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L^p ESTIMATES FOR THE WAVE EQUATION AND APPLICATIONS

CHRISTOPHER D. SOGGE

In the last few years there has been a lot of work in proving estimates for Fourier integrals arising in studying the wave equation. The purpose of this paper is to go over some recent developments and applications.

We shall start out by going over L^p-Sobolev space estimates for Fourier integral operators. These arise naturally in harmonic analysis, for instance, in the study of maximal operators, as well as in study of eigenfunctions and eigenvalues on manifolds. Sharp $L^p \to L^p$ estimates for Fourier integrals are usually harder to obtain than estimates involving different norms. The arguments involved often involve making a "plane wave" decomposition of the operator, obtaining simpler operators which lend themselves to harmonic analysis techniques.

We shall also go over some recent joint work with H. Lindblad which involves mixed-norm estimates for the wave equation and applications to semilinear wave equations. These estimates strengthen earlier ones used by Grillakis and others. They are related to Strichartz's local smoothing estimates and their proof is based on a proof of his restriction theorem using real interpolation, rather than the more standard analytic interpolation arguments. The arguments rely on estimates for "dyadic pieces" of the fundamental solution for the wave equation, which in turn follow from stationary phase. Using the mixed-norm estimates we obtain sharp local existence theorems for semilinear wave equations with rough data. Some of the existence results were also obtained independently by L. Kapitanski.

1. $L^p \to L^p$ inequalities and harmonic analysis

The type of Fourier integrals that arise in solving the wave equation (microlocally) take the form

$$(\mathcal{F}f)(z) = \int_{\mathbb{R}^n} e^{i\varphi(z,\xi)} a(z,\xi) \hat{f}(\xi)\, d\xi\,, \tag{1}$$

where $a \in S^{\mu}_{\text{comp}}(\mathbb{R}^d \times \mathbb{R}^n \backslash 0)$ is a symbol of order μ (and type $(1,0)$) and the phase φ is real, in $C^\infty(\mathbb{R}^d \times \mathbb{R}^n \backslash 0)$ and satisfies

$$\text{rank } \varphi''_{z\xi} = n\,. \tag{2}$$

This of course forces the dimension d of the target space to be $\geqslant n$. The usual convention is that the order of $\mathcal{F}$ is $\mu + (n-d)/4$.

It has been known for some time that if $d = n$, then, in general, zero-order elliptic operators are not bounded on L^p for p different from 2. For instance, if $\varphi = z \cdot \xi + |\xi|$, $\mathcal{F}$ is unbounded on L^p, $p \neq 2$. This fact due to W. Littman [20] and one can extend it and show that if $d = n$ and

$$\text{rank } \varphi''_{\xi\xi} = k \tag{3}$$

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somewhere, then $\mathcal{F}$ is not bounded on L^p if it is elliptic and $\mu > -k|1/p - 1/2|$. Since, generically, $k = n-1$ in (3), usually one must lose $(n-1)|1/p - 1/2|$ derivatives in L^p, which is in sharp contrast to the case of pseudo-differential operators, where there is no loss of derivatives in L^p if $1 < p < \infty$. One does not lose more than this, as seen in the following result of Seeger, Stein and the author [26].

Theorem 1. *Let $d = n$ and suppose that (2) holds. Then, if $1 < p < \infty$, $\mathcal{F} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ if $\mu = -(n-1)|1/p - 1/2|$. Also, if rank $\varphi''_{\xi\xi} \equiv k$, one can take $\mu = -k|1/p - 1/2|$. For elliptic operators, all of these results are sharp.*

If rank $\varphi''_{\xi\xi} \equiv k$, then the singular support of the kernel of $\mathcal{F}$ is a submanifold of $\mathbb{R}^n \times \mathbb{R}^n$ of codimension $n-k$, so this result says that the loss of regularity is related to the "size" of the singular support.

If the rank of $\varphi''_{\xi\xi}$ is not constant then the singular support of the kernel can be quite complicated, and, for related reasons, it can be very hard to analyze the kernel using standard stationary phase methods. One can get around this obstacle though by using plane wave ideas which have been used extensively in harmonic analysis, going back to early work of Fefferman on the ball multiplier and related problems.

In the present context, the idea is to break up the complicated operator $\mathcal{F}$ into pieces which are much simpler and behave essentially as (translated) non-isotropic pseudo-differential operators of type-1/2. Specifically, one first breaks up the operator dyadically. If $\beta \in C_0^\infty(\mathbb{R}_+)$ satisfies $\sum_{-\infty}^{\infty} \beta(\tau/2^j) = 1$, $\tau > 0$, one lets

$$(\mathcal{F}_\lambda f)(z) = \int e^{i\varphi(z,\xi)} \beta(|\xi|/\lambda) a(z,\xi) \hat{f}(\xi)\, d\xi\,, \quad \lambda = 2^j \geqslant 1\,,$$

so that $\mathcal{F} = \sum_{j=1}^{\infty} \mathcal{F}_{2^j} + \mathcal{F}_0$, where $\mathcal{F}_0$ is smoothing. The kernels of the $\mathcal{F}_\lambda$ are essentially supported on a λ^{-1} neighborhood of the singular support, but they can be quite complicated as λ gets large. To get around this fact, one makes a further decomposition. Specifically, for a given λ, let $\{\chi_\lambda^\nu\}_{\nu=1}^{\lambda^{(n-1)/2}}$ be homogeneous of degree zero, in $C^\infty(\mathbb{R}^n \backslash 0)$, and satisfy $\sum_\nu \chi_\lambda^\nu(\xi) = 1$, $\xi \neq 0$, as well as

$$\chi_\lambda^\nu(\xi) = 0 \text{ if } |\xi - \xi_\lambda^\nu| \geqslant C\lambda^{-1/2}, \text{ some } \xi_\lambda^\nu \in S^{n-1} \text{ and } |D^\alpha \chi_\lambda^\nu(\xi)| \leqslant C_\alpha \lambda^{|\alpha|/2}\, \forall \alpha\,, \text{ if } \xi \in S^{n-1}\,.$$

Thus, the χ_λ^ν should be thought of as conic cutoff functions supported in cones of aperture $\approx \lambda^{-1/2}$. Using these functions, we let

$$(4) \qquad (\mathcal{F}_\lambda^\nu f)(z) = \int e^{i\varphi(z,\xi)}\, a_\lambda^\nu(z,\xi) \hat{f}(\xi)\, d\xi\,, \quad a_\lambda^\nu(z,\xi) = \beta(|\xi|/\lambda)\chi_\lambda^\nu(\xi) a(z,\xi)\,.$$

The point of this decomposition is that, in the right scale, $\xi \to \varphi$ is essentially linear on supp a_λ^ν. Because of this and the fact that a_λ^ν has ξ-support in a rectangle with $n-1$ sides of length $\approx \lambda^{1/2}$ and one of length λ, one can show that, for fixed y, the kernel, $K_\lambda^\nu(z,y)$, of $\mathcal{F}_\lambda^\nu$ is essentially supported in a rectangle of dual dimensions, that is, one with $n-1$ sides of length $\lambda^{-1/2}$ and one of length λ^{-1}.

Using this decomposition one can show that

$$\mathcal{F} : \mathcal{H}^1 \to L^1 \quad \text{if} \quad \mu = -(n-1)/2\,,$$

if $\mathcal{H}^1$ is the standard Hardy space. Since a classical theorem of Hörmander says that zero order Fourier integrals of this type are bounded on L^2, the first part of Theorem 1 follows from applying the Hardy space interpolation of Fefferman and Stein. The other part of the theorem follows from similar arguments.

If we take $x = z$, and $\varphi = x \cdot \xi \pm t|\xi|$, with t fixed, then using Theorem 1 we can get sharp estimates for the Cauchy problem

$$\begin{cases} \Box u(t,x) = F(t,x) \\ u(0,x) = f(x)\,, \;\; \partial_t u(0,x) = g(x)\,, \end{cases} \tag{5}$$

if $\Box = (\partial/\partial t)^2 - \Delta$ is the d'Alembertian in $\mathbb{R}^{1+n}$. Specifically, if $0 < t < T$, with $T < \infty$, and $1 < p < \infty$, then

$$\begin{aligned} (6)\quad & \|u(t,\cdot)\|_{L^p(\mathbb{R}^n)} \\ & \leqslant C_T \left(\|f\|_{L^p_{\mu_p}(\mathbb{R}^n)} + \|g\|_{L^p_{\mu_p-1}(\mathbb{R}^n)} + \int_0^t \|F(s,\cdot)\|_{L^p_{\mu_p-1}(\mathbb{R}^n)}\, ds \right), \; \mu_p = (n-1)|1/p - 1/2|\,. \end{aligned}$$

Here L^p_μ denotes the L^p-Sobolev space with μ derivatives. This special case of Theorem 1 goes back to Peral [23] and Beals [1].

The same result holds for the wave equation outside of a convex obstacle and this is due to Smith and the author [27]. The plane wave decomposition and the analysis, though, is necessarily harder due to the effects of diffraction. It would be interesting to know whether this estimate and the estimates in the next section carry over to the setting of the wave equation inside a convex obstacle. Recent results of Grieser [8] lead one to suspect that in this case the sharp estimates could be considerably worse than those in the Euclidean or the diffractive case.

For applications it is often useful to have space-time estimates for u, rather than fixed time estimates as in (6). It turns out that if $p \leqslant 2$, the sharp local space-time estimates are no better than the ones obtained trivially from (6) via Minkowski's integral inequality. On the other hand, if $p > 2$, one can use the above decompositions and "geometric arguments," exploiting the curvature of the underlying light-cones in (7) below, to see that there is a gain of regularity, "local smoothing," if one measures the regularity in space-time.

Theorem 2. *Let $n \geqslant 2$. Then if $p > 2$ there is an $\varepsilon_p > 0$ so that if $S = [0,1] \times \mathbb{R}^n$ is the unit strip and if μ_p is as above then*

$$\|u\|_{L^p(S)} \leqslant C\left(\|f\|_{L^p_{\mu_p-\varepsilon}(\mathbb{R}^n)} + \|g\|_{L^p_{\mu_p-1-\varepsilon}(\mathbb{R}^n)} + \int_0^1 \|F(s,\cdot)\|_{L^p_{\mu_p-1-\varepsilon}(\mathbb{R}^n)}\, ds \right),$$

provided that $\varepsilon < \varepsilon_p$.

In $(1+2)$-dimensions this was first obtained by the author [29]. The proof was later simplified and improved greatly in papers by Mockenhaupt, Seeger and the author [21], [22]. These results say that in the important case of $n = 2$ one can take $\varepsilon_p = 1/2p$ for $p \geqslant 4$, and $\varepsilon_p = \frac{1}{2}(\frac{1}{2} - \frac{1}{p})$ for $2 < p < 4$. Slightly better results hold in higher dimensions. One might expect that one should be able to take $\varepsilon_p = 1/p$ for $p \geqslant 2n/(n-1)$. This would be a sharp result and would have a number of important applications in harmonic analysis.

This local smoothing phenomenon applies to a broader class of Fourier integral operators. If $d = n + 1$, and if $\mathcal{F}$ is as above, there is local smoothing if (2) holds and if we have the "cone condition," which says that the n-dimensional cones

$$\Gamma_z = \{ (\xi, \varphi'_z(z,\xi)) \} \subset T^*_z \mathbb{R}^{1+n} \backslash 0 \tag{7}$$

have the maximum possible number, $n-1$, of non-vanishing principal curvatures.

The main applications of these results concern maximal operators. For instance, if one uses the Sobolev embedding theorem they immediately give Bourgain's circular maximal theorem [3]:

$$\| \sup_{t>0} | \int_{S^1} f(x+ty)\, d\sigma(y) | \|_{L^p(\mathbb{R}^2)} \leqslant C_p \|f\|_{L^p(\mathbb{R}^2)}, \quad p>2, \quad f \in \mathcal{S}.$$

If the singular support of $\mathcal{F}$ is a hypersurface, the above local smoothing estimates are non-trivial. On the other hand, if the singular support is a submanifold of higher codimension, one presently does not have results which improve those obtained trivially from an application of Theorem 1. In particular, one would like to prove that there is local smoothing for the conormal Fourier integral operator of order $-1/2-1/4$ which which sends functions, f, of $\mathbb{R}^3$ to the 4-dimensional space of lines $\ell \in G_1(\mathbb{R}^3)$ in $\mathbb{R}^3$:

$$(\mathcal{F}f)(\ell) = \int_\ell \rho f d\sigma, \quad \rho \in C_0^\infty(\mathbb{R}^3).$$

Locally (after making a change of variables), this operator can be written as in (1) with $\mu=-1/2$, and both (2) and (7) will be satisfied. The L^4 local smoothing theorem for averaging over families of curves in the plane suggests that there should be an improvement of up to $1/8$ of a derivative over the easy consequence of Theorem 1 that $\mathcal{F}: L^4(\mathbb{R}^3) \to L^4_{1/4}(G_1(\mathbb{R}^3))$. This $1/8$ local smoothing would improve Bourgain's lower bound for the Hausdorff dimension of Besicovitch $(3,1)$ sets, i.e., compact measurable sets in $\mathbb{R}^3$ containing a unit line segment in every direction. Specifically, the best estimate now [4] is that such a set must have dimension $\geqslant 3-2/3$, while the $1/8$ local smoothing estimate would have as an immediate corollary that these sets always have dimension $\geqslant 3-1/2$. An early work of John [13] says that this local smoothing question is related to the problem of showing that there is local smoothing for solutions of the ultrahyperbolic equation. These problems seem to be harder than their hyperbolic counterparts because of the fact that the cones in (7) that arise are homogeneous extensions of non-convex, ruled surfaces.

2. Mixed-norm inequalities and semilinear wave equations

In this section I would like to describe some ongoing joint work [19] with Hans Lindblad concerning mixed norm estimates for the wave equation and applications to existence problems for semilinear wave equations.

It has been known for some time that non-trivial space-time estimates for the wave equation lead to good existence and scattering theorems for semilinear equations. This goes back, among other places, to pioneering work of Segal, Strauss and Strichartz. See Strauss [32] and Struwe [35] for historical background. In the case of the Laplacian one has the very favorable estimates of Hardy-Littlewood and Sobolev:

$$\|u\|_{L^q(\mathbb{R}^n)} \leqslant C_{p,q} \|\Delta u\|_{L^p(\mathbb{R}^n)}, \quad 1<p<q<\infty, \quad n(\tfrac{1}{p}-\tfrac{1}{q})=2, \quad u \in \mathcal{S}.$$

Additionally, if $x'=(x_2,\dots,x_n)$ and

$$\|f\|_{L^s_{x_1}L^q_{x'}(\mathbb{R}^n)} = \Big(\int_{-\infty}^{\infty} \Big(\int_{\mathbb{R}^{n-1}} |f(x_1,x')|^q\, dx'\Big)^{s/q} dx_1\Big)^{1/s},$$

there are related mixed-norm estimates for the Laplacian:

$$\|u\|_{L^s_{x_1}L^q_{x'}(\mathbb{R}^n)} \leqslant C_{q,s,p,r} \|\Delta u\|_{L^r_{x_1}L^p_{x'}(\mathbb{R}^n)},$$
$$(n-1)(\tfrac{1}{p}-\tfrac{1}{q})+\tfrac{1}{r}-\tfrac{1}{s}=2, \quad 1<r<s<\infty, \quad 1<p<q<\infty, \quad u \in \mathcal{S},$$

(cf. [30, p. 25-26]). Unfortunately, nothing this strong holds for the d'Alembertian. However, we do have control over certain mixed-norms and this leads to a sharp existence theorem for semilinear equations.

Theorem 3. *Let $n \geqslant 2$ and let u solve the inhomogeneous Cauchy problem (5). Then there is a constant C_q depending only on q so that*

$$\|u\|_{L^s_t L^q_x([0,T]\times\mathbb{R}^n)} + \|u(T,\cdot)\|_{\dot{\gamma}} \leqslant C_q\big(\|F\|_{L^r_t L^p_x([0,T]\times\mathbb{R}^n)} + \|f\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|g\|_{\dot{H}^{\gamma-1}(\mathbb{R}^n)}\big),$$

if we have the gap condition $n(\frac{1}{p}-\frac{1}{q})+\frac{1}{r}-\frac{1}{s}=2$, and

$$(i) \qquad \tfrac{1}{p}-\tfrac{1}{q}=\tfrac{2}{n+1},\ s=\tfrac{4q}{(n-1)(q-2)},\ \gamma=\tfrac{n+1}{2}\big(\tfrac{1}{2}-\tfrac{1}{q}\big),\ \text{when } \begin{cases} \big|\tfrac{1}{2}-\gamma\big|<\tfrac{1}{n-1},\ n\geqslant 3 \\ \big|\tfrac{1}{2}-\gamma\big|<\tfrac{1}{4},\ n=2. \end{cases}$$

If $n \geqslant 4$ the inequality also holds if the gap condition holds and

$$(ii) \qquad r=2,\ s=\tfrac{4q}{(n-1)(q-2)},\ \gamma=\tfrac{n+1}{2}\big(\tfrac{1}{2}-\tfrac{1}{q}\big),\ 0\leqslant\gamma<\tfrac{n-3}{2(n-1)},$$

or

$$(iii) \qquad s=2,\ r=\tfrac{4p}{4p-(n-1)(2-p)},\ \gamma=\tfrac{n-1}{2}-\tfrac{n}{q},\ \tfrac{n+1}{2(n-1)}<\gamma\leqslant 1.$$

Also, if $n \geqslant 2$, $\frac{2(n+1)}{n-1}\leqslant q<\infty$ and $\gamma=\frac{n}{2}-\frac{n+1}{q}$

$$\|u\|_{L^q([0,T]\times\mathbb{R}^n)} + \big\|\,|D_x|^{\gamma-1/2}u\,\big\|_{L^{\frac{2(n+1)}{n-1}}([0,T]\times\mathbb{R}^n)} + \|u(T,\cdot)\|_{\dot{\gamma}}$$
$$\leqslant C_q\Big(\|f\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|g\|_{\dot{H}^{\gamma-1}(\mathbb{R}^n)} + \big\|\,|D_x|^{\gamma-1/2}F\,\big\|_{L^{\frac{2(n+1)}{n+3}}([0,T]\times\mathbb{R}^n)}\Big).$$

Here $\dot{H}^\gamma$ denotes the homogeneous Sobolev space with norm $\|f\|_{\dot{H}^\gamma}=\|\,|D_x|^\gamma f\|_{L^2}$, and

$$\|u(t,\cdot)\|^2_{\dot{\gamma}} = \|u(t,\cdot)\|^2_{\dot{H}^\gamma(\mathbb{R}^n)} + \|\partial_t u(t,\cdot)\|^2_{\dot{H}^{\gamma-1}(\mathbb{R}^n)}.$$

One could restate the conditions on γ in the three cases in terms of a condition on q (or p). In case (i) it would read:

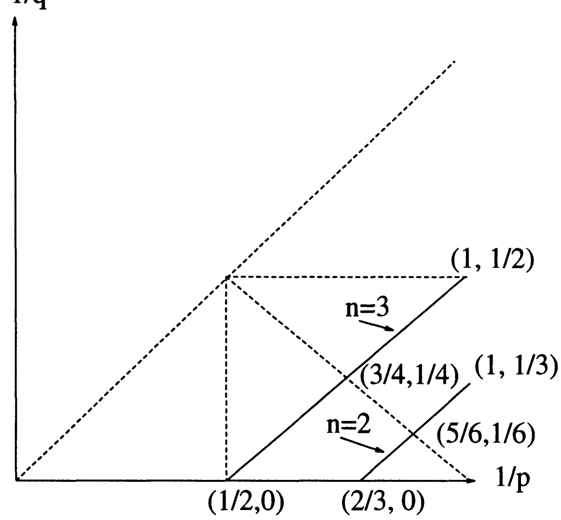
$$\begin{cases} 3<q<\infty,\ n=2 \\ 2<q<\infty,\ n=3 \\ \frac{2(n+1)(n-1)}{(n-1)^2+4}<q<\frac{2(n-1)}{n-3}, \end{cases}$$

while in the other two cases it would become $2\leqslant q<\frac{2(n+1)(n-1)}{(n-1)^2+4}$ and $\frac{2(n-1)}{n-3}<q\leqslant\frac{2n}{n-3}$, respectively.

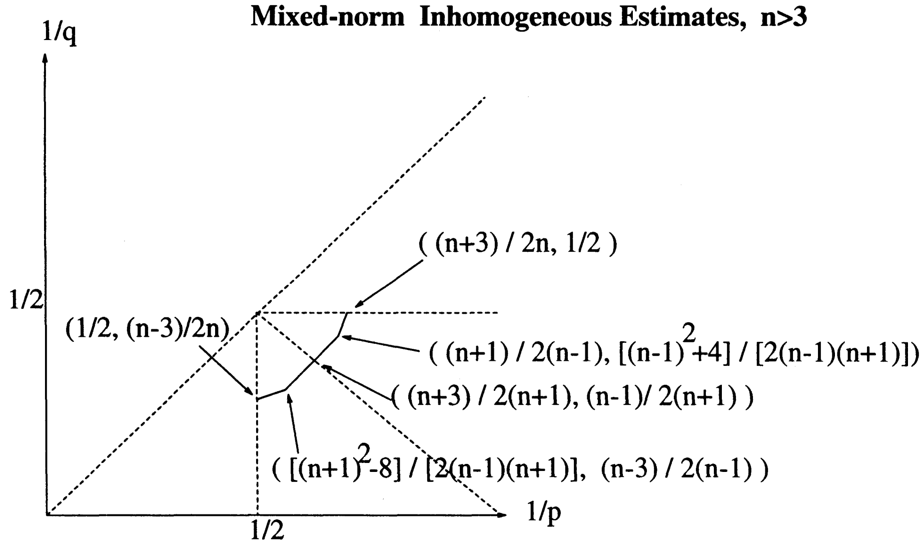
The special case where $\gamma=\frac{1}{2}$, $p=r=\frac{2(n+1)}{n+3}$ and $q=s=\frac{2(n+1)}{n-1}$ is due to Strichartz [33], [34]. When $n\geqslant 4$ the inequality corresponding to case (iii) in the corollary is slightly stronger than Grillakis' inequality [9, Corollary 1.4]. Using these strengthened estimates one can use his arguments to show that in dimension $n=6$ one does not need to assume radial symmetry to obtain classical solutions to wave equations with critical (repulsive) nonlinearities and smooth data. This result and, in fact, the existence of smooth solutions for $n\leqslant 7$, though, was also obtained by Shatah and Struwe [25].

In the following two figures we graph $(1/p, 1/q)$, where p and q are pairs of exponents occurring in the above mixed-norm estimates.

Mixed-norm Inhomogeneous Estimates, n=2, 3



Mixed-norm Inhomogeneous Estimates, n>3



Using the form of the fundamental solution of $\square$ one sees that, in order to prove Theorem 3, it suffices to make appropriate mixed-norm estimates for operators of the form

$$(W^\alpha F)(t,x) = \iint_{\mathbb{R}^{1+n}} e^{ix\cdot\xi + i(t-s)|\xi|} \hat{F}(s,\xi) \frac{d\xi}{|\xi|^\alpha} \, ds \,, \ \alpha < n \,,$$

or for related operators sending functions of n-variables to functions of $(n+1)$-variables. The proofs only use the Hardy-Littlewood theorem for fractional integrals, the M. Riesz interpolation theorem and pointwise estimates for the dyadic parts of the kernels:

$$K_j^\alpha(t,x)=\int_{\mathbb{R}^n} e^{ix\cdot\xi+it|\xi|}\beta(|\xi|/2^j)\,\frac{d\xi}{|\xi|^\alpha}\,,$$

if $\beta\in C_0^\infty(\mathbb{R}^n{}_+)$ is as before. The pointwise estimates, which are related to Hüygen's principle, are the following:

$$|K_j^\alpha(t,x)|\leqslant C_N\,\lambda^{\frac{n+1}{2}}\,|t|^{-\frac{n-1}{2}}\,\big(1+\lambda\big|\,|t|-|x|\,\big|\big)^{-N}\,, N=1,2,\dots\,,\ \lambda=2^j\,.$$

These follow easily from stationary phase. The large negative power of $|t|$ in higher dimensions, which is related to the fact that the fundamental solution becomes more and more singular as n increases, accounts for why the favorable range in case (i) gets smaller and smaller as n grows.

Using Theorem 3, Lindblad and the author obtain sharp results concerning semilinear Cauchy problems of the form

$$\begin{cases}\Box u=F_\kappa(u)\\ u(0,x)=f(x),\quad \partial_t u(0,x)=g(x)\,,\end{cases}\tag{8}$$

where, for a given $\kappa>1$, F_κ is assumed to be a C^1 function satisfying

$$|F_\kappa(u)|\leqslant C|u|^\kappa\,,\qquad |F_\kappa'(u)|\leqslant C|u|^{\kappa-1}\,.\tag{9}$$

With this notation, our main result is the following.

Theorem 4. *Let $n\geqslant 2$ and set*

$$\kappa_0=\tfrac{(n+1)^2}{(n-1)^2+4}\,,\ \textit{if}\ n\geqslant 3\,,\ \textit{and}\ \kappa_0=3\ \textit{for}\ n=2\,.$$

Assume that F_κ satisfies (9) for $\kappa_0<\kappa<\infty$ if $n=2$ or 3, or $\kappa_0<\kappa\leqslant\frac{n+1}{n-3}$ for $n\geqslant 4$. If $n\geqslant 4$ and $\kappa>\frac{n+1}{n-3}$ we may also take $F_\kappa=\pm u^\kappa$, provided that κ is an integer. Suppose that the initial data satisfy $f\in\dot H^\gamma(\mathbb{R}^n)$, $g\in\dot H^{\gamma-1}(\mathbb{R}^n)$, with γ satisfying

$$\gamma=\gamma(\kappa)=\begin{cases}\frac{n+1}{4}-\frac{1}{\kappa-1}\,, & \kappa_0<\kappa\leqslant\frac{n+3}{n-1}\,,\\ \frac{n}{2}-\frac{2}{\kappa-1}\,, & \text{if }\ \kappa\geqslant\frac{n+3}{n-1}\,.\end{cases}\tag{10}$$

Then there is a $T_*>0$ and a unique (weak) solution u to (8) verifying

$$u\in L^\infty\big([0,T_*];\dot H^\gamma(\mathbb{R}^n)\big)\cap C^{0,1}\big([0,T_*];\dot H^{\gamma-1}(\mathbb{R}^n)\big)\cap L_t^sL_x^q([0,T_*]\times\mathbb{R}^n)\,,$$

where $q=\frac{n+1}{2}(\kappa-1)$ and $s=q$ if $\kappa\geqslant\frac{n+3}{n-1}$, while $s=\frac{4q}{(n-1)(q-2)}$ if $\kappa<\frac{n+3}{n-1}$. For a given $\kappa<\frac{n+3}{n-1}$, T_* is depends only on the size of the norm of the initial data, while for $\kappa\geqslant\frac{n+3}{n-1}$ this is not the case. On the other hand, if $\kappa\geqslant\frac{n+3}{n-1}$, one can take $T_*=\infty$ provided that norm of the data is small, i.e.,

$$\|f\|_{\dot H^\gamma(\mathbb{R}^n)}+\|g\|_{\dot H^{\gamma-1}(\mathbb{R}^n)}<\varepsilon\,,$$

where $\varepsilon > 0$ depends only on κ and the constant in (9).

An equivalent way of stating the local existence part is that for data $f \in \dot H^{\gamma}(\mathbb{R}^3)$, $g \in \dot H^{\gamma-1}(\mathbb{R}^3)$ there is local existence for (8), provided that

$$(10') \qquad \kappa = \begin{cases} 1 + \frac{4}{(n+1)-4\gamma}, & \text{if } \gamma_0 < \gamma \leqslant 1/2, \\ 1 + \frac{4}{n-2\gamma}, & \text{if } \gamma \geqslant 1/2, \end{cases}$$

with $\gamma_0 = \frac{n-3}{2(n-1)}$, $n \geqslant 3$ and $\gamma_0 = 1/4$ for $n = 2$. The different behavior for γ smaller and bigger than $1/2$ is related to the Strichartz-1/2 local smoothing estimate for the wave equation we referred to before. It is interesting to note that, since $1/2 = \gamma(\kappa) \iff \kappa = \frac{n+3}{n-1}$, the change of behavior takes place at the conformally invariant nonlinearity (e.g. $F_\kappa(u) = c|u|^{\frac{n+3}{n-1}-1}u$).

For the local existence results, we can also take $\gamma > \gamma(\kappa)$ in (10) if we use assume that the data belong to the inhomogeneous Sobolev spaces H^{γ} and $H^{\gamma-1}$, respectively. Under this assumption we can also of course assume that κ is smaller than the number given in (10′). Moreover, if we assume that the data belong to the inhomogeneous Sobolev spaces we need only assume that (9) holds when $|u| \geqslant 1$. On the other hand, assuming that the data belong to the homogeneous Sobolev spaces and that (9) holds for small u as well is necessary for the global existence results. Finally, if $n \geqslant 4$ and $\kappa > \frac{n+1}{n-3}$, we have to assume that κ is an integer and that F_κ is a pure power for technical reasons based on the fact that our proof requires a certain amount of regularity of F_κ if κ is larger than $\frac{n+3}{n-1}$.

These results generalize those in Lindblad [18]. There it was shown that for $n = 3$ and $\Box u = F_\kappa(u)$ with $\kappa = 2$ there is local existence if $\gamma > 0$. On the other hand, it was shown that if $F_\kappa = u^2$ this problem is not well-posed for L^2, i.e. $\gamma = 0$, so it is interesting that for $n = 3$ and $\kappa > 2$ one can obtain sharp endpoint results. It is also worth noting that in the range $\kappa \geq \frac{n+3}{n-1}$ the existence is just what is given by the trivial scaling argument, whereas in the lower range $\kappa < \frac{n+3}{n-1}$ one needs more regularity than predicted by the scaling argument. This argument relies on the fact that if u solves (8) with data f, g, then $u_\varepsilon = \varepsilon^{-\frac{2}{\kappa-1}}u(t/\varepsilon, x/\varepsilon)$ solves the same equation with data $f_\varepsilon = \varepsilon^{-\frac{2}{\kappa-1}}f(x/\varepsilon)$, $g_\varepsilon = \varepsilon^{-\frac{2}{\kappa-1}-1}g(x/\varepsilon)$. If, say the lifespan of u were T, then the lifespan of u_ε would be $T_\varepsilon = \varepsilon T$. On the other hand,

$$\|f_\varepsilon\|_{\dot H^\gamma}/\|f\|_{\dot H^\gamma} = \|g_\varepsilon\|_{\dot H^{\gamma-1}}/\|g\|_{\dot H^{\gamma-1}} = \varepsilon^{\frac{n}{2}-\frac{2}{\kappa-1}-\gamma},$$

and so if γ were smaller than $\frac{n}{2} - \frac{2}{\kappa-1}$, one would have both the norm of the data and the lifespan going to zero with ε. If f and g had compact support one could thus add up suitable translates of dilates of the data, obtaining new data for which there is no local existence.

Proving that the problem $\Box u = |u|^\kappa$ is ill-posed in $\dot H^{\gamma(k)-\varepsilon}$, $\varepsilon > 0$, is more delicate if $\kappa_0 < \kappa < \frac{n+3}{n-1}$. However, it turns out that there are sequences of data $(f_j, g_j) \in C_0^\infty$ with fixed compact support so that the $\dot H^{\gamma(k)-\varepsilon} \times \dot H^{\gamma(k)-\varepsilon-1}$ norms of (f_j, g_j) go to zero, while at the same time $T_j \to 0$, if T_j is the supremum over all times T such that there is a solution $u_j \in C^\infty([0,T] \times \mathbb{R}^n)$ with this data. Somewhat stronger versions involving the lifespans of $\dot H^{\gamma(k)-\varepsilon}$ extensions of the u_j hold as well. T_j

Some of these results were also obtained independently by Kapitanski [15] using a different proof. Also following [18], he obtained the local existence results in Theorem 4 when $n \geqslant 3$ and $\kappa_0 < \kappa \leqslant \frac{n+2}{n-2}$. The results for $\kappa > \frac{n+2}{n-2}$ and the global existence results are new. Moreover, the uniqueness in L^q and the ill-posedness results are new.

Our results also improve some in Beals and Bezard [2]. In this paper, they showed that for $n \geqslant 5$ there is local existence for $F_\kappa(u) = u^2$ provided that $\gamma = (n-3)/2$, while our results show that this is the case if $\gamma = 1/4$, when $n = 4$, or $(n-4)/2$ if $n \geqslant 5$. Notice that for this quadratic nonlinearity the existence results improve in some sense as the dimension increases, since when $n < 5$ the nonlinearity is in the "subconformal" range, while for $n \geqslant 5$ it is in the "superconformal range". This applies to other nonlinearities too since the "superconformal range" $[\frac{n+3}{n-1}, \infty) \to (1, \infty)$.

In higher dimensions there is a third range of κ. Here the relationship between κ and γ is less favorable than the above one corresponding to $\kappa_0 < \kappa \leqslant \frac{n+3}{n-1}$. This is related to the fact that in Theorem 3, the $\dot{H}^\gamma(\mathbb{R}^n)$ estimates for the (linear) wave equation are less favorable for γ smaller than $\gamma_0 = \frac{n-3}{2(n-1)}$, compared to $\gamma > \gamma_0$.

Theorem 5. *Let $n \geqslant 4$ and suppose that $\frac{n+3}{n} \leqslant \kappa < \kappa_0 = \frac{(n+1)^2}{(n-1)^2+4}$. Suppose further that $f \in \dot{H}^\gamma(\mathbb{R}^n)$, $g \in \dot{H}^{\gamma-1}(\mathbb{R}^n)$, with γ satisfying*

$$\gamma = \gamma(\kappa) = \frac{n+1}{4} - \frac{(n+1)(n+5)}{4} \cdot \frac{1}{2n\kappa - (n+1)}\,. \tag{11}$$

Then there is a $T_* > 0$, depending only on the size of the initial data, and a unique (weak) solution u to (8) verifying

$$u \in L^\infty\big([0,T_*]; \dot{H}^\gamma(\mathbb{R}^n)\big) \cap C^{0,1}\big([0,T_*]; \dot{H}^{\gamma-1}(\mathbb{R}^n)\big) \cap L^s_t L^q_x([0,T_*] \times \mathbb{R}^3)\,,$$

with $q = \frac{4n\kappa - 2(n+1)}{n+5}$ and $s = \frac{4q}{(n-1)(q-2)}$ and $s = \frac{4q}{(n-1)(q-2)}$.

As before, we can restate things in terms of γ. Specifically, if $f \in \dot{H}^\gamma(\mathbb{R}^n)$ and $g \in \dot{H}^{\gamma-1}(\mathbb{R}^n)$, there is local existence for (8) provided that

$$\kappa = \frac{n+1}{n}\Big(1 + \frac{2(1+\gamma)}{(n+1) - 4\gamma}\Big)\,, \quad \text{if } 0 \leqslant \gamma < \frac{n-3}{2(n-1)}\,. \tag{11'}$$

Also, if $\gamma > \gamma(\kappa)$, we have local existence and uniqueness if we assume that the data belong to the inhomogeneous Sobolev spaces H^γ and $H^{\gamma-1}$, respectively. Finally, for the border case where $\kappa = \kappa_0$, if $\gamma > \gamma(\kappa_0) = \frac{n-3}{2(n-1)}$, there is local existence and uniqueness for $f \in H^\gamma(\mathbb{R}^n)$ and $g \in H^{\gamma-1}(\mathbb{R}^n)$.

Theorem 5 is stronger than corresponding results in Kapitanski [15]. In particular, for L^2 data, i.e. $\gamma = 0$, he shows that there is local existence if $\kappa < \frac{n+1}{n-1}$, which is smaller than our power $\frac{n+3}{n}$.

To close, let us sketch the proof of the $\dot{H}^{1/2}$ existence theorem for the conformally invariant equation in $\mathbb{R}^{1+3}$,

$$\Box u = \pm u^3\,,$$

with data $f \in \dot{H}^{1/2}$, $g \in \dot{H}^{-1/2}$. To prove that there is local existence or global existence for small data for this equation, we use standard iteration arguments and Strichartz's estimates.

More specifically, we first set $u_{-1} \equiv 0$, and then define u_m, $m = 0, 1, 2, \ldots$, by

$$\begin{cases} \Box u_m = \pm u_{m-1}^3 \\ u_m(0,x) = f(x)\,,\ \partial_t u_m(0,x) = g(x)\,. \end{cases}$$

Then, we need to show that there is a $0 < T_* \leqslant \infty$ and a function u as in Theorem 4 so that

$$u_m \to u, \text{ and } u_m^3 \to u^3 \text{ in } \mathcal{D}'(S_{T_*})\,. \tag{12}$$

Here $S_{T_*} = [0, T_*] \times \mathbb{R}^3$ if T_* is finite and $\mathbb{R}^{1+3}$ if it is infinite. This of course implies that u is a weak solution of the Cauchy problem.

The main step in proving this is to use Strichartz's special case of Theorem 4 that we referred to before to see that the nonlinear mapping sending u_m to u_{m+1} is a contraction in $L^4(S_T)$ if either T or the size of the data is small enough. To see this we use this estimate and Hölder's inequality to get

$$\|u_{m+1} - u_{j+1}\|_{L^4(S_T)} \leqslant C\|u_m^3 - u_j^3\|_{L^{4/3}(S_T)} \leqslant C\|u_m - u_j\|_{L^4(S_T)} \left(\|u_m\|^2_{L^4(S_T)} + \|u_j\|^2_{L^4(S_T)}\right).$$

Taking $j = -1$ yields

$$\|u_{m+1}\|_{L^4(S_T)} \leqslant \|u_{m+1} - u_0\|_{L^4(S_T)} + \|u_0\|_{L^4(S_T)} \leqslant C\|u_m\|^3_{L^4(S_T)} + \|u_0\|_{L^4(S_T)}. \tag{13}$$

The first estimate in Theorem 4 implies that the u_0 is in L^4 with a norm which is dominated by a fixed constant times the norm of the data. Therefore if we assume that the latter is small or if we take T to be small enough we can assume that

$$\|u_0\|_{L^4(S_T)} \leqslant \varepsilon_0,$$

with $\varepsilon_0 > 0$ as small as we like. Moreover if this number is small enough, (13) and induction imply that

$$\|u_{m+1}\|_{L^4(S_T)} \leqslant \tfrac{1}{2}\|u_m\|_{L^4(S_T)} + \|u_0\|_{L^4(S_T)},$$

yielding $\|u_m\|_{L^4(S_T)} \leqslant 2\varepsilon_0$. On account of this we get the first part of (12) since if $2C\varepsilon_0^2 < 1/2$, and if we take $j = m - 1$ above we find that

$$\|u_{m+1} - u_m\|_{L^4(S_T)} \leqslant \tfrac{1}{2}\|u_m - u_{m-1}\|_{L^4(S_T)},$$

which of course implies that u_m converges to a limit u in L^4 and hence in $\mathcal{D}'$. The fact that u_m^3 converges to u^3 follows from this and

$$\|u^3 - u_m^3\|_{L^{4/3}(S_T)} \leqslant C\|u - u_m\|_{L^4(S_T)} \cdot \left(\|u\|^2_{L^4(S_T)} + \|u_m\|^2_{L^4(S_T)}\right).$$

In fact, since we have just seen that the L^4 norms of u and the u_m are bounded by a fixed constant, this inequality together with the convergence of u_m to u in L^4 implies that u_m^3 converges to u^3 in $L^{4/3}$ and hence in $\mathcal{D}'$. Finally, the remaining fact that the $\dot{H}^{1/2}$ norms of $u(t, \cdot)$ are uniformly bounded for $0 < t < T$, follows from another application of the special case of Theorem 3 which is due to Strichartz.

The proofs of the other existence results follow similar lines. In each case, using Theorem 3, one shows that, if $\|u_0\|_{L^q(S_T)}$ is small enough, with $q = \frac{n+1}{2}(\kappa - 1)$, then the mapping sending u_m to u_{m+1} is a contraction in the space $L_t^s L_x^q(S_T)$, with $s = q$ in the superconformal range, and $s = 4q/(n-1)(q-2)$ in the subconformal range.

The proof that there is a unique solution with the above properties follows the same lines. For instance, if u and $\tilde{u}$ both solve the conformally invariant equation in $S_{T_*} \subset \mathbb{R}^{1+3}$ with data in $\dot{H}^{1/2}(\mathbb{R}^3)$, one argues as above to see that

$$\|u - \tilde{u}\|_{L^4(S_T)} \leqslant \tfrac{1}{2}\|u - \tilde{u}\|_{L^4(S_T)},$$

provided that $T > 0$ is small enough. This of course implies that $u = \tilde{u}$ in S_T, leading to the uniqueness.

The same argument gives a more general result concerning uniqