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GRIGORI V. ROZENBLUM

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Index Formulas for Pseudodifferential Operators with Discontinuous Symbols

G.V.Rozenblum
St.Petersburg University for Telecommunications*,
Aalborg University

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The famous Atiyah-Singer theorem expresses the index of elliptic pseudodifferential operators in the terms of some topological objects connected with the symbol of the operator. However this theoren requires the symbol to be continuous. The case of discontinuous symbols may be sometimes treated as a pseudodifferential boundary value problem however usually some additional conditions appear, in particular, the transmission property (cf.[1]). In this paper we present an approach for obtaining index formulas for zero order pseudodifferential operators with discontinuous symbols without transmission property being imposed. The approach is based on the study of C^*-algebra generated by such operators and uses some ideas and methods in operator algebra K-theory. It was originated in the paper [2] by B.A.Plamenevsky and the author. The details and proofs may be found in [3].

Let X be a compact n-dimensional manifold without boundary, Y be an m-dimensional smooth submanifold, $0 \leq m < n$. The symbol $a(x, \xi)$ is supposed to be a smooth zero order positively homogeneous function on the cotangent bundle $T^0(X \setminus Y)$ (with zero section removed) over $X \setminus Y$. At Y the symbol may have discontinuities: in local coordinates $x = (y, z), y \in Y$ the function $a(x, \xi)$ has limits when $z \to 0$ but these limits may depend on the direction from which z approaches 0:

$$\Phi(y, \phi, \xi) = \lim_{t \to 0} a(y, t\phi, \xi), (y, \phi) \in SN(Y), \tag{1}$$

where $SN(Y)$ is the spheric normal bundle to Y. We suppose that the limits in (1) and in what follows are uniform with respect to all variables. To symbols satisfying (1) one associates zero order pseudodifferential operators in the usual

*On leave from August, 1993, to July, 1994 way, by means of partitions of unity on X and the Fourier transform; as in the smooth case, the resulting operator does not depend, modulo compact operators, on the choice of partitions of unity and local charts. We denote such operator by $OPS_X a(x,\xi)$. In an obvious way definition of our class of symbols and of operators is carried over to matrix symbols and to operators acting on vector-functions on X and also to the case of bundles over X.

Operators with such symbols do not behave themselves in an usual way under multiplication or under taking the adjoint operator; for example,

$$OPS_X a OPS_X b - OPS_X(ab) \tag{2}$$

is not a compact operator. This is, in particular, the reason why invertibility of the symbol a is not sufficient for the operator to be Fredholm in $L_2(X)$. As usual, the natural way to deal with this difficulty is to consider the C^*-algebra generated with such operators; then factoring this algebla by the ideal $\mathcal{K}$ of compact operators in L_2 we get an algebra which is natural to regard as the algebra of symbols, and these symbols describe properties of operators. Roughly speaking, to do this, one must add various operators of the form (2).The corersponding study, from the point of view of representation theory, was made in [4].

As a result of this operation, one obtains the algebra $\boldsymbol{S}$ of symbols consisting of pairs $(a,\mathcal{A})$ where a is a continuous function on $S^*(X\setminus Y)$ (this means having limits of the form (1)) and $\mathcal{A}$ is a function on T^*Y with values being bounded operators in $L_2(R^d), d=n-m$. We describe here some dense subalgebra of symbols; the whole algebra is obtained by closing in the norm of $C(S^*(X\setminus Y))\times C(T^*Y, \mathcal{B}L_2(R^d))$. The operator part of the symbol must be skew-homogeneous:

$$\mathcal{A}(y,t\eta) = U_t^{-1}\mathcal{A}(y,\eta)U_t, \tag{3}$$

where U_t is the unitary group of dilations in R^d. When $\eta\to 0$ the operator has a limit $\mathcal{A}(y,0)=\lim\mathcal{A}(y,\eta)$ in a rather weak sense:

$$\|(\mathcal{A}(y,\eta)-\mathcal{A}(y,0))\chi\|\to 0, \eta\to 0, \chi\in C_0^\infty. \tag{4}$$

