have (see Fig.4):

Proposition 2.3 *For $0 \neq \lambda \in \mathbb{R}$ the integral curves of ${}^{\mathrm{sc}}H^{2,0}_{j_{2,0}(\Delta-\lambda^2)}$ in $\Sigma_{\mathrm{sc}}(\Delta - \lambda^2)$ are the points of $R_\pm(\lambda)$ and the curves of the form*

$$\tau = |\lambda| \cos(s + s_0), \ \mu = |\lambda| \sin(s + s_0)\hat{\mu}, \ (y, \hat{\mu}) = \exp((s + s_0)H_{\frac{1}{2}h})(y', \mu'), \tag{2.4}$$

where $s_0 \in (0, \pi)$, $s \in [-s_0, \pi - s_0]$, $(y', \mu') \in T^*\partial X$, $h(y', \mu') = 1$, $ds/dt = \frac{1}{2}h(y, \mu)$.

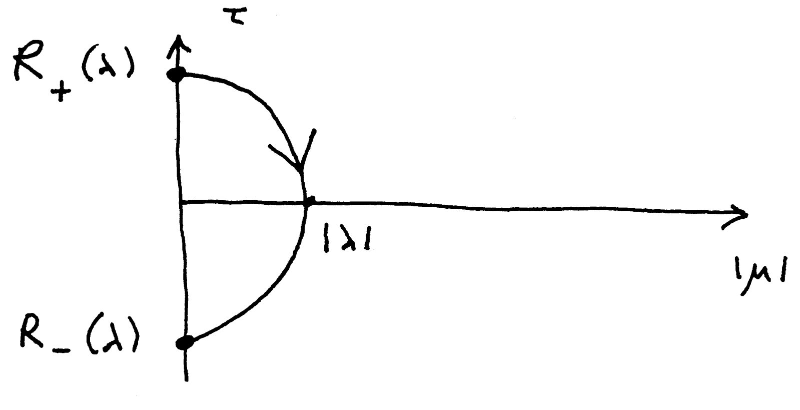

Figure 4: The radial points for the scattering bicharacterstic flow.

Thus the distance on the boundary between radial points on bicharacteristics is π – which is precisely the reason for its appearance in Theorem in Sect.1. The original motivation for the conjecture was more geometric: if a sequence of maximally extended geodesics in X^o approaches ∂X uniformly then it has a subsequence converging to a geodesic on ∂X of length π. This can also be seen from Proposition 2.3.

For propagation results at radial points we refer to Sect.9 of [4] and we will present here only a simple but hopefully explanatory case:

Proposition 2.4 *If $0 \neq \lambda \in \mathbb{R}$ and $u \in \mathcal{D}'(X)$ then for $s < -1/2$*

$$WF_{\mathrm{sc}}^{*,s}(u) \subset R_+(\lambda) \cup R_-(\lambda),\ WF_{\mathrm{sc}}^{*,s+1}((\Delta-\lambda^2)u) \cap R_\pm(\lambda) = \emptyset \implies WF_{\mathrm{sc}}^{*,s}(u) \cap R_\pm(\lambda) = \emptyset, \tag{2.5}$$

and for $s \geq -1/2$

$$WF_{\mathrm{sc}}^{*,-\frac{1}{2}}(u) \cap R_\pm(\lambda) = \emptyset,\ WF_{\mathrm{sc}}^{*,s+1}((\Delta-\lambda^2)u) \cap R_\pm(\lambda) = \emptyset \implies WF_{\mathrm{sc}}^{*,s}(u) \cap R_\pm(\lambda) = \emptyset. \tag{2.6}$$

Thus for slowly decaying solutions of $(\Delta - \lambda^2)u = 0$ the singularities at infinity *cannot* concentrate at radial points and the break-down occurs at $s = -\frac{1}{2}$:

$$WF_{\mathrm{sc}}^{*,s}\left(x^{\frac{n-1}{2}} \exp(\pm i\lambda/x)\right) = \begin{cases} \emptyset, & s < -\frac{1}{2} \\ \{(0, y; \mp\lambda, 0)\}, & s \geq -\frac{1}{2}. \end{cases}$$

