JOURNÉES ÉQUATIONS AUX DÉRIVÉES PARTIELLES

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Journées Équations aux dérivées partielles (1993), p. 1-9
<http://www.numdam.org/item?id=JEDP_1993____A9_0>

ASYMPTOTIC BEHAVIOR OF FOURIER-LAPLACE TRANSFORMS

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1. Introduction. This lecture is a summary of a recent manuscript by Ragnar Sigurdsson and myself. It is mainly concerned with complex analysis, but since the original motivation comes from differential operators with constant coefficients and convolution equations, it might still be appropriate to present it here.

If $f \in \mathcal{E}'(\mathbf{R}^n)$ and $F = \hat{f}$ is the Fourier-Laplace transform, then the Paley-Wiener theorem states that

$$|F(\zeta)| \leq C(1+|\zeta|)^N \exp(H(\operatorname{Im}\zeta)), \qquad \zeta \in \mathbf{C}^n, \tag{1.1}$$

where C and N are constants and H is the supporting function of $K = \operatorname{supp} f$. If f is just a hyperfunction then

$$|F(\zeta)| \leq C_\varepsilon \exp(H(\operatorname{Im}\zeta) + \varepsilon|\zeta|), \qquad \zeta \in \mathbf{C}^n, \tag{1.1$'$}$$

for every $\varepsilon > 0$. Both statements have a converse.

If F is any entire analytic function such that

$$|F(\zeta)| \leq e^{C+A|\zeta|}, \qquad \zeta \in \mathbf{C}^n, \tag{1.2}$$

then $u = \log|F|$ belongs to the set $\operatorname{PSH}(\mathbf{C}^n)$ of plurisubharmonic functions in $\mathbf{C}^n$, and

$$u(\zeta) \leq C + A|\zeta|, \qquad \zeta \in \mathbf{C}^n. \tag{1.3}$$

For any $u \in \operatorname{PSH}(\mathbf{C}^n)$ the estimate (1.3) is valid if and only if the forward orbit $\{T_t u; t \geq 1\}$ of the dilation group

$$T_t u(\zeta) = t^{-1}u(t\zeta), \qquad \zeta \in \mathbf{C}^n, \quad t > 0, \tag{1.4}$$

is relatively compact in the topology of $L^1_{\text{loc}}(\mathbf{C}^n)$. For $u \in \operatorname{PSH}(\mathbf{C}^n)$ satisfying (1.3) we define the *limit set* $L_\infty(u)$ at infinity as the set of all limit points $v \in \operatorname{PSH}(\mathbf{C}^n)$ of $T_t u$ as $t \to +\infty$. They all vanish at the origin and satisfy (1.3) with $C = 0$. A complete description of the subsets M of $\operatorname{PSH}(\mathbf{C}^n)$ such that $M = L_\infty(u)$ for some $u \in \operatorname{PSH}(\mathbf{C}^n)$ was given in [5]. If u also satisfies a condition corresponding to (1.1)$'$,

$$u(\zeta) \leq C_\varepsilon + A|\operatorname{Im}\zeta| + \varepsilon|\zeta|, \qquad \zeta \in \mathbf{C}^n,\ \varepsilon > 0, \tag{1.5}$$

then $v \leq 0$ in $\mathbf{R}^n$ if $v \in L_\infty(u)$. The *indicator function* j_u of u, defined as the upper semicontinuous regularization of $\overline{\lim}_{t\to\infty} T_t u$, is a plurisubharmonic function homogeneous of degree 1, and it is equal to $\sup_{v\in L_\infty(u)} v$.

Theorem 1.1. *If $u \in \mathrm{PSH}(\mathbf{C}^n)$ satisfies (1.5) then the indicator function j_u vanishes in $\mathbf{R}^n$, and $j_u(\zeta) \leq A|\operatorname{Im}\zeta|$, $\zeta \in \mathbf{C}^n$. More precisely,*

$$H(\eta) = \sup_{\xi \in \mathbf{R}^n} j_u(\xi + i\eta), \tag{1.6}$$

is a convex positively homogeneous function, the supporting function of a closed convex subset of the ball $\{x \in \mathbf{R}^n; |x| \leq A\}$, and $j_u(\zeta) = H(\operatorname{Im}\zeta)$ if $\zeta \in \mathbf{C}\mathbf{R}^n$. For every $\varepsilon > 0$ there is a constant C_ε such that

$$u(\zeta) \leq C_\varepsilon + H(\operatorname{Im}\zeta) + \varepsilon|\zeta|, \quad \zeta \in \mathbf{C}^n, \tag{1.7}$$

which remains true for $\varepsilon = 0$ if $C_0 = \sup_{\xi \in \mathbf{R}^n} u(\xi) < \infty$.

