

# JOURNÉES ÉQUATIONS AUX DÉRIVÉES PARTIELLES

MICHAEL SOLOMYAK

**Eigenvalue estimates for a class of operators related to self-similar measures**

*Journées Équations aux dérivées partielles* (1994), p. 1-6

[http://www.numdam.org/item?id=JEDP\\_1994\\_\\_\\_\\_A11\\_0](http://www.numdam.org/item?id=JEDP_1994____A11_0)

© Journées Équations aux dérivées partielles, 1994, tous droits réservés.

L'accès aux archives de la revue « Journées Équations aux dérivées partielles » (<http://www.math.sciences.univ-nantes.fr/edpa/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/legal.php>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme  
Numérisation de documents anciens mathématiques  
<http://www.numdam.org/>

# EIGENVALUE ESTIMATES FOR A CLASS OF OPERATORS RELATED TO SELF-SIMILAR MEASURES

Michael Solomyak

The Weizmann Institute of Science  
Rehovot 76100, Israel

**Abstract.** We obtain the sharp order of growth of the eigenvalue distribution function for the operator in the Sobolev space  $H_0^1(X)$ , generated by the quadratic form  $\int_X |u|^2 d\mu$ , where  $X \subset \mathbb{R}^d$  is a domain and  $\mu$  is a probability self-similar fractal measure on  $X$ .

**Acknowledgement:** Most of the results presented were obtained in cooperation with E. Verbitsky (1-dimensional case) and K. Naimark ( $d$ -dimensional case,  $d \geq 2$ ); see [SV] and [NS].

1. The notions of self-similar set and self-similar measure were introduced by Hutchinson [H]. Let  $\mathcal{S} = \{S_1, \dots, S_m\}$  be a set of contractive similitudes on  $\mathbb{R}^d$ ,  $h_1, \dots, h_m$  – their coefficients of contraction. Also, let a system of positive numbers (“weights”)  $\mathbf{p} = \{p_1, \dots, p_m\}$  be given, such that  $p_1 + \dots + p_m = 1$ .

Then there exists a unique non-empty compact set  $\mathcal{C} = \mathcal{C}(\mathcal{S}) \subset \mathbb{R}^d$  such that  $\mathcal{C} = \bigcup_{k=1}^m S_k \mathcal{C}$ , and a unique boundedly supported probability Borelian measure  $\mu = \mu(\mathcal{S}, \mathbf{p})$  which satisfies the self-similarity property

$$\mu = \sum_{k=1}^m p_k \mu \circ S_k^{-1}. \quad (1)$$

Furthermore,  $\mathcal{C}(\mathcal{S}) = \text{supp } \mu(\mathcal{S}, \mathbf{p})$ . Measures  $\mu(\mathcal{S}, \mathbf{p})$ , corresponding to the same set  $\mathcal{S}$  but to different families  $\mathbf{p}$ , are mutually singular.

It is  $\mu(\mathcal{S}, \mathbf{p})$  that we are interested in. For the rest of the paper  $\mu = \mu(\mathcal{S}, \mathbf{p})$ .

We always suppose that  $\mathcal{S}$  meets the “open set condition” (see [H]). This means that there exists a bounded open set  $\Omega \subset \mathbb{R}^d$ , such that

$$1. \bigcup_{k=1}^m S_k(\Omega) \subset \Omega, \quad 2. S_k(\Omega) \cap S_\ell(\Omega) = \emptyset \text{ for } k \neq \ell. \quad (2)$$

Evidently, (2) implies that  $\mathcal{C}(\mathcal{S}) \subset \overline{\Omega}$ . It follows also that  $\sum_{k=1}^m h_k^d \leq 1$ . In addition, we suppose that

$$\mu(\partial\Omega) = 0. \quad (3)$$

It is known that otherwise  $\mu(\Omega) = 0$  (see [Lau W]). Assumption (3) is of a rather technical character and can often be withdrawn.