Proposition 2.4 is proved using a positive commutator argument and the difference between $s < -1/2$ and $s \geq -1/2$ appears very naturally. In fact, for $B \in \Psi_{\mathrm{sc}}^{-\infty,r}(X, {}^{\mathrm{sc}}\Omega^{\frac{1}{2}})$

$$[\Delta - \lambda^2, B] = -ixC,\ \ j_{-\infty,r}(C) = {}^{\mathrm{sc}}H^{2,0}_{j_{2,0}(\Delta-\lambda^2)}(j_{-\infty,r}(B)).$$

Since to obtain (2.5) we want a positive commutator behaving like x^{-2s}, that is $C \sim x^{-2s-1}$, a simple choice would be $B \sim x^{-2s-1}$ since by noting that the ∂_x compontent of ${}^{\mathrm{sc}}H^{2,0}_{j_{2,0}(\Delta-\lambda^2)}$ is $\tau x \partial_x$

$$j_{-\infty,-2s-1}(C) \sim -\tau(2s+1)x^{-2s-1}.$$

The sign changes at $s = -\frac{1}{2}$ and to control the terms coming from the necessary cut-offs we need $s < -\frac{1}{2}$ unless assumptions about $WF_{\mathrm{sc}}^{*,-\frac{1}{2}}(u)$ at $R_\pm(\lambda)$ are made.

In some sense this is a complicated description, in great generality, of well-known facts from classical scattering, in particular of Sommerfeld radiation conditions. Their microlocal version takes the following form. Let $\dot{C}^\infty(X)$ denote $C^\infty(X)$ functions vanishing to infinite order at ∂X.

Proposition 2.5 *If $0 \neq \lambda \in \mathbb{R}$ and $u \in \mathcal{D}'(X)$ satisfies the* microlocal λ-outgoing *condition:*

$$WF_{\mathrm{sc}}^{*,-\frac{1}{2}}(u) \cap R_+(\lambda) = \emptyset$$

and $(\Delta - \lambda^2)u \in \dot{C}^\infty(X)$ then

$$x^{-\frac{n-1}{2}} \exp(-i\lambda/x)u \in C^\infty(X).$$

Conversely, given $g \in C^\infty(\partial X)$ there exists $w \in C^\infty(X)$ such that $w|_{\partial X} = g$ and $u = x^{(n-1)/2}\exp(i\lambda/x)w$ satisfies $(\Delta - \lambda^2)u \in \dot{C}^\infty(X)$; moreover w is determined by g up to a term in $\dot{C}^\infty(X)$.

This proposition is a key component in the proof of (1.2) which in turn gives the definition of the absolute scattering matrix (1.3). It is interesting to note here the analogy with the Neumann operator for elliptic problems: the scattering matrix relates the boundary data for the solutions of $(\Delta - \lambda^2)u = 0$.

3 An example.

We will now sketch a direct argument for an oversimplified example: $X = \mathbb{R}_+ \times \mathbb{R}^{n-1}$, $g = dx^2/x^4 + dy^2/x^2$, which is a model for a cone (which arises when $\mathbb{R}^{n-1}$ is replaced by a sphere of radius different from 1 – there is no essential difference however except for complexity of computations) – see Fig.5.