The limiting operator $\mathcal{A}(y,0)$ must be invariant with respect to dilations and

$$(\mathcal{A}(y,\eta_1)-\mathcal{A}(y,\eta_2))\chi\in\mathcal{K}, \chi\in C_0^\infty, \eta_j\neq 0.$$

We shall call the function a the scalar symbol and $\mathcal{A}$ the operator symbol and the pair $\Xi=(a,\mathcal{A})$ is the complete symbol. The same terms will be used while considering matrix situation. Scalar and operator symbols must be consistent. The consistency condition can be written explicitly only for some subalgebra of 'smooth' symbols. This condition arises from the requirement that the operator on X glued together by means of a partition of unity from the usual pseudodifferential operator on $X\setminus Y$ with symbol a and from the pseudodifferential operator in $L_2(N(Y))$ with operator-valued symbol $\mathcal{A}$ does not depend (modulo compacts) on the choice of the partition of unity. In the explicit form the consistency condition requires that

$$(\mathcal{A}(y,0) - OPS_z\Phi(y, z/|z|, 0, \tau))\zeta \in \mathcal{K}, \zeta \in C_0^\infty(R^d \setminus 0), y \in Y, \tag{5}$$

and in addition

$$(\mathcal{A}(y,\eta) - OPS_z\Phi(y, z/|z|, \eta, \tau))\zeta' \in \mathcal{K}, (y,\eta) \in T^*Y, \tag{6}$$

with any function ζ' which equals zero near the origin and is unity outside some compact (here OPS_z denotes pseudodifferential operator in $L_2(R^d)$ with respect to z variable).

Remarks.1. In the special case of zero-dimensional discontinuity manifold Y the operator symbol is just a single dilation-invariant operator for each discontinuity point.

2. To be more precise, some conditions concerning behavior of the operator symbol under differentiation with respect to η and y must be added, guaranteeing independence, up to a compact operator, of the resulting pseudodifferential operator on X on the choice of the partition of unity along Y, in other words, guaranteeing some sort of operator pseudo-locality. The main difficulty here lies in the fact that the operator symbols are not smooth at $\eta = 0$ and, generally, the usual for scalar case operation of smoothing the symbol at this point by multiplying by of some cut-off function does not fit here since such a cut-off leads to a non-compact perturbation of the resulting pseudodifferential operator on X. Exact conditions, implying a special sort of calculus of operator-valued pseudodifferential operators, are given in [3,5]. These conditions are not preserved under closure of the above dense subulgebra.

Denote by $\boldsymbol{S}$ the algebra of complete symbols. Algebraic operations in $\boldsymbol{S}$ are component-wise and the metrics is $C(S^*(X \setminus Y)) \times C(T^*Y, \mathcal{B}L_2(R^d))$.

Theorem 1[4]. For the operator with complete (matrix) symbol $\Xi = (a, \mathcal{A}) \in M(\boldsymbol{S})$ to be Fredholm it is necessary and sufficient that Ξ be invertible in $\boldsymbol{S}$, in other words, that both 'scalar symbol' a and operator symbol $\mathcal{A}$ be invertible (this is usually denoted by $\Xi \in GL(\boldsymbol{S})$ where the latter is the linear group over $\boldsymbol{S}$).

The previous remark about subtlery in calculus of operator symbols is reflected in particular in this theorem: invertibility of the operator symbol is required not only for η large enough, as the analogy with the scalar pseudodifferential calculus would invite, but for all η, including $\eta = 0$.

The general method for finding the index for operators with symbols in a given algebra consists in performing a (stable) homotopy of the symbol to some symbol having a canonical (more simple form). This, actually, was the way of treating pseudodifferential boundary balue problems in [1]. In our case such a form will consist of several components. Two of them will correspond to a special ideal in the algebra $\boldsymbol{S}$.