When $u = \log|F|$ where F is the Fourier-Laplace transform of a hyperfunction f of compact support, then the supporting function which occurs in Theorem 1.1 is the supporting function of $\operatorname{supp} f$.

Definition 1.2. If $u \in \mathrm{PSH}(\mathbf{C}^n)$ satisfies (1.5), then the supporting function H_u of u is the convex positively homogeneous function defined by (1.6) where j_u is the indicator function.

Remark. If $v \in L_\infty(u)$ then $v \leq j_u$ so $H_v \leq H_u$. Since j_u is the supremum of all $v \in L_\infty(u)$ and $v(\zeta) \leq H_v(\operatorname{Im}\zeta)$ by (1.7) with $\varepsilon = C_0 = 0$, we have $j_u(\zeta) \leq \sup_{v \in L_\infty(u)} H_v(\operatorname{Im}\zeta)$, hence

$$H_u = \sup_{v \in L_\infty(u)} H_v. \tag{1.8}$$

From [5, Theorem 0.2], [6, Theorem 1.3.1] and [2, Theorem 15.1.5] we obtain:

Theorem 1.3. *If $u \in \mathrm{PSH}(\mathbf{C}^n)$ satisfies (1.5), then $M = L_\infty(u)$ is compact, connected, and T invariant, $v \leq 0$ in $\mathbf{R}^n$ for $v \in M$, and T is chain recurrent on M. Conversely, for every set $M \subset \mathrm{PSH}(\mathbf{C}^n)$ with these properties one can find $u \in \mathrm{PSH}(\mathbf{C}^n)$ with $L_\infty(u) = M$, and u satisfies (1.5) then. One can choose $u = \log|F|$ where F is an entire analytic function. Thus $F = \hat{f}$ where f is a hyperfunction with compact support; the supporting function of f is then equal to the supporting function of $\sup_{v \in M} v$.*

The notion of chain recurrence is explained in [5]. As in [6, Theorem 1.2.7] there is an analogue for k tuples of plurisubharmonic functions, and it has the following consequence (cf. [6, Proposition 2.1.4]):

Proposition 1.4. *Let K_1, K_2, K_3 be compact convex subsets of $\mathbf{R}^n$. Then there exist hyperfunctions f_j with $\operatorname{ch supp} f_j = K_j$, $j = 1, 2, 3$, and $f_3 = f_1 * f_2$ if and only if*

$$K_3 \subset K_1 + K_2, \tag{1.9}$$

and with the notation

$$K = \{(x_1, x_2) \in K_1 \times K_2; x_1 + x_2 \in K_3\} \tag{1.10}$$

the projections $\pi_j : K \to K_j$ *are surjective for* $j = 1, 2$.

Proof. The necessity of (1.9) is obvious. Let H_j be the supporting function of $\operatorname{ch}\operatorname{supp} f_j$. Then H_1 is the supremum of the supporting functions h_1 of elements $v_1 \in L_\infty(\log|\hat{f}_1|)$, so K_1 is the convex hull of the corresponding convex sets. If x_1 is an extreme point of K_1 it follows in view of the compactness of $L_\infty(\log|\hat{f}_1|)$ that one can find h_1 such that $\langle x_1, \cdot\rangle \le h_1$. We have $(v_1, v_2) \in L_\infty(\log|\hat{f}_1|, \log|\hat{f}_2|)$ for some $v_2 \in L_\infty(\log|\hat{f}_2|)$. If h_2 is the supporting function of v_2, it follows from Theorem 1.5 below that $h_1 + h_2$ is the supporting function of $v_1 + v_2 \in L_\infty(\log|\hat{f}_3|)$. Hence $h_1 + h_2 \le H_3$. When $\langle x_2, \cdot\rangle \le h_2$ it follows that $x_2 \in K_2$ and that $x_1 + x_2 \in K_3$. Thus π_1 is surjective, and the surjectivity of π_2 follows in the same way.