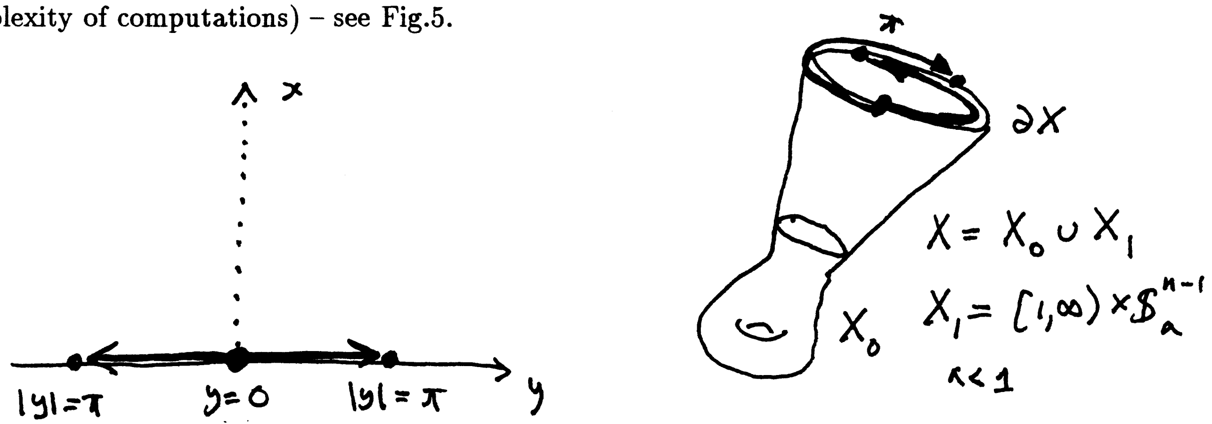

Figure 5: The simple example as a model for the cone.

To understand the microlocal structure of the scattering matrix we need to construct the Poisson operator, $\mathcal{P}_\lambda$, which for f and u in (1.2) is the map

$$\mathcal{P}_\lambda \;:\; \mathcal{C}^\infty(\partial X) \ni f \longmapsto u \in \mathcal{C}^\infty(X^o) \tag{3.1}$$

The scattering matrix is the outgoing boundary value of $\mathcal{P}_\lambda$ in the sense of (1.3). This is practically equivalent to constructing 'plane wave solutions' which, somewhat formally, have δ-functions as their boundary values – see (1.5) for the Euclidean case.

For the model $X = \mathbb{R}_+ \times \mathbb{R}^{n-1}$ the Laplacian is

$$\Delta = (x^2 D_x)^2 + ix(n-1)x^2 D_x + x^2 D_y^2.$$

For simplicity we put $\lambda = 1$ and we want to find the plane wave solutions

$$\begin{cases} (\Delta - 1)u = 0 \\ \lim_{x\to 0} x^{-\frac{n-1}{2}} e^{\frac{i\lambda}{x}} \int u(x,y)\phi(y)dy = \phi(0), \end{cases} \tag{3.2}$$

$\phi \in \mathcal{C}_0^\infty(\mathbb{R}^{n-1})$ supported in a small neighbourhood of 0 – in the boundary condition we only need a spacial localization due to the geometric simplicity of the example.

The 'ansatz' motivated by the Euclidean case is

$$u_1(x,y) = e^{i\Phi(y)/x} a(x,y) \tag{3.3}$$

where Φ satisfies the eikonal equation

$$\Phi^2 + |\partial_y \Phi|^2 - 1 = 0, \;\; \Phi(y) \sim -1 + c|y|^2, \; y \sim 0, \; c \neq 0 \tag{3.4}$$

where the initial condition guarantees the second part of (3.2) for some amplitude a. The solution to (3.4) is immediate: $\Phi(y) = -\cos|y|$ – as we will see in Sect.4 this is a special case of parametrization of scattering Legendrian submanifolds.

If we put $a(x,y) = a_0(y) + xa_1(y) + \cdots$, the first transport equation takes the form

$$(\partial_y \Phi \cdot \partial_y - \Phi(y) x \partial_x + \frac{n-1}{2}\Phi(y) + \frac{1}{2}\partial_y^2 \Phi) a_0 = 0, \tag{3.5}$$

which near 0 is solved by $a_0 = 1 + \mathcal{O}(y)$. However, (3.5) becomes degenerate when $\partial_y \Phi = 0$, that is when $|y| = \pi$ and we need to try another 'ansatz' in place of (3.3). Its systematic explanation will be given in Sect.4 and what works near $|y| = \pi$ is

$$u_1(x,y) = x^{-\frac{1}{2}} \int_0^\infty e^{i\Psi(y,s)/x} b\left(\frac{x}{s}, y, s, x\right) ds, \tag{3.6}$$

where $b \in \mathcal{C}_0^\infty([0,\infty) \times \mathbb{R}^{n-1} \times [0,\infty) \times [0,\infty))$ and Ψ is given by

$$\Psi(y,s) = 1 - f(y)s + \frac{1}{2}s^2, \quad \cos|y| = 1 - \frac{1}{2}f(y)^2, \ |y| \sim \pi, \tag{3.7}$$

that is $f(y) = \pi - |y|$, which is a smooth function near $|y| = \pi$. We note that Ψ also solves an eikonal equation

$$|\Psi|^2 + |\partial_y \Psi|^2 - 1 = 0, \quad \text{when} \quad \partial_s \Psi = 0, \tag{3.8}$$

which now corresponds to a special type of parametrization of a Legendrian submanifold – see Sect.4.