If, for example,  $d = 1$ ,  $m = 2$ ,  $S_1(x) = \frac{x}{3}$ ,  $S_2(x) = 1 - \frac{x}{3}$  and  $p_1 = p_2 = \frac{1}{2}$ , then  $\mathcal{C}(\mathcal{S})$  is the classical Cantor set, and  $\mu(\mathcal{S}, \mathbf{p})$  is the Cantor measure. Here (2) is satisfied, with  $\Omega = (0, 1)$ . Evidently, (3) is satisfied too.

**2.** Let  $X \subset \mathbb{R}^d$  be a bounded domain, such that  $\Omega \subseteq X$ . In the Sobolev space  $H_0^1(X)$ , with the Dirichlet integral  $\int_X |\nabla u|^2 dx$  as the metric form, we consider the quadratic functional

$$Q_\mu[u] = \int_X |u|^2 d\mu, \quad \text{where } \mu = \mu(\mathcal{S}, \mathbf{p}).$$

If  $d = 1$ , then  $Q_\mu$  is always bounded and the corresponding embedding  $H_0^1(X) \subset L_2(X, \mu)$  is compact. To ensure the same for  $d \geq 2$ , additional requirements on  $\mathcal{S}$  and  $\mathbf{p}$  are necessary. Denote

$$A_k = h_k^{2-d} p_k, \quad \overline{A}(\mathcal{S}, \mathbf{p}) = \max_k A_k.$$

**Lemma.** *Let (2) be satisfied. Then the imbedding  $H^1(\Omega) \subset L_2(\Omega, d\mu)$  is compact if and only if*

$$\overline{A}(\mathcal{S}, \mathbf{p}) < 1. \quad (4)$$

Necessity is quite straightforward and does not need (2). Sufficiency can be easily derived from [M], §8.8.

Note that (2) and (4) imply that the Hausdorff dimension of  $\mu$  satisfies  $\beta := \dim \mu(\mathcal{S}, \mathbf{p}) > d - 2$ .

If (4) is satisfied, then  $Q_\mu[u]$  generates in  $H_0^1(X)$  a compact, self-adjoint and positive operator, say  $T_\mu$  or, in a more detailed notation,  $T_{\mu,X}$ . We are interested in the behaviour of its eigenvalues  $\lambda_k(T_\mu)$ . As usual, we express this behaviour in terms of the corresponding distribution function

$$n(t, T_\mu) = \#\{k : \lambda_k(T_\mu) > t\}.$$

If  $u \in H_0^1(X)$  is an eigenfunction of  $T_\mu$  and  $\lambda$  is the corresponding eigenvalue, then for any  $v \in H_0^1(X)$ ,

$$\lambda \int_X \nabla u \cdot \nabla \bar{v} dx = \int_X u \bar{v} d\mu. \quad (5)$$

For general self-similar  $\mu$ , it is impossible to rewrite (5) in terms of any classical boundary value problem. Notice only that always  $u(x) = 0$  outside  $\mathcal{C}(\mathcal{S})$ .

**3.** We write  $\Omega \in \mathcal{P}$  if the Poincaré inequality

$$\int_\Omega |u|^2 dx \leq K \left( \int_\Omega |\nabla u|^2 dx + \left| \int_\Omega u dx \right|^2 \right) \quad (6)$$

is satisfied for any  $u \in H^1(\Omega)$ . Any bounded  $\Omega$  whose boundary is “not too bad” belongs to  $\mathcal{P}$ ; see [M] for the general description of domains  $\Omega \in \mathcal{P}$ .

**Theorem 1.** *Suppose that  $\Omega \in \mathcal{P}$  and  $\mu = \mu(\mathcal{S}, \mathbf{p})$ . Let (2) and (4) be satisfied. Define  $\delta > 0$  as the unique solution of*

$$\sum_{k=1}^m A_k^\delta = \sum_{k=1}^m (h_k^{2-d} p_k)^\delta = 1. \quad (7)$$

*Let  $X$  be an arbitrary open set in  $\mathbb{R}^d$  such that  $X \supset \Omega$ . Then there exists a number  $\mathcal{C} = \mathcal{C}(\mathcal{S}, \mathbf{p}, X)$  such that*

$$n(t, T_{\mu,X}) \leq \mathcal{C} t^{-\delta}, \quad \text{any } t > 0. \quad (8)$$

*If, besides, (3) is satisfied, then there exist also  $c > 0$  and  $t_0 > 0$ , such that*

$$n(t, T_{\mu,X}) \geq c t^{-\delta}, \quad \text{any } t \in (0, t_0). \quad (9)$$

**Remark.** If (3) is not satisfied and  $X = \Omega$ , then  $T_\mu = 0$  and (9) fails. For open sets  $X \supset \overline{\Omega}$ , (9) probably remains valid without assumption (3).