Suppose now that (1.9) holds and that π_j is surjective, $j = 1, 2$. By Theorem 1.3 for pairs we can choose $f_j \in \mathcal{E}'$, $j = 1, 2$, with

$$L_\infty(\log|\hat{f}_1|, \log|\hat{f}_2|) = \{(\langle x_1, \cdot\rangle, \langle x_2, \cdot\rangle); (x_1, x_2) \in K\}.$$

The surjectivity of π_j implies that $\operatorname{ch}\operatorname{supp} f_j = K_j$, $j = 1, 2$. If $f_3 = f_1 * f_2$ then

$$L_\infty(\log|\hat{f}_3|) = \{\langle x_1 + x_2, \cdot\rangle; (x_1, x_2) \in K\} = \{\langle x, \cdot\rangle; x \in K_3\},$$

by (1.9) and (1.10), which proves that $\operatorname{ch}\operatorname{supp} f_3 = K_3$.

Now we turn to functions in $\operatorname{PSH}(\mathbf{C}^n)$ satisfying (1.3) and a weak analogue of (1.1),

$$\int_{\mathbf{R}^n} \frac{u^+(\xi)\, d\xi}{(1 + |\xi|)^{n+1}} < \infty, \tag{1.11}$$

where $u^+ = \max(u, 0)$. This is equivalent to the apparently stronger condition

$$\int_{\mathbf{R}^n} \frac{|u(\xi)|\, d\xi}{(1 + |\xi|)^{n+1}} < \infty, \tag{1.11$'$}$$

and implies the weaker condition

$$t^{-n-1} \int_{\xi \in \mathbf{R}^n, |\xi| < t} |u(\xi)|\, d\xi \to 0, \qquad t \to +\infty. \tag{1.12}$$

When u satisfies (1.3) and (1.12), then every $v \in L_\infty(u)$ vanishes in $\mathbf{R}^n$, and that is the only restriction on $L_\infty(u)$ in addition to the properties listed in Theorem 1.3. When (1.3) and (1.11) are fulfilled one can say much more:

Theorem 1.5. *If* $u \in \operatorname{PSH}(\mathbf{C}^n)$ *satisfies* (1.3) *and* (1.11), *then*

$$\int_K |u(\zeta + tz\eta)/t - H(\operatorname{Im} z\eta)|\, d\lambda(z) \to 0, \quad t \to +\infty, \tag{1.13}$$

for almost all $(\zeta, \eta) \in \mathbf{C}^n \times (\mathbf{R}^n \setminus \{0\})$, *if* K *is a compact subset of* $\mathbf{C}$ *and* H *is the supporting function of* u. *For every* $v \in L_\infty(u)$ *we have*

$$v(\zeta) = H(\operatorname{Im}\zeta), \quad \zeta \in \mathbf{C}\mathbf{R}^n, \tag{1.14}$$

and

$$u(t\zeta)/t - H(\operatorname{Im}\zeta) \to 0 \quad \text{in } L^1_{\mathrm{loc}}(\mathbf{C}\mathbf{R}^n \setminus \{0\}), \text{ as } t \to +\infty, \tag{1.15}$$

$$u(te^{i\theta}\xi)/t - H(\xi \sin\theta) \to 0 \quad \text{in } L^1_{\mathrm{loc}}(\mathbf{R}^n), \text{ as } t \to +\infty, \quad \theta \in \mathbf{R}. \tag{1.16}$$

The result motivates the following:

Definition 1.6. If H is a supporting function in $\mathbf{R}^n$, that is, H is convex and positively homogeneous of degree 1, then P_H will denote the set of $u \in \mathrm{PSH}(\mathbf{C}^n)$ such that

$$(1.17) \qquad u(\zeta) \leq H(\operatorname{Im}\zeta), \quad \zeta \in \mathbf{C}^n; \quad u(\zeta) = H(\operatorname{Im}\zeta), \quad \zeta \in \mathbf{C}\mathbf{R}^n.$$

The conclusion (1.14) in Theorem 1.5 was that $L_\infty(u) \subset P_H$ if $u \in \mathrm{PSH}(\mathbf{C}^n)$ satisfies (1.3) and (1.11), and H is the supporting function of u. It is clear that P_H is a convex compact T invariant subset of $\mathrm{PSH}(\mathbf{C}^n)$. We shall discuss the properties of P_H in the following section.