In writing down the transport equation we will use the 'blow-up' coordinates s and $X = x/s$. Putting $\beta(X, y, s) = b(X, y, s, Xs)$ it becomes

$$\begin{cases} (L_0 + sXL_1)\beta(X,y,s) \equiv 0 \bmod (X^\infty) \\ \beta(X,y,s)|_{s=\epsilon} \equiv f(X,y) \bmod (X^\infty) \end{cases} \tag{3.9}$$

where

$$L_0 = s\partial_s - \frac{n-2}{2} - \frac{1}{2}s^2\partial_y, \tag{3.10}$$

L_1 is a second order differential operator in $\partial_y, s\partial_s$ and $X\partial_X$, and the initial condition f comes from matching with the solution for $|y| < \pi$ obtained using (3.3). By writing β and f in power series in X with coefficients depending on y and s we can solve (3.10). The subprincipal term $(n-2)/2$ determines the structure of the solution and in fact we obtain $\beta = s^{(n-2)/2}\tilde{\beta}$ with

$$(L_0 + sXL_1)s^{(n-2)/2}\tilde{\beta}(X,y,s) \equiv 0 \bmod (s^{(n-2)/2}X^\infty) \tag{3.11}$$

Thus we have u_1 of the form (3.3) for $|y| < \pi$ and (3.6) for $|y| \sim \pi$, satisfying the incoming boundary condition in (3.2) and $(\Delta - 1)u_1 = f_1$, where $\chi f_1 \in \mathcal{C}^\infty$ (that is $\mathcal{O}(x^\infty)$), for $\chi \in \mathcal{C}^\infty$ supported away from $|y| = \pi$ and near $|y| = \pi$

$$f_1(x,y) = x^{-\frac{1}{2}} \int_0^\infty e^{i\Psi(y,s)/x} x s^{\frac{n-2}{2}} \alpha\left(\frac{x}{s}, y, s\right) ds, \tag{3.12}$$

where $\alpha \in X^\infty \mathcal{C}_0^\infty([0,\infty) \times \mathbb{R}^{n-1} \times [0,\infty))$. This easily shows that

$$f_1 \in x^{(n+1)/2} \exp(i\lambda/x) \mathcal{C}^\infty(\mathbb{R}_+ \times \mathbb{R}^{n-1})$$

and, by the methods referred to in Sect.2, we can find

$$v_1 \in x^{(n-1)/2} \exp(i\lambda/x) \mathcal{C}^\infty(\mathbb{R}_+ \times \mathbb{R}^{n-1})$$

such that $(\Delta - 1)v_1 = f_1$.

Hence the solution to (3.2) is given by $u = u_1 + v_1$ and to study the singular part of the outgoing boundary data of u we only need to study the u_1 term. Since Φ in (3.3) is non-degenerate for $0 < |y| < \pi$, the outgoing boundary value comes from the term of the form (3.6). Thus for $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^{n-1})$ supported near $|y| = \pi$ we want to investigate the limit

$$\begin{aligned}
&\lim_{x\to 0} e^{-\frac{i\lambda}{x}} x^{-\frac{n-1}{2}} \int_{\mathbb{R}^{n-1}} u_1(x,y)\phi(y)dy \\
&= \lim_{x\to 0} e^{-\frac{i\lambda}{x}} x^{-\frac{n-1}{2}} x^{-\frac{1}{2}} \int_{\mathbb{R}^{n-1}} \int_0^\infty e^{i(1-f(y)s+\frac{1}{2}s^2)/x} s^{\frac{n-2}{2}} \tilde{\beta}\left(\frac{x}{s}, y, s\right) \phi(y)dy \\
&= \lim_{x\to 0} \int_{\mathbb{R}^{n-1}} \int_0^\infty e^{-if(y)\eta + i\frac{1}{2}x\eta^2} \eta^{\frac{n-2}{2}} \tilde{\beta}\left(\frac{1}{\eta}, y, x\eta\right) \phi(y) d\eta dy \\
&= \int_{\mathbb{R}^{n-1}} \int_0^\infty e^{-if(y)\eta} \eta^{\frac{n-2}{2}} \tilde{\beta}\left(\frac{1}{\eta}, y, 0\right) \phi(y) d\eta dy \\
&= \langle T(\delta_0), \phi \rangle,
\end{aligned}$$