Consider the commutator ideal in the algebra $\boldsymbol{S}$: $\boldsymbol{J} = [\boldsymbol{S}, \boldsymbol{S}]$. Since scalar parts of the symbol form a commutative algebra , elements in $\boldsymbol{J}$ have zero scalar symbol and the operator symbol $\mathcal{A}$ must be consistent with the zero scalar symbol. Explicit description of symbols $(0, \mathcal{R}) \in \boldsymbol{J}$ may be given only on a dense set: the operator-valued function $\mathcal{R}$ must be skew-homogeneous in the sense of (3),

$$\mathcal{R}(\dagger, \eta)\zeta' \in \mathcal{K}, \eta \neq /, \tag{7}$$

with ζ' as in (5), there is convergence as in (4), the operator $\mathcal{R}$ is dilation-invariant, the operator $\mathcal{R}(\dagger, /)\zeta$ is compact for $\zeta \in C_0^\infty(R^d \setminus 0)$ and, finally,

$$\|\mathcal{R}(\dagger, \eta)\zeta'_\sqcup\| \to /, \sqcup \to \infty,$$

with $\zeta'_t(z) = \zeta'(z/t)$ and ζ' as in (7).

The properties given above are simple consequencies of usual formulas of commutation of pseudodifferential operators (even with discontinuous symbols) and multiplication by smooth functions. They imply, im particular, that the difference $\mathcal{R}(\dagger, \eta_\infty) - \mathcal{R}(\dagger, \eta_\in)$ is a compact operator for any $\eta_1, \eta_2 \neq 0$. In what follows we omit the first (zero) component of the symbol $(0, \mathcal{R}) \in \boldsymbol{J}$ and write simply $\mathcal{R} \in \boldsymbol{J}$. As an example of symbols in $\boldsymbol{J}$ may serve the pseudodifferential operator

$$(\mathcal{R}(\dagger, \eta)\sqsubseteq)(\ddagger) = \mathcal{F}^{-\infty}_{\tau \to \ddagger}\nabla(\dagger, \ddagger, \eta, \tau)\mathcal{F}_{\ddagger' \to \tau}\sqsubseteq(\ddagger')$$

with symbol $r(y, z, \eta, \tau) = |z|^{-1/2}(|\eta|^2 + |\tau|^2)^{-1/4}$ in R^1. This example shows that the elements in $\boldsymbol{J}$ are natural to be considered as symbols of lower order. In fact, being generated by symbols of the form (2), they have negative differential order. Nevertheless, pseudodifferential operators on X defined by such symbols are not compact (the singularity at $z = 0$ prevents it) and being added to some Fredholm operator, these operators may change the index. Therefore it's natural to understand symbols in $\boldsymbol{J}$ as a generalization of singular Green operators in Boutet-de-Monvel calculus.

As it is usually made in the case of algebras without unit, by $GL(\boldsymbol{J})$ we denote the set of invertible matrices of the form $I + \mathcal{R}$ where I is a unit matrix and $\mathcal{R}$ is a matrix with elements in $\boldsymbol{J}$.

The index of pseudodifferential operators on X with symbols in $GL(\boldsymbol{J})$ is found in the following way. Let ζ be a function in $C_0^\infty(R^d)$ which equals 1 near the origin. For $I + \mathcal{R} \in \mathcal{GL}(\boldsymbol{J})$ we consider the operator-valued symbol

$$(I + \mathcal{B}(\dagger, \eta)) = (\mathcal{I} + \mathcal{R}(\dagger, /))^{-\infty}(\mathcal{I} + \mathcal{R}(\dagger, \eta))\zeta + (\infty - \zeta).$$

According to the properties of the ideal $\boldsymbol{J}$, the operator-function $\mathcal{B}$ assumes compact operators as its values and the operator $I + \mathcal{B}(y, \eta)$ is invertible for η large enough. In this way this function defines an element in the group $K_1(C(S^*Y) \otimes \mathcal{K})$ (in the sense of operator algebra K-theory); this group is isomorphic to $K_1(C(S^*Y)) = K^1(S^*Y)$ (where the latter group is the ordinary K-group in topology). We denote corresponding element in $K^1(S^*Y)$ by $d(I +$

$\mathcal{R}$). Let $\partial : K^1(S^*Y) \to K^0(B^*Y, S^*Y)$ be the connecting homomorphism in the exact K-theoretical sequence of the pair $K^1(S^*Y)$ and $i_{AS} : K^0(B^*Y, S^*Y) \to \boldsymbol{Z}$ be the classical Atiyah-Singer index homomorphism.