The limit (1.15) was established by Vauthier [7] when $u \in \mathrm{PSH}(\mathbf{C}^n)$ is bounded above in $\mathbf{R}^n$, hence when $u \in \mathrm{PSH}(\mathbf{C}^n)$ is the difference of two such functions. The proof is somewhat easier then. The support theorem of Titchmarsh and Lions is an immediate consequence of (1.13); recall that it states that

$$(1.18) \qquad \operatorname{ch}\operatorname{supp}(f_1 * f_2) = \operatorname{ch}\operatorname{supp} f_1 + \operatorname{ch}\operatorname{supp} f_2, \quad \text{if } f_1, f_2 \in \mathcal{E}'(\mathbf{R}^n).$$

By Proposition 1.4 this is not true for hyperfunctions.

2. Properties of P_H. The set P_H always contains the function $\zeta \mapsto H(\operatorname{Im}\zeta)$. In the rotationally symmetric case where $H(\xi) = |\xi|$, it is easy to see that also the function

$$(2.1) \qquad \zeta \mapsto |\operatorname{Im}\langle\zeta,\zeta\rangle^{\frac{1}{2}}|$$

is in P_H, and that it is strictly smaller than $H(\operatorname{Im}\zeta)$ for every $\zeta \notin \mathbf{C}\mathbf{R}^n$, which is a cone of real dimension $n+1$ and smooth except at the origin. The function (2.1) occurs as the indicator function of $\log|\hat{f}|$ when $f \in \mathcal{E}'$ is rotationally symmetric. The interest of the strict inequality is explained by the following result from [6, Theorem 2.3.1]:

Theorem 2.1. *Let f be a hyperfunction in $\mathbf{R}^n$ with compact support, and let H be the supporting function of* $\operatorname{supp} u$. *Let $\xi^0, \eta^0 \in \mathbf{R}^n \setminus \{0\}$. Then*

$$(2.2) \qquad j_{\log|\hat{f}|}(\xi^0 + i\eta^0) = H(\eta^0)$$

if and only if $(x, \xi^0) \in WF_A(f)$ for some $x \in \operatorname{supp} f$ with $\langle x, \eta^0\rangle = H(\eta^0)$.

If we take for f the characteristic function of the unit ball, then $WF_A(f)$ is the conormal bundle of the boundary and we get the property of (2.1) just mentioned. For a general supporting function H we shall denote by M_H the set of all $\zeta \in \mathbf{C}^n$ such that $u(\zeta) = H(\operatorname{Im}\zeta)$ for every $u \in P_H$. Thus $M_H = \mathbf{C}\mathbf{R}^n$ if $H(\xi) = |\xi|$. On the other hand, it is easy to see that $M_H = \mathbf{C}^n$ if H is the supporting function of a polyhedron. This is related to flatness of the supporting function as proved by the following two theorems. We write $\mathbf{C}_+$ for the open upper half plane in $\mathbf{C}$.

Theorem 2.2. *For any $u \in P_H$ and $\eta \in \mathbf{R}^n \setminus 0$, $\theta \in \mathbf{R}^n$, the subharmonic function*

$$(2.3) \qquad v(z) = u(\theta + z\eta) - H(\operatorname{Im} z\eta) = u(\theta + z\eta) - \operatorname{Im} z H(\eta), \quad z \in \mathbf{C}_+,$$

is ≤ 0 in $\mathbf{C}_+$, and as $t \to +\infty$

$$(2.4) \qquad tv(tz) = t(u(\theta + tz\eta) - \operatorname{Im} tzH(\eta)) \to \operatorname{Im}(1/z)q_u(\eta,\theta), \quad \text{in } L^1_{\text{loc}}(\overline{\mathbf{C}}_+),$$

for some $q_u(\eta,\theta) \geq 0$, *unless the left-hand side* $\to -\infty$ *uniformly on every compact set in* $\mathbf{C}_+$; *in that case we define* $q_u(\eta,\theta) = +\infty$. *We have* $q_u(\eta,\theta) = \langle \operatorname{Im}\zeta, \Delta v/\pi\rangle_{\mathbf{C}_+}$,

$$(2.5) \qquad q_u(\eta, \theta + c\eta) = q_u(\eta,\theta), \ c \in \mathbf{R}, \quad q_u(c\eta,\theta) = q_u(\eta,\theta)/c, \ c > 0,$$

and $q_{u_s}(\eta,\theta) = q_u(s\eta, s\theta)/s$, $s > 0$, *if* $u_s = T_s u$. *If* $q_u(\eta,\theta) < \infty$ *then* (2.4) *also holds with convergence in* $L^1_{\text{loc}}(i\mathbf{R}_+)$; *if* $0 \leq \chi \in C_0^\infty(\mathbf{R}_+)$ *and* $\int \chi(s)\,ds/s = 1$, *it follows that*