where $T \in I^0(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \mathcal{G}'_\pi)$ is a zeroth order Fourier Integral Operator associated to the canonical transformation

$$\mathcal{G}_\pi \ : \ (y, \eta) \longmapsto (y + \pi\eta/|\eta|, \eta).$$

In fact, its kernel is given by

$$\int e^{-i(|x-y|-\pi)\eta} \eta^{\frac{n-2}{2}} \tilde{\beta}\left(\frac{1}{\eta}, x - y, 0\right) d\eta.$$

4 Scattering Legendrian distributions.

To extend the construction presented in Sect.3 to arbitrary X we need to develop a calculus of *scattering Legendrian distributions* which generalize the distributions given by (3.3) and (3.6). In some sense this calculus is analogous to the calculus of distributions associated to cleanly intersecting Lagrangians [7]. Now however one of the Lagrangians will not only have boundary but can also have conic singularities. These singularities occur when the geodesic flow on ∂X has conjugate points – they were not present in the simple case discussed in Sect.3.

We start with a discussion of geometry. Let $(x, y; \tau, \mu)$ be the local coordinates on ${}^{sc}T^*X$ as introduced in Sect.2. The boundary face, ${}^{sc}T^*_{\partial X}X$, of ${}^{sc}T^*X$, has a natural contact structure with the canonical form, ${}^{sc}\chi$, given by the pull-back to ${}^{sc}T^*_{\partial X}X$ of the form

$$d\tau + \mu \cdot dy \tag{4.1}$$

We note that if the boundary defining function is changed to $\tilde{x} = \alpha x$ then the corresponding form, ${}^{sc}\tilde{\chi}$, satisfies ${}^{sc}\tilde{\chi} = \alpha\,{}^{sc}\chi$ and consequently the contact line bundle is completely natural.

We recall that a submanifold is called *Legendrian* if the canonical form vanishes on it and if it has a maximal dimension, in this case dim $X - 1$. A relevant example is given by

$$\begin{aligned}
G_y(\lambda) = \{(y'; \tau, \mu) \in {}^{sc}T^*_{\partial X}X \ : \\
\tau^2 + h(y, \mu) = \lambda^2, \mu \neq 0, \lim_{t\to -\infty} \exp(t^{sc}H^{2,0}_{j_{2,0}(\Delta)})(y', \tau, \mu) = (y; |\lambda|, 0)\}
\end{aligned} \tag{4.2}$$

which is a smooth open Legendrian submanifold. Using Proposition 2.3 we see that

$$\overline{G_y(\lambda)} = G_y(\lambda) \cup \{(y; |\lambda|, 0) : y \in \partial X\} \cup \mathcal{G}_y(\lambda), \tag{4.3}$$

where

$$\mathcal{G}_y(\lambda) = \{(y', |\lambda|, 0) \in {}^{sc}T^*_{\partial X}X \ : \ \exists\, \eta, \eta' \ (y', \eta') = \exp(\pi H_{\frac{1}{2}h})(y, \eta)\}.$$

This closure has at most a conic singularity in μ at occuring at the last term in (4.3). The Legendrian $\overline{G_y(\lambda)}$ was implicit in Sect.3 and the phases $\Phi(y)/x$ and $\Psi(y,s)/x$ in (3.3) and (3.6) respectively parametrized it.