Theorem 2. The index of the pseudodifferential operator $1 + R$ on X corresponding to the operator symbol $I + \mathcal{R}$ equals

$$\operatorname{ind}(1 + R) = i_{AS}\partial d(I + \mathcal{R}).$$

This formula calculates the index for a class of operator-valued pseudodifferential operators. It may be explained in the following way. Having a compact-valued function $\mathcal{B}(y, \eta)$ on S^*Y with invertible $I + \mathcal{B}$, one approximates it by a finite-rank operator function $\mathcal{B}_\epsilon(y, \eta)$ in such a way that the range of $\mathcal{B}_\epsilon(y, \eta)$ is contained for all $(y, \eta) \in S^*Y$ in a fixed finite-dimensional subspace and the norm of $\mathcal{B}_\epsilon(y, \eta) - \mathcal{B}(y, \eta)$ is small enough. Then the index of the operator $1 + R$ equals the index of usual pseudodifferential operator on Y with symbol $\mathcal{B}_\epsilon(y, \eta)$.

The formula must be changed in the case of isolated singularity x^0, i.e. $m = 0, d = n$. In this case (where only the operator $\mathcal{R}(x^0, 0)$ in R^n exists and $I+\mathcal{R}$ is invertible) the operator $I+\mathcal{R}$ defines, by means of the Mellin transform, a Toeplitz operator with operator-valued symbol. For such operators the formula for the index was found in [2] by B.Plamenevsky and the author.

The second component in our special form will be some smooth symbol on X, for which the index may be found by Atiyah-Singer formula immediately. Denote by $\boldsymbol{I}_X$ the set of symbols $(a, \mathcal{A})$ where a is a continuous function on S^*X which vanish on Y, with zero operator symbols (the consistency conditions are fulfilled in an obvious way). This set forms an ideal in $\boldsymbol{S}$ and elements in $GL(\boldsymbol{I}_X)$ describe invertible matrix symbols on X which are unit matrices on Y. There's no problem in finding the index for corresponding operators.

The last of our terms will be a special discontinuous matrix symbol on $X \setminus Y$ with an operator symbol canonically corresponding to it . This construction is by far performed in three cases: $m = 0, m = n - 1$ and $m = 1$.

We shall study what are the forms to which limit values of "scalar parts" of symbols in $GL(\boldsymbol{S})$ may be transformed by means of a stable homotopy in $GL(\boldsymbol{S})$. To do this, we consider the ideal $\boldsymbol{I} = \boldsymbol{J} \oplus \boldsymbol{I}_X$ in $\boldsymbol{S}$. To this ideal there corresponds the exact sequense of K-groups of C^*-algebras :

$$\cdots K_1(\boldsymbol{I}) \xrightarrow{j_*} K_1(\boldsymbol{S}) \xrightarrow{p_*} K_1(\boldsymbol{B}) \cdots \tag{8}$$

The sense of the homomorphisms j_*, p_* here is the following. The homomorphism j_* consists in considering an invertible symbol in the ideal $\boldsymbol{I}$ as a symbol in the whole algebra $\boldsymbol{S}$. The homomorphism p_* assoiciates to the symbol in $GL(\boldsymbol{S})$ the limit values of the scalar part of the symbol at the manifold Y. The algebra $\boldsymbol{B}$ is the factor-algebra $\boldsymbol{B} = \boldsymbol{S}/\boldsymbol{I}$ and it coincides with the algebra of continuous functions on the manifold $M = S^*X|_Y + SNY$ which is the Whitney sum of the cospheric bundle over X restricted to Y and of the spheric normal bundle to Y.

Suppose we are so lucky that the range Q of p_* in $K_1(\boldsymbol{B})$ is zero. This would imply, due to exactness of the sequence (8), that j_* is a surjection, which means that any symbol in $GL(\boldsymbol{S})$ is stable homotopic to a symbol in $GL(\boldsymbol{I})$, in other words, to the direct sum of invertible symbols in $\boldsymbol{J}$ and $\boldsymbol{I}_X$. For both terms here the index is already known which gives us the index for original symbol.Otherwise, if Q is not zero, we may find some 'good' representatives for elements in Q and corresponding 'good' representatives in preimages of these elements in $K_1(\boldsymbol{S})$. 'Good' here means that for such discontinuous symbols one can still calculate the index.