$$(2.6) \qquad t\int_{\mathbf{R}_+} (tsH(\eta) - u(\theta + its\eta))\chi(s)\,ds \to q_u(\eta,\theta), \quad t \to +\infty,$$

whether $q_u(\eta,\theta)$ *is finite or not. If* $q_u(\eta,\theta) = 0$ *then* $u(\theta + z\eta) = H(\operatorname{Im} z\eta)$ *if* $z \in \mathbf{C}_+$, *and* $q_u(\eta,\theta) = 0$ *for every* $u \in P_H$ *if and only if* $\theta + i\eta \in M_H$. *The non-negative function* q_u *is lower semicontinuous.*

The limit $q_u(\eta,\cdot)$ is usually finite in the direction θ if H is almost twice differentiable at η in that direction:

Theorem 2.3. *If* $\eta,\theta \in \mathbf{R}^n$ *are linearly independent and* $u \in P_H$, *then*

$$(2.7) \quad \sup_{T>0} T^{-3}\int_{-T}^{T} q_u(\eta,\tau\theta)\,d\tau \leq C \varliminf_{(\tau,\tilde\eta,\tilde\theta)\to(0,\eta,\theta)} (H(\tilde\eta + \tau\tilde\theta) + H(\tilde\eta - \tau\tilde\theta) - 2H(\tilde\eta))/\tau^2.$$

Here C *is independent of* u *and* H. *Hence* $q_u(\eta,\tau\theta) < \infty$ *for almost all* $\tau \in \mathbf{R}$, *if the right-hand side of* (2.7) *is finite, and* $\mathbf{R}\theta + \mathbf{C}_+\eta \subset M_H$ *if*

$$(2.8) \qquad \varliminf_{(\tau,\tilde\eta,\tilde\theta)\to(0,\eta,\theta)} (H(\tilde\eta + \tau\tilde\theta) + H(\tilde\eta - \tau\tilde\theta) - 2H(\tilde\eta))/\tau^2 = 0.$$

When H has Lipschitz continuous first derivatives in $\mathbf{R}^n \setminus \{0\}$, one can prove much more. To simplify the statement we shall make a somewhat stronger assumption:

Theorem 2.4. *Let* $u \in P_H$ *where* $H \in C^2$ *in* $\mathbf{R}^n \setminus \{0\}$. *Then the non-negative function* $q_u(\eta,\theta)$ *in* $(\mathbf{R}^n \setminus \{0\}) \times \mathbf{R}^n$ *defined by* (2.4) *is locally bounded; in fact,*

$$(2.9) \qquad 0 \leq q_u(\eta,\theta) \leq \tfrac{1}{2}\langle H''(\eta)\theta,\theta\rangle.$$

The difference

$$(2.10) \qquad Q_u(\eta,\theta) = \tfrac{1}{2}\langle H''(\eta)\theta,\theta\rangle - q_u(\eta,\theta)$$

is a convex function of θ *with values in* $L^\infty(S^{n-1})$.

When $u \in P_H$ is a smooth function, positively homogeneous of degree 1 in a neighborhood of $\mathbf{C}\mathbf{R}^n \setminus \mathbf{R}^n$, thus $H(\eta) = u(i\eta)$ is smooth for $\eta \neq 0$, it is easy to improve Theorem

2.4 by differential calculus. If $\theta, \eta \in \mathbf{R}^n \setminus \{0\}$, then the subharmonic function v defined by (2.3) is ≤ 0. By Taylor's formula

$$tv(tz) = t^2(u(\theta/t + z\eta) - u(z\eta)) \to \tfrac{1}{2}\langle u''(z\eta)\theta, \theta\rangle, \quad t \to +\infty, \quad \text{where} \tag{2.11}$$
$$u''(\xi + i\eta) = \partial^2 u(\xi + i\eta)/\partial\xi\partial\xi.$$