If a Legendrian G is smooth near $(\bar{y}; \bar{\tau}, \bar{\mu}) \in G$ then we say that $\phi(y, u)$ defined near $(\bar{y}, 0) \in \partial X \times \mathbb{R}^k$ parametrizes G near $(\bar{y}; \bar{\tau}, \bar{\mu})$ if

$$\begin{aligned} &(i)\ \phi(\bar{y}, \bar{u}) = -\bar{\tau},\ d_y\phi(\bar{y}, \bar{u}),\ d_u\phi(\bar{y}, \bar{u}) = 0 \\ &(ii)\ d_{y,u}\left(\frac{\partial \phi}{\partial u_j}\right),\ j = 1, \cdots k, \text{ are independent at } (\bar{y}, \bar{\mu}) \\ &(iii)\ C_\phi = \{(y,u) : d_u\phi = 0\} \ni (y,u) \mapsto (y; -\phi, d_y\phi) = (y; \tau, \mu) \end{aligned} \tag{4.4}$$

is a diffeomorphism from a neighbourhood of $(\bar{y}, 0)$ in C_ϕ to a neighbourhood of $(\bar{y}; \bar{\tau}, \bar{\mu}) \in G$. This definition follows of course the standard definition for conic Lagrangians (see Sect.21.2 of [2]) and the existence of $\phi(y, u)$ is obtained in much the same way as in that case. A difference occurs when we allow Legendrians with conic singularities at $\mu = 0$ – as in $\overline{G_y(\lambda)}$ above.

Let G be a Legendrian submanifold which is smooth in polar coordinates at $G \cap \{\mu = 0\}$. That means that in a conic neighbourhood of $(\bar{y}; \bar{\tau}, 0)$, $\mu/|\mu| \sim \hat{\mu}_0$, G is given by

$$\{(\tau, y, \hat{\mu}|\mu|) \ : \ \tau = f(y, \hat{\mu}, |\mu|),\ g_j(y, \hat{\mu}, |\mu|) = 0,\ j = 1, \cdots, n\},\ \ |\mu| = h(y, u), \tag{4.5}$$

where $d_{y,\hat{\mu}}g_j$, $j = 1, \cdots, n$ are independent at the base point $(\bar{y}, \hat{\mu}_0)$. In polar coordinates $(y; \tau, \hat{\mu}, |\mu|)$ we will denote the 'blown-up' Legendrian by $\widehat{G}$. We note that since ${}^{sc}\chi$ given by (4.1), vanishes on G :

$$\tau = \text{ const on } \widehat{G}_0 = \widehat{G} \cap \{|\mu| = 0\}$$

and that

$$\hat{\Lambda} = \{(y, \eta) \ : \ (y; \tau, t\eta, 0) \in \widehat{G}_0,\ t \in \mathbb{R}\} \subset T^*\partial X \setminus 0$$

is a conic Lagrangian. It is in fact the Lagrangian which appeared in the distributional limit $\lim_{x \to 0} e^{-i\lambda/x} x^{-(n-1)/2} u_1(x, y)$ in Sect.3.

By a parametrization of G, or rather, $\widehat{G}$ near $(\bar{y}; \bar{\tau}, \hat{\mu}_0, 0)$ we mean a $\mathcal{C}^\infty$ function $\phi(y, s, u)$ defined near $(\bar{y}; 0, 0) \in \partial X \times [0, \infty) \times \mathbb{R}^k$ such that

$$\begin{aligned} &(i)\ \phi(y, s, u) = -\bar{\tau} + s\psi(y, s, u) \\ &(ii)\ d_{y,s,u}\psi \text{ and } d_{y,s,u}\left(\frac{\partial \psi}{\partial u_j}\right),\ j = 1, \cdots, k, \text{ are independent at } (\bar{y}, 0, 0) \\ &(iii) C_\phi = \{(y, s, u) : \phi'_s = 0, \phi'_u = 0\} \ni (y, s, u) \mapsto (y; -\phi, d_y\phi/|d_y\phi|, |d_y\phi|) \in \widehat{G} \end{aligned} \tag{4.6}$$

is a diffeomorphism onto a neighbourhood of $(\bar{y}; \bar{\tau}, \hat{\mu}_0, 0)$ in $\widehat{G}$.