So the question arises, on description of elements in Q which is a subgroup in, possibly rather large, group $K_1(\boldsymbol{B})$.

Case $m = 0$.

Theorem 3. The subgroup Q is trivial for n odd. If n is even, Q has no more than two generators for each of points of discontinuity. Any element in Q has a representative, a matrix-function on M which has near each of discontinuity points the block-diagonal form

$$\Phi(\phi, \xi) = \operatorname{diag}(\Phi_1(\xi), \Phi_2(\phi)). \tag{9}$$

In other words, in the odd-dimensional case any discontinuity is homotopically trivial, it may be removed by means of a stable homotopy in $GL(\boldsymbol{S})$. In the even-dimensional case there may exist homotopically nontrivial discontinuities, but they can be transformed near Y to the block-diagonal form (9), where one block, Φ_1, is a continuous symbol and the other one, Φ_2, is discontinuous, but it corresponds to the matrix multiplication, containing "no differentiation".

We denote by LD the set of invertible matrix functions $a(x, \xi)$ on $S^*(X \setminus Y)$ having near Y the form (9). It's clear that the operator symbol $\mathcal{A} = OPS_z\Phi(\phi, \xi)$ is an invertible operator in $L_2(R^n)$, so the pair $(a, \mathcal{A})$ represents an element in $GL(\boldsymbol{S})$. Denote by $\boldsymbol{S}_{LD}$ the subset in $GL(\boldsymbol{S})$ obtained by this way from elements in LD. Theorem 3 , together with exactness of the sequence (8), implies that any elliptic matrix symbol $\Xi \in GL(\boldsymbol{S})$ can be transformed by a stable homotopy to the direct sum $\Xi_0 \oplus \Xi_1$ where $\Xi_0 \in GL(\boldsymbol{I})$ and $\Xi_1 \in \boldsymbol{S}_{LD}$ (and this homotopy may be found explicitly).

So, in order to find the index for Ξ we have to find the index for $\Xi_1 = (a, \mathcal{A}) \in \boldsymbol{S}_{LD}$. This is achieved by the following surgery. Let $\mathcal{E}$ be the (trivial) vector bundle over X where the symbol a acts and $\operatorname{diag}(\Phi_1(\xi), \Phi_2(\phi))$ be the block-diagonal form of the symbol a near a point $x^0 \in Y$. Cut a small hole D out of X around x^0 and glue together along ∂D the bundles $\mathcal{E}|_{\mathcal{D}}$ and $\mathcal{E}|_{\mathcal{X}\setminus\mathcal{D}}$ by means of the transition map $\operatorname{diag}(1, \Phi_2(\phi))$. We obtain some other, in general, nontrivial bundle $\mathcal{E}'$ over X. Define the new symbol $a'(x, \xi) = \operatorname{diag}(\Phi_1(\xi), 1) \in \operatorname{Hom}(\pi^*\mathcal{E}'|_D, \pi^*\mathcal{E}|_D)$ (where π is the natural projection $\pi : T^*X \to X$) and $a'(x, \xi) = a(x, \xi)$ over $X \setminus D$. Then a' becomes a continuous symbol in $\operatorname{Hom}(\pi^*\mathcal{E}', \pi^*\mathcal{E})$.

Theorem 4. The index of the operator with symbol $(a, \mathcal{A}) \in \boldsymbol{S}_{LD}$ equals the index of usual pseudodifferential operator on X with symbol a'.

Thus the task of finding the index for the operator with a discontinuous symbol is reduced to finding the index for a certain continuous symbol (however acting in different bundles) and the latter question is again answered by the Atiyah-Singer formulas.

Case $m = n - 1$. Here the algebra $\boldsymbol{B}$ of limit values of the scalar symbol consists of pairs $\Phi(y, \phi, \xi) = (\Phi(y, +1, \xi), \Phi(y, -1, \xi))$ corresponding to $\phi = \pm 1$ i.e. to two sides of the manifold Y.