4. A more detailed analysis is possible for  $d = 1$ . Suppose for simplicity that  $X = \Omega = (0, 1)$ . Denote

$$c_k = \log A_k = \log(h_k p_k), \quad k = 1, \dots, m.$$

We suppose  $m \geq 2$ , because otherwise  $\mu(\mathcal{S}, \mathbf{p})$  is the Dirac measure supported at a point  $x_0 \in [0, 1]$ , and the problem considered is meaningless.

**Theorem 2.** *Let  $d = 1$ ,  $X = \Omega = (0, 1)$ ,  $m \geq 2$ , and (2) be satisfied. If, besides, at least one of the ratios  $\frac{c_k}{c_\ell}$  is irrational, then there exists a positive constant  $B$ , such that*

$$n(t; T_\mu) \sim Bt^{-\delta} \quad \text{as } t \rightarrow 0. \quad (10)$$

*If all the ratios  $\frac{c_k}{c_\ell}$  are rational, then there exists a uniformly positive, bounded periodic function  $\psi(s)$  on  $\mathbb{R}$ , such that*

$$n(t; T_\mu) \sim \psi\left(\log \frac{1}{t}\right)t^{-\delta} \quad \text{as } t \rightarrow 0. \quad (11)$$

## 5. Discussion

(1) The proofs of both Theorems 1, 2 use the variational approach. We reduce Theorem 1 to the analysis of a certain combinatorial problem. This one is investigated with the help of the “renewal equation”, see e.g. [F]. If  $d = 1$ , then a direct reduction of the initial problem to renewal equation turns out to be possible. This allows to prove Theorem 2.

Renewal equation was applied to spectral problems (of another nature) involving fractals in [K Lap], [Lap] and [Lev V], see also references there. For the usage of the renewal theory in other problems related to fractals, see [Lau W] and [St 1,2].

(2) The Hausdorff dimension of  $\mu$  equals (see [St1])

$$\beta = \left( \sum_{k=1}^m p_k \ln p_k \right) / \left( \sum_{k=1}^m p_k \ln h_k \right).$$

The Hausdorff dimension  $\alpha$  of  $\mathcal{C}(\mathcal{S})$  is determined by the equation  $\sum_{k=1}^m h_k^\alpha = 1$  (see [H]). Always  $\beta \leq \alpha$ , and the equality takes place if and only if  $p_k = h_k^\alpha$ ,  $k = 1, \dots, m$ . These  $p_k$ 's are called "natural weights".

The exponent  $\delta$  meets the following correlations:

$$\frac{\beta}{\beta + 1} \leq \delta \leq \frac{\alpha}{\alpha + 1} \quad \text{for } d = 1, \quad (12)$$

$$\delta = 1 \quad \text{for } d = 2,$$

$$\frac{\alpha}{\alpha - (d - 2)} \leq \frac{\beta}{\beta - (d - 2)} \leq \delta \quad \text{for } d \geq 3. \quad (13)$$

If  $d \geq 3$ , the value of  $\delta$  can be arbitrarily large. The equalities in (12), (13) take place only in the case of natural weights.

(3) The sign "=" in the inequality  $\sum_{k=1}^m h_k^d \leq 1$  is rather rare. It corresponds to the case when  $S_k(\Omega)$ ,  $k = 1, \dots, m$ , constitute a tiling of  $\Omega$  by a family of subdomains similar to  $\Omega$ . If, in addition,  $p_k$  are the natural weights,  $\mu(\mathcal{S}, \mathbf{p})$  is nothing but the  $d$ -dimensional Lebesgue measure on  $\Omega$ . Here, clearly,  $\delta = \frac{d}{2}$ .

(4) Similar results are true for the operators on  $H^1(\Omega)$ , i.e. for "natural boundary conditions".