Again the existence of such parametrizations follows the method familiar in the setting of conic Lagrangians, with modifications due to the singularity. Although it may not be at first apparent, the special form of the phase in *(i)* comes from the fact that it parametrizes *jointly* $\widehat{G}$ and the smooth Legendrian $\{(y; \bar{\tau}, 0)\}$ intersecting $\widehat{G}$ cleanly at $\widehat{G}_0$. We refer to [8] for the discussion of invariance and equivalence of phases – as presented here it is implicit in the definitions below.

In the discussions of spaces of distributions we start with those associated to smooth Legendrian submanifolds (see (3.3) for an example). Thus let $G \subset {}^{\mathrm{sc}}T^*_{\partial X}X$ be a smooth Legendrian submanifold which could be open. Near each point $p \in G$ there exists a local parametrization $\phi \in \mathcal{C}^\infty(U \times \mathbb{R}^k)$, $\pi(p) \in U \subset \partial X$, in the sense of (4.4). The Legendrian distributions of order m defined with respect to this local parametrization, are functions of the form

$$v(x,y) = \int e^{i\phi(y,u)/x} a(x,y,u) x^{m-\frac{k}{2}+\frac{n}{4}} du, \quad a \in \mathcal{C}_0^\infty([0,\epsilon) \times U \times \mathbb{R}^k). \tag{4.7}$$

Definition. *For a smooth Legendrian submanifold $G \subset {}^{\mathrm{sc}}T^*_{\partial X}X$, $u \in I^m_{\mathrm{sc}}(X, G, {}^{\mathrm{sc}}\Omega^{\frac{1}{2}}) \subset \mathcal{D}'(X, {}^{\mathrm{sc}}\Omega^{\frac{1}{2}})$, if for any $\psi \in \mathcal{C}_0^\infty(X)$*

$$\psi u = u_0 + \sum_{j=1}^{N} v_j \nu_j$$

where $u_0 \in \dot{\mathcal{C}}^\infty\ (X, {}^{\mathrm{sc}}\Omega^{\frac{1}{2}})$, $\nu_j \in \mathcal{C}^\infty(X, {}^{\mathrm{sc}}\Omega^{\frac{1}{2}})$ and v_j's are functions of the form (4.7) for local parametrizations of G.

The analogy with Lagrangians distributions of Hörmander (see [2], Sect.25.1) which is clear from (4.7) is in fact exact for smooth Legendrians. As in Sect.2 we present it for $X = \mathbb{S}^n_+$. Thus we define

$${}^{\mathrm{sc}}\mathcal{F} \ : \ \dot{\mathcal{C}}^\infty(\mathbb{S}^n_+, {}^{\mathrm{sc}}\Omega^{\frac{1}{2}}) \longrightarrow \mathcal{S}(\mathbb{R}^n, \Omega^{\frac{1}{2}})$$

as $\mathcal{F} \circ (\mathrm{SP}^{-1})^*$ where $\mathcal{F}$ is the Euclidean Fourier transform acting on half densities. We also define the *Legendre diffeomorphism*:

$$L \ : \ {}^{\mathrm{sc}}T^*_{\partial\mathbb{S}^n_+}\mathbb{S}^n_+ \longrightarrow S^*\mathbb{R}^n, \quad L(\theta,\tau,\mu) = (-\mu + \tau\theta, -\theta).$$

We then have

Proposition 4.1 *If G is a smooth Legendrian submanifold of ${}^{\mathrm{sc}}T^*_{\partial\mathbb{S}^n_+}\mathbb{S}^n_+$ and $\Lambda \subset T^*\mathbb{R}^n \setminus 0$ is a homogeneous Lagrangian submanifold such that $\Lambda \cap S^*\mathbb{R}^n = L(G)$, then the Fourier transform gives an isomorphism*

$${}^{\mathrm{sc}}\mathcal{F} \ : \ I^m_{\mathrm{sc}}(\mathbb{S}^n_+, G, {}^{\mathrm{sc}}\Omega^{\frac{1}{2}}) \longrightarrow I^{-m}(\mathbb{R}^n, \Lambda, \Omega^{\frac{1}{2}}) \cap \mathcal{E}'(\mathbb{R}^n, \Omega^{\frac{1}{2}}) + \mathcal{S}(\mathbb{R}^n, \Omega^{\frac{1}{2}}).$$

Invariance properties and Proposition 4.1 allow a definition of the symbol map following that for Lagrangian distributions. We refer to [8] for the precise and slightly complicated definition recalling only that the natural symbol bundle is

$$E_m(G) = |N^*\partial X|^{m-\frac{n}{4}} \otimes \Omega^{\frac{1}{2}}_G \otimes M_G$$

where M_G is a modified Maslov bundle. We have

Proposition 4.2 *The symbol map*

$$\sigma_{\mathrm{sc},m} \ : \ I^m_{\mathrm{sc}}(X, G, {}^{\mathrm{sc}}\Omega^{\frac{1}{2}}) \longrightarrow \mathcal{C}^\infty(G, E_m(G))$$

gives a short exact sequence

$$0 \longrightarrow I^{m+1}_{\mathrm{sc}}(X, G, {}^{\mathrm{sc}}\Omega^{\frac{1}{2}}) \longrightarrow I^m_{\mathrm{sc}}(X, G, {}^{\mathrm{sc}}\Omega^{\frac{1}{2}}) \longrightarrow \mathcal{C}^\infty(G, E_m(G)) \longrightarrow 0.$$

As we saw in Sect.3 smooth Legendrians are not sufficient for the study of propagation on the boundary at infinity. Let G be a Legendrian submanifold with a conic singularity at $\mu = 0$, that is with a local description given by (4.5). Let $\widehat{G}$ denote the 'blown-up' Legendrian as introduced above. We define a union of intersecting Legendrian submanifolds

$$\widetilde{G} = G \cup \bigcup_{(*;\tau,0)\in\widehat{G}} \operatorname{graph}\left(\tau \cdot \frac{dx}{x^2}\right). \tag{4.8}$$

As we noted already, τ is constant on each component of $\widehat{G} \cap \{|\mu| = 0\}$ so that the second union is over disjoint smooth Legendrians intersecting $\widehat{G}$ (which may repeat in the union).
Definition *For $\widetilde{G}$ given by (4.8) and $m, p \in \mathbb{R}$ satisfying $p - m > \frac{1}{2}$ we define the space $I^{m,p}_{\mathrm{sc}}(X, G, {}^{\mathrm{sc}}\Omega^{\frac{1}{2}})$ as consisting of $u \in \mathcal{D}'(X, {}^{\mathrm{sc}}\Omega^{\frac{1}{2}})$ of the form*

$$u = u_0 + u_1 + \sum_{j=1}^{N} v_j \nu_j$$

where

$$u_0 \in \sum_{(*,\tau,0)\in\widehat{G}} I^p_{\mathrm{sc}}(X, \operatorname{graph}(\tau \cdot dx/x^2), {}^{\mathrm{sc}}\Omega^{\frac{1}{2}}), \quad u_1 \in I^m_{\mathrm{sc}}(X, G^o, {}^{\mathrm{sc}}\Omega^{\frac{1}{2}}),$$

$\nu_j \in \mathcal{C}^\infty(X, {}^{\mathrm{sc}}\Omega^{\frac{1}{2}})$ and v_j 's are of the form

$$v(x,y) = \int_0^\infty \int_{\mathbb{R}^k} e^{i\phi(y,u,s)/x} a\left(\frac{x}{s}, y, s, u, x\right) \left(\frac{x}{s}\right)^{m+\frac{n}{4}-\frac{k+1}{2}} s^{p+\frac{n}{4}-1} ds du, \tag{4.9}$$

where $a \in \mathcal{C}^\infty_0([0,\infty) \times \partial X \times [0,\infty) \times \mathbb{R}^k \times [0,\epsilon))$ and ϕ parametrizes $\widehat{G}$ locally in the sense of (4.6).

The condition $p - m > 1/2$ was imposed to guarantee absolute integrability in (4.9) – otherwise the integral needs to be interpreted as a distribution. In the case of scattering Legendrian distributions there is no simple analogue of Proposition 4.1, unless G is smooth through $\mu = 0$ in which case ${}^{\mathrm{sc}}\mathcal{F}$ maps them into Lagrangian distributions associated to intersecting Lagrangians [7]. Nevertheless, the symbol map can be defined and the subsequent calculus allows a generalization of the procedure from Sect.3 yielding Theorem in Sect.1

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