Theorem 5. Suppose that the Euler characteristic of the manifold Y equals 0. Then $Q = p_* K_1(\boldsymbol{S}) \subset K_1(\boldsymbol{B})$ contains stable homotopy classes only of those matrices $\Phi(y, \phi, \xi)$ which have block-diagonal form

$$\Phi(y, \phi, \xi) = \operatorname{diag}(\Phi_1(y, \xi), \Phi_2(y, \phi)).$$

We have here the situation similar to Theorem 3. Elliptic symbols are stable homotopic near Y to direct sum of two symbols, of which one is continuous on Y , $\Phi_1(y, +1, \xi) = \Phi_1(y, -1, \xi)$ and the other one may be discontinuous but it does not depend on ξ, i.e. the corresponding pseudodifferential operator becames simply multiplication by a matrix.

We make here some additional explanations. For the pair of invertible matrices $(\Phi(y, +1, \xi),\ \Phi(y, -1, \xi))$ to represent an element in Q it is necessary and sufficient that they are limit values of some invertible symbol in $GL(\boldsymbol{S})$. This leads to two types of conditions. The first one, connected with the global topology of X and Y, requires that Φ can be continuously prolongated from $S^*X|_Y + SN(Y)$ to $S^*(X \setminus Y)$ as an invertible matrix. We do not discuss this purely topological condition here. The second condition requires that there must exist an inverible operator symbol on T^*Y, consistent with Φ. It is this condition which implies Theorem 5.

Now, similar to the case $m = 0$, we define classes LD and $\boldsymbol{S}_{LD}$ and come to the problem of finding the index for symbols in $\boldsymbol{S}_{LD}$. This is achieved by an analogous surgery.

Cut the manifold X (and the bundle $\mathcal{E}$) along Y and glue together along Y the new boundary of $\mathcal{E}|_{X \setminus Y}$ by means of the transition matrix $\operatorname{diag}(1, \Phi_2(y, +1)$ $\Phi_2(y, -1)^{-1})$. This produces a new bundle $\mathcal{E}'$ over X such that the symbol $a(x, \xi)$, considered now as a symbol $a'(x, \xi) \in \operatorname{Hom}(\pi^*\mathcal{E}', \pi^*\mathcal{E})$ becomes continuous and the index of the corresponding operator may be found by means of the Atiyah-Singer formula.

Theorem 6. Under the conditions of Theorem 5 the index of the pseudodifferential operator with symbol $(a, \mathcal{A}) \in \boldsymbol{S}_{LD}$ equals the index of the operator with the continuous symbol a'.

The vanishing condition for the Euler characteristic of Y plays a technical part in the proof of Theorem 5 and is not actually restricting. It is always satisfied, in particular, if the dimension of the manifold X is even (so that $m = \dim Y$ is odd). If $\dim X$ is odd we can apply the usual procedure of tensor multiplication with a certain pseudodifferential operator on the circle, thus obtaining a new operator on $X \times S^1$, with symbol having discontinuity along $Y \times S^1$.

Remark. In the cases considered here the decomposition of the symbol into three parts and corresponding decomposition of the index into three terms is not canonical. However it is possible to associate to the symbol $\Xi \in GL(S)$ an invariant (with respect to the arbitrariness in construction) object, a class in the K-homological group $K_0(X)$ which is a natural generalization of the topological index of the symbol. On the other hand, Fredholm pseudodifferential operators with symbols in $GL(S)$ are abstract elliptic operators on X, i.e. their commutants with multiplications with continuous functions are compact operators. An equivalence class of such operators defines a class in $K_0(X)$ which is called the analytical index of the operator (cf.[6]). It is proved that for both cases above the analytical index of the operator coincides with the topological index of the symbol. This fact generalizes the K-theoretical form of the Atiyah-Singer index theorem.

Case $m = 1$.

In this case, proving to be more complicated than the previous ones, it turns out to be impossible to reduce the index problem completely to the repeated use of the Atiyah-Singer theorem and we have to apply less general results. The dimension of Y equals 1 so Y is a circle.

Theorem 7. Let n be even. Then $Q = p_*(K_1(S)) \subset K_1(B)$ contains stable homotopy classes only of such matrices which have block-diagonal form $\Phi = \mathrm{diag}(\Phi_1(y,\xi), \Phi_2(y,\phi), \Phi_3(\phi,\xi))$.