(5) Consider the case  $X = \mathbb{R}^d$ , which is possible for  $d \geq 3$ . Here we deal with the "homogeneous Sobolev space"  $\mathcal{H}^1(\mathbb{R}^d)$ , instead of  $H_0^1(X)$ . The space  $\mathcal{H}^1(\mathbb{R}^d)$  is defined as the completion of  $C_0^\infty(\mathbb{R}^d)$  with respect to the metric given by the Dirichlet integral. Equation (5) (with  $u$  and  $v$  from  $\mathcal{H}^1(\mathbb{R}^d)$ ) is still satisfied for any eigenpair  $(u, \lambda)$ .

**Theorem 3.** *The result of Theorem 1 remains valid for  $X = \mathbb{R}^d$ , provided  $d \geq 3$ .*

(6) Under the assumptions of Theorem 3, the quadratic form

$$a_{\mu, \alpha}[u] = \int_{\mathbb{R}^d} |\nabla u|^2 dx - \alpha \int_{\mathbb{R}^d} |u|^2 d\mu, \quad u \in H^1(\mathbb{R}^d), \quad (14)$$

is bounded from below and closed in  $L_2(\mathbb{R}^d)$ . In (14)  $\alpha > 0$  is the "coupling parameter". The self-adjoint operator  $A_{\mu, \alpha}$  in  $L_2(\mathbb{R}^d)$ , generated by the quadratic form (14), can be treated as Schrödinger operator with self-similar measure  $\mu$  as a potential. Due to the "Birman-Schwinger principle" (see e.g. [R Sim]), Theorem 3 implies the following result.

**Theorem 4.** *Let the assumptions of Theorem 3 be satisfied. Then, for any  $\alpha > 0$ , the number  $N_-(\mu; \alpha)$  of bound states of  $A_{\mu, \alpha}$  satisfies*

$$N_-(\mu, \alpha) \leq C_{(\mu)} \alpha^\delta, \quad \text{any } \alpha > 0,$$

where  $\delta$  is given by (7).

#### REFERENCES

- [F] W. Feller, *An introduction to the probability theory and its applications*, Vol. II, John Wiley & Sons, Inc., New York-London-Sydney-Toronto, 1971.
- [H] J.E. Hutchinson, "Fractals and self-similarity", *Indiana Univ. Math. J.* **30** (1981) 713–747.
- [K Lap] J. Kigami, M.L. Lapidus, "Weyl's problem for the spectral distribution of Laplacians on p.c.f. self-similar fractals", *Commun. Math. Phys.* **158** (1993) 93–125.
- [Lap] M. Lapidus, "Vibrations of fractal drums, the Riemann hypothesis, waves in fractal media and the Weyl-Berry conjecture", *Ordinary and partial differential equations*, Vol. 4, B.D. Sleeman and R.J. Jarvis (editors), *Pitman Research Notes in Mathematics* **289** (1993) 126–209.
- [Lau W] K.-S. Lau, J. Wang, "Mean quadratic variations and Fourier asymptotics of self-similar measures", *Monat. Math.* **115** (1993) 99–132.
- [Lev V] M. Levitin, D. Vassiliev, "Spectral asymptotics, renewal theorem and the Berry conjecture for a class of fractals" (1994) preprint.
- [M] V. Maz'ja, *Sobolev spaces*, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1985.
- [NS] K. Naimark, M. Solomyak, "On the eigenvalue behaviour for a class of operators related to self-similar measures on  $\mathbb{R}^d$ ", submitted to *C.R. Acad. Sci. Paris*.
- [R Sim] M. Reed, B. Simon, *Methods of modern mathematical physics*, Vol. IV, Academic Press, New York-San Francisco-London, 1978.
- [SV] M. Solomyak, E. Verbitsky, "On a spectral problem related to self-similar measures", *Bull. of London Math. Soc.* (to appear).
- [St1] R.S. Strichartz, "Self-similar measures and their Fourier transforms I", *Indiana Univ. Math. J.* **39** (1990) 797–817.
- [St2] R.S. Strichartz, "Self-similarity in harmonic analysis", preprint.