Comparison with (2.4) shows that

$$\tfrac{1}{2}\langle u''(z\eta)\theta, \theta\rangle = -\frac{\operatorname{Im} z}{|z|^2} q_u(\eta, \theta), \quad z \in \mathbf{C}_+, \ \eta \in \mathbf{R}^n \setminus \{0\}, \tag{2.12}$$

so $q_u(\eta, \theta)$ is a non-negative quadratic form in θ with $q_u(\eta, \eta) = 0$. The Levi form

$$\mathcal{L}_u(\zeta; w) = \sum_{j,k=1}^{n} \frac{\partial^2 u(\zeta)}{\partial\zeta_j \partial\bar\zeta_k} w_j \bar w_k = \frac{\partial^2 u(\zeta + zw)}{\partial z \partial \bar z}\Big|_{z=0}, \tag{2.13}$$

can be calculated on $\mathbf{CR}^n \setminus \mathbf{R}^n$ by means of H and q_u,

$$\mathcal{L}_u(z\eta; w) = \tfrac{1}{4}\langle (H''(\operatorname{Im} z\eta) - q''_{u\,\theta\theta}(\eta, 0)/\operatorname{Im} z)w, \bar w\rangle. \tag{2.14}$$

Since $q_u(\eta, \theta)/\operatorname{Im} z = q_u(\operatorname{Im} z\eta, \theta)$ by (2.5), the plurisubharmonicity of u gives again

$$0 \leq q_u(\eta, \theta) \leq \tfrac{1}{2}\langle H''(\eta)\theta, \theta\rangle, \quad \eta \in \mathbf{R}^n \setminus \{0\}, \ \theta \in \mathbf{R}^n. \tag{2.15}$$

Since $\mathcal{L}_u(\theta + z\eta; \eta)$ is non-negative and vanishes when $\theta = 0$ and $\operatorname{Im} z\eta \neq 0$, it must vanish of second order then. In fact, we shall prove that it vanishes of fourth order. Since

$$tv(tz) = t^2(u(\theta/t + z\eta) - \operatorname{Im} zH(\eta)) \to \operatorname{Im}(1/z) q_u(\eta, \theta), \quad t \to +\infty,$$

we obtain by applying $\partial^2/\partial z \partial \bar z$, noting that $\operatorname{Im} z$ and $\operatorname{Im}(1/z)$ are harmonic,

$$t^2 \mathcal{L}_u(\theta/t + z\eta; \eta) \to 0, \quad t \to +\infty.$$

This proves that $\mathcal{L}_u(\theta + z\eta; \eta)$ vanishes of third order when $\theta = 0$, and since $\mathcal{L}_u \geq 0$ it must vanish of fourth order. We do not know if there is an analogue of this fact for general functions in P_H.

The following theorem suggests that Theorem 2.4 lists the most important properties of q_u when $u \in P_H$ and u is homogeneous.

Theorem 2.5. *Let H be a supporting function in $C^3(\mathbf{R}^n \setminus \{0\})$, and let $Q(\eta, \theta)$, $\theta \in \mathbf{R}^n$, $\eta \in \mathbf{R}^n \setminus \{0\}$ be convex, even and homogeneous of degree 2 in θ, positively homogeneous of degree -1 in η, and assume that*

$$Q(\eta, \theta + \tau\eta) = Q(\eta, \theta), \quad \tau \in \mathbf{R},$$
$$0 < Q(\eta, \theta) < \tfrac{1}{2}\langle H''(\eta)\theta, \theta\rangle, \quad \text{if } \eta, \theta \text{ are linearly independent.}$$

Assume that $Q(\eta,\theta) \in C^2$ when η and θ are linearly independent. Then one can find $f \in \mathcal{E}'(\mathbf{R}^n)$ such that for the indicator function u of $\log|\hat{f}|$ we have when $\operatorname{Im} z > 0$, $\operatorname{Re} z = 0$,

$$t(u(\theta + tz\eta) - H(\operatorname{Im} tz\eta)) \to (\tfrac{1}{2}\langle H''(\eta)\theta,\theta\rangle - Q(\eta,\theta))\operatorname{Im}(1/z), \quad t \to \infty.$$

Thus $u \in P_H$ and $q_u(\eta,\theta) = \frac{1}{2}\langle H''(\eta)\theta,\theta\rangle - Q(\eta,\theta)$.

The core of the construction is an examination of the indicator function of $\log|\hat{f}|$ when f is the characteristic function of a convex set with analytic boundary. (These generalize (2.1).) Unfortunately, the homogeneity assumed in the theorem is essential in this proof.