Thus, in addition to two terms we already encountered, there may appear a term of the form $\Phi_3(\phi,\xi)$, i.e. a matrix not depending on the coordinate y along Y. One can have more information on the matrix $\Phi_3(\phi,\xi)$: there exists a special matrix $\Phi_0(\phi,\xi)$ such that any matrix Φ_3 is the direct sum of several copies of Φ_0 or of Φ_0^{-1}. In other words, "bad" subgroup in Q not admitting splitting in the sense of Theorems 3 and 5 is one-dimensional.

The same sort of surgery, as before, disposes of the discontinuity caused by Φ_2. The index formula for the discontinuity only of the form Φ_3 is obtained in the form, similar to one found by Fedosov [7] for the case of continuous symbols.

After suitable homotopy, we can assume that the symbol $a(x,\xi)$ having a discontinuity of the form Φ_3 is zero order positively homogeneous near Y in the variable z normal to Y. We suppose also that a is smooth for $x \in X \setminus Y, \xi \neq 0$.

Recalling Fedosov's formula, we must consider here not only homogeneous symbols but formal λ -symbols:

$$\tilde{a}(x,\xi,\lambda) \sim \sum_{k=0}^{-\infty} a_k(x,\xi)\lambda^k$$

where $a_k(x,\xi)$ is homogeneous in ξ with order k for ξ large; the symbols must be smooth for all $(x,\xi) \in T^*(X \setminus Y)$ (including the zero section). Such a λ-symbol may be obtained from a given symbol $a \in C^\infty(S^*(X \setminus Y))$ by smoothing it near $\xi = 0$ and by augmenting by some terms $a_k(x,\xi)\lambda^k, k < 0$. These lower-order terms will not be invariant objects on $T^*(X \setminus Y)$ but being defined in coordinate neighbouhoods, they may be chosen in such way that the λ–symbols are consistent with respect to the passage from one neighbourhood to the other. The formulas for change of variables and for compositions for λ-symbols have the same form as in the usual theory of pseudodifferential operators, only with differentiation with respect to ξ replaced by $\lambda^{-1}\partial_\xi$.

In our case of discontinuous symbols it is important to note that while augmenting the symbol a smoothed near the zero section by lower order symbols we get lower order terms having singularities near Y since their construction involves differentiation with respect to x.

Fedosov's expression for the index for pseudodifferential operators [7] has the form

$$\mathrm{ind}A = (2\pi)^{-n} \int_{T^*X} h(x,\xi)dxd\xi \tag{10}$$

where $h(x,\xi) = (tr(\tilde{a} \circ \tilde{r} - 1)_{-n} - tr(\tilde{r} \circ \tilde{a} - 1)_{-n})|_{\lambda=1}$, $\tilde{r}$ is the parametrix for the λ-symbol $\tilde{a}$ in the calculus of λ-symbols, the subscript $-n$ means that one has to take the term with λ^{-n} in the formal compositions of symbols and tr denotes the matrix trace.

In our discontinuous case the formula (10) cannot be applied directly since, due to the singularity in lower-order terms, the integral would have a logarithmic divergence. However, let us fix a global coordinate system in X near the circle Y and define the integral

$$J(\epsilon) = (2\pi)^{-n} \int_{T^*X \setminus \{|z|<\epsilon\}} h(x,\xi)dxd\xi.$$

This integral has an expansion $J(\epsilon) = J_{-1}\log\epsilon + J_0 + J_1\epsilon + \cdots$ as $\epsilon \to 0$.

Theorem 9. Suppose that $(a, \mathcal{A}) \in \boldsymbol{S}_{LD}$ and its discontinuous part is zero order homogeneous in z near Y. Then the term J_0 does not depend on the choice of coordinates and $\mathrm{ind}A = J_0$.

Remark. For the case $\dim Y = 0$ a similar result, with the integral of the type (10) being understood in the sense of principal value, was obtained in [8].

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Dept. Math., Aalborg University, Fredrik Bajers Vej, 7E, Aalborg, 9220, Denmark, *e-mail greg@iesd.auc.dk*