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DISCRETE SPECTRUM OF THE PERIODIC ELLIPTIC OPERATOR WITH A DIFFERENTIAL PERTURBATION

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Let A be an elliptic operator of the second order with periodic coefficients in $L_2(\mathbb{R}^d)$, $d \geq 2$. We study its perturbation by a non-negative operator of the second order with decreasing coefficients. We discuss the asymptotics (in large coupling constant) for the discrete spectrum of the perturbed operator, appearing in the spectral gaps of the operator A. Similar questions have been analysed recently in the paper [1]. The purpose of the present note is to answer a question of professor P.A. Deift which arose during our discussion of the work [1] (see subsection 4 below).

1. Let a be a posive real $(d \times d)$-matrix-valued function, p be a real-valued function; $a, p \in L_\infty(\mathbb{R}^d)$,

$$a(x+n) = a(x), \quad p(x+n) = p(x), \quad n \in \mathbb{Z}^d.$$

Introduce in $L_2(\mathbb{R}^d)$, $d \geq 2$ the quadratic form

$$\mathfrak{A}[u] = \int (a(x)\nabla u \overline{\nabla u} + p(x)|u(x)|^2)dx, \quad u \in H^1(\mathbb{R}^d). \tag{1}$$

The form (1) is semi-bounded from below and closed. The associated selfadjoint operator in $L_2(\mathbb{R}^d)$ is an elliptic operator of divergence form:

$$A = \nabla^* a \nabla + pu.$$

The spectrum of A has a band structure and, as is well known, can be described in the following way. Let $\mathbb{Q}^d \subset \mathbb{R}^d$ be a unit cube, $\mathbb{T}^d$ be a flat torus. Define the family of forms

$$\mathfrak{A}_\xi \langle u \rangle = \int_{\mathbb{Q}^d} (a\nabla u \overline{\nabla u} + p|u|^2)dx, \quad u \in H^1(\mathbb{R}^d),$$
$$u(x)\exp(-ix\xi) = u(x+n)\exp(-i(x+n)\xi), \quad \xi \in \mathbb{T}^d.$$

The form $\mathfrak{A}_\xi$ generates an elliptic "quasi-periodic" operator $A(\xi)$ in $L_2(\mathbb{Q}^d)$ with purely discrete spectrum. Denote by

$$E_1(\xi) \leq \cdots \leq E_r(\xi) \leq \ldots$$

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its consecutive eigenvalues and by

$$\psi_l(\xi, x), \quad l = 1, \dots, r, \dots, \qquad \int_{\mathbb{Q}^d} |\psi_l(x)|^2 dx = 1,$$

associated normalized eigenfunctions. The spectrum $s(A)$ of A consists of intervals (bands), which are images of the continuous mappings $E_l : \mathbb{T}^d \to \mathbb{R}$.

The set $s(A)$ may have gaps. Below we fix one of those gaps $\Lambda = (\lambda_-, \lambda_+)$, where

$$\max_\xi E_r(\xi) = \lambda_-, \quad \min_\xi E_{r+1}(\xi) = \lambda_+.$$

2. Let now $v \in L_\infty(\mathbb{R}^d)$ be a real $(d \times d)$-matrix-valued function, $v(x) \geq 0$ and the asymptotic relation hold:

$$v(x) \sim g(x) b(x), \quad |x| \to \infty,$$

where the factor g is scalar,

$$g(x) = F(\theta)|x|^{-2\sigma}, \quad F(\theta) \geq 0, \; \theta = x/|x|, \; \sigma > 0,$$

and $b(x) \geq 0$ is a real matrix-valued function,

$$b(x+n) = b(x), \quad n \in \mathbb{Z}^d.$$

The perturbed operator

$$A(\alpha) = A + \alpha V, \quad V = \nabla^* v \nabla, \quad \alpha > 0,$$

is defined as a sum of quadratic forms.

In the gap Λ the operator $A(\alpha)$ has discrete spectrum, which can accumulate at the point λ_- only (generically, for $\sigma > 1$ there is no accumulation). When a increases, the eigenvalues of $A(\alpha)$ do not decrease.

Let us fix an "observation point" λ, $\lambda_- \leq \lambda < \lambda_+$. Denote by $N(\alpha, \lambda)$ the number of eigenvalues of the operator $A(t)$ crossing the point λ while t grows from 0 to α.

3. We study the asymptotics of $N(\alpha, \lambda)$ as $\alpha \to \infty$. Let us fix notation. Set for $l, m = 1, \dots, r$

$$\beta_{lm}(\xi) = \int_{\mathbb{Q}^d} b(x) \nabla \psi_l(\xi, x) \overline{\nabla \psi_m(\xi, x)} dx, \tag{2}$$
$$\gamma_{lm}(\xi) = \beta_{lm}(\xi) \big(\lambda - E_l(\xi)\big)^{-1/2} \big(\lambda - E_m(\xi)\big)^{-1/2},$$

and define the matrix

$$\Gamma(\xi) = \{\gamma_{lm}(\xi)\}_{l,m=1}^r.$$

Let

$$\varkappa = \frac{d}{2\sigma}.$$

The main result of the paper is given by the following

Theorem.

(1) *Let $\lambda \in \Lambda$. Then*

$$N(\alpha,\lambda) \sim \frac{\alpha^{\varkappa}}{d(2\pi)^d} \int_{\mathbb{T}^d} (\operatorname{tr}\Gamma(\xi)^{\varkappa}) d\xi \int_{\mathbb{S}^{d-1}} F(\theta)^{\varkappa} dS(\theta), \ \alpha \to \infty. \tag{4}$$

(2) *Let $d \geq 3$ and $\sigma > 1$. Assume also that* $\max_\xi E_{r-1}(\xi) < \lambda_-$ *and the equality $E_r(\lambda) = \lambda_-$ occurs at finitely many points $\xi \in \mathbb{T}^d$ each of which is a point of non-degenerate maximum. Then the asymptotics (4) holds for $\lambda = \lambda_-$.*

Let us discuss an important special case. Suppose that $b = a$ and $p = 0$. Then the equalities (2), (3) transform into

$$\beta_{lm}(\xi) = E_l(\xi)\delta_{lm},$$
$$\Gamma(\xi) = \operatorname{diag}\left\{\frac{E_l(\xi)}{\lambda - E_l(\xi)}\right\}_{l=1}^{r}.$$

Therefore the asymptotics (4) takes the form

$$N(\alpha,\lambda) \sim \frac{\alpha^{\varkappa}}{d(2\pi)^d} \int_{\mathbb{S}^{d-1}} F(\theta)^{\varkappa} dS(\theta) \sum_{l \leq r} \int_{\mathbb{T}^d} \left(\frac{E_l(\xi)}{\lambda - E_l(\xi)}\right)^{\varkappa} d\xi, \ \alpha \to \infty. \tag{5}$$

Emphasize that the r.h.s. of (5) does not contain the functions $\psi_l(\xi, x)$. Further, contributions from different bands enter (5) additively. These two facts do not take place, generally speaking, in case of the general formula (4). In particular this applies to the most natural case $b = \mathbb{I}$, $a \neq (const)\mathbb{I}$, $p = 0$, and to the case $b = a, p \neq 0$.

4. In the paper [1] the asymptotics of $N(\alpha,\lambda)$, $\alpha \to \infty$ was studied under conditions

$$b = a, \ p = 0, \ \lambda \in \Lambda, \ F = const.$$

The operator A was not supposed to be periodic. It was assumed only that for the operator A one could define a *density of states.* In the periodic case the asymptotic formula from [1] coincides with (5) (for $F = const$). Restrictions $b = 0$, $p = 0$ in [1] seemed to be of technical nature. The question has arisen, whether the formula (5) is true without these assumptions. Asymptotics (4) clearly shows that the result changes dramatically.

5. Methods used in [1] and here are principally different. Method of [1] is essentially variational. As was mentioned above, this fact allowed the authors not to require any periodicity of A, but assume only the existence of the density of states. The variational approach, however, deals usually with the spectrum of an operator, but not with any other spectral characteristics. On the other hand we saw that (4) contained the eigenfunctions ψ_l responsible for spectral projectors of the operator A. The variational technique can hardly provide any general formula of the

form (4). Our method is based on the Floquet theory and the expansion theorem for the operator A. This is the reason why the periodicity of A is important for us. At the same time our approach allows one to analyse the case $\lambda = \lambda_-$ (cf. [2], where this situation is treated when A is perturbed by a positive decreasing potential).

6. I am grateful to professor P.A. Deift, who has drawn my attention to the subject discussed above.

References

1. S. Alama, M. Avellaneda, P.A. Deift, R. Hempel, *On the existence of eigenvalues of a divergence form operator $A + \lambda B$ in a gap of $\sigma(A)$*, Asymptotic Analysis **8(4)** (1994), 311–344.
2. M.Sh. Birman, *Discrete spectrum of the periodic Schrödinger operator for non-negative perturbations*, Operator Theory: Advances and Applications, vol. 70, Birkhäuser, Basel, 1994.

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