In the smooth case we have seen that the Levi form degenerates of fourth order on $\mathbf{CR}^n$. That there is no higher degeneration in general is shown by the explicit example

$$u = \tfrac{1}{\sqrt{2}}(u_1^2 + u_2^2), \quad u_1(\zeta) = |\operatorname{Im}\zeta|,\ u_2(\zeta) = |\operatorname{Im}\langle\zeta,\zeta\rangle^{\frac{1}{2}}|.$$

The Levi form $\mathcal{L}_u$ has the lower bound

$$\mathcal{L}_u(\zeta;w) \ge L(\zeta;w)/32, \quad \zeta \in \mathbf{CR}^n \setminus \mathbf{R}^n,\ w \in \mathbf{C}^n, \tag{2.16}$$

$$L(\zeta;w) = \frac{d(\zeta,\mathbf{CR}^n)^4|w_\eta\eta|^2}{|\operatorname{Im}\zeta|^3|\zeta|^2} + \frac{|w_\eta^\perp|^2}{|\operatorname{Im}\zeta|}, \quad w = w_\eta\eta + w_\eta^\perp,\ \langle w_\eta^\perp,\eta\rangle = 0. \tag{2.17}$$

Starting from this example one can also prove:

Proposition 2.6. *Let $H \in C^3(\mathbf{R}^n \setminus \{0\})$ be a supporting function such that $H''(\xi)$ has rank $n-1$ for every $\xi \ne 0$. Then one can find $u_0 \in P_H$ positively homogeneous of degree 1 and a constant $C_0 > 0$ such that $u_0 \in C^3(\mathbf{C}^n \setminus \mathbf{R}^n)$, and*

$$C_0^{-1}L(\zeta;w) \le \mathcal{L}_{u_0}(\zeta;w) \le C_0 L(\zeta;w), \quad \zeta \in \mathbf{C}^n \setminus \mathbf{R}^n,\ w \in \mathbf{C}^n, \tag{2.18}$$

$$\mathcal{L}_{u_\delta}(\zeta;w) \le (1 + C_0\delta)\mathcal{L}_{u_0}(\zeta;w), \quad \zeta \in \mathbf{C}^n \setminus \mathbf{R}^n,\ w \in \mathbf{C}^n, \tag{2.19}$$

when δ is small. Here u_δ is the regularization of u_0 defined by

$$u_\delta(\zeta) = \int u(\zeta + \delta A\zeta)\varphi(A)\,dA, \tag{2.20}$$

where $0 \le \varphi \in C_0^\infty$ in the n^2 dimensional vector space of real $n \times n$ matrices, and the integral of φ with respect to the Lebesgue measure equals 1.

The regularization in (2.20) makes u_δ a C^∞ function outside $\mathbf{CR}^n$ but not in $\mathbf{CR}^n$, since $(1+\delta A)\mathbf{CR}^n \subset \mathbf{CR}^n$ for the maps involved. There is a natural metric G in $\mathbf{C}^n \setminus \mathbf{CR}^n$ defined as follows: If $\zeta \in \mathbf{C}^n \setminus \mathbf{CR}^n$ we can write $\zeta = e^{i\theta}(a+ib)$ where $a,b \in \mathbf{R}^n$, $\langle a,b\rangle = 0$ and $|a| \ge |b|$; we define

$$G_\zeta(w) = \frac{|\operatorname{Re}(e^{-i\theta}w)|^2}{|a|^2} + \frac{|\operatorname{Im}(e^{-i\theta}w)|^2}{|b|^2}, \quad \zeta \in \mathbf{C}^n \setminus \mathbf{CR}^n,\ w \in \mathbf{C}^n.$$

The metric is actually smooth,

$$G_\zeta(w) = \frac{2(|\zeta|^2|w|^2 - \operatorname{Re}(\langle\zeta,\zeta\rangle\overline{\langle w,w\rangle}))}{|\zeta|^4 - |\langle\zeta,\zeta\rangle|^2}.$$

Its importance in connection with the regularization is that $u_\delta(\zeta)$ is an average of u_0 over a G_ζ ball of radius $\sim \delta$ with center at ζ, which means that u_δ behaves as a symbol corresponding to the metric G_ζ.

3. Constructions of plurisubharmonic and entire functions. The fact that the Levi form of every function in P_H is highly degenerate at $\mathbf{CR}^n$ makes it hard to construct a function with a given limit set $M \subset P_H$ unless the functions in M are fairly close to each other at $\mathbf{CR}^n$. The following theorem gives a sufficient condition:

Theorem 3.1. *Let H be a supporting function, let $M \subset P_H$ be compact, connected and T invariant, and let T be chain recurrent on M. Assume that there is a positively homogeneous function $u_0 \in P_H$ satisfying* (2.18) *and* (2.19), *and constants $C > 0$, $\delta > 0$, such that with $E_\zeta = \{w \in \mathbf{C}^n; G_\zeta(w - \zeta) < \delta^2\}$*

$$\int_{E_\zeta} |v(w) - u_0(w)|\, d\lambda(w)/\lambda(E_\zeta) \leq C d(\zeta, \mathbf{CR}^n)^4/|\operatorname{Im}\zeta|^3, \tag{3.1}$$

for all $v \in M$ and $\zeta \in \mathbf{C}^n \setminus \mathbf{CR}^n$. Also assume that

$$M \times M \ni (u_1, u_2) \mapsto \sup_{\zeta \in \mathbf{C}^n \setminus \mathbf{CR}^n} \frac{|\operatorname{Im}\zeta|^3}{d(\zeta, \mathbf{CR}^n)^4} \int_{E_\zeta} |u_1(w) - u_2(w)|\, d\lambda(w)/\lambda(E_\zeta), \tag{3.2}$$

which is bounded by (3.1), *is a continuous function. Then there is a function $u \in P_H$ with $L_\infty(u) = M$. For any given functions $\kappa(r) \downarrow 0$ and $\lambda(r) \uparrow \infty$ as $\mathbf{R}_+ \ni r \to \infty$ one can choose u so that $\mathcal{L}_u(\zeta; w) \geq \kappa(|\zeta|) L(\zeta; w)$ for large $|\zeta|$ when $\zeta \in \mathbf{C}^n \setminus \mathbf{R}^n$ and for any closed cone $\Gamma \subset \mathbf{C}^n \setminus \mathbf{CR}^n$*

$$|D^\alpha(u - u_0)(\zeta)| \leq C_{\alpha,\Gamma}|\zeta|^{1-|\alpha|}\lambda(|\zeta|)^{2n+|\alpha|}, \quad \textit{when } \zeta \in \Gamma \textit{ and } |\zeta| \textit{ is large.}$$

For the proof one chooses a sequence u_ν which is dense in M such that with sequences $\alpha_\nu \downarrow 0$ and $\omega_\nu \uparrow \infty$ the distance from $T_{\omega_\nu} u_\nu$ to $T_{\alpha_{\nu+1}} u_{\nu+1}$ converges to 0. The existence of such sequences expresses that M is connected and that T is chain recurrent on M. Next one moves a regularization of u_ν far away by a dilation T_{1/τ_ν} where $\tau_0 = 1$, $\sigma_\nu = \omega_\nu \tau_\nu = \alpha_{\nu+1}\tau_{\nu+1}$, which defines τ_ν and σ_ν inductively. With u chosen in this way for $2\sigma_{\nu-1} \leq |\zeta| \leq \sigma_\nu$ one switches to the next function when $\sigma_\nu \leq |\zeta| \leq 2\sigma_\nu$. However, to be able to compensate for the error terms in the Levi form caused by the switch one actually has to replace the regularization of u_ν by an average with a regularization of u_0, taking a steadily diminishing weight for u_0. The technical details cannot be given here.

The last statements in Theorem 3.1 allow one to modify constructions in Sigurdsson [6] to prove:

Corollary 3.2. *If the hypotheses of Theorem 3.1 are fulfilled and $u_0 \in C^3(\mathbf{C}^n \setminus \mathbf{CR}^n)$, then one can find $f \in \mathcal{E}'(\mathbf{R}^n)$ such that H is the supporting function of the convex hull of $\operatorname{supp} f$, and $L_\infty(\log|\hat{f}|) = M$.*

A complete characterization of limit sets of plurisubharmonic functions u satisfying (1.3) and (1.11) would require either a substantial improvement of the properties of $L_\infty(u)$ given in Section 1, or a much more efficient construction method than that used to prove Theorem 3.1, or both. A basic question is if q_v is independent of $v \in L_\infty(u)$, which is implied by the hypotheses we had to make in Theorem 3.1. In particular, that would mean that $q_v(\eta, \theta)$ is homogeneous in θ of degree 2. However, it is an open question whether that is true. Note that if q_v is in fact independent of $v \in L_\infty(u)$, then we would have a uniquely defined map $\mathcal{E}' \ni f \mapsto q_v$, $v \in L_\infty(\log|\hat{f}|)$, which would satisfy the analogue of the theorem of supports (1.18). However, if q_v does depend on $v \in L_\infty(u)$, then one would have a situation similar to that in Proposition 1.4, which we proved to indicate the consequences of variations in the limit set.

Further references and background material can be found in the references below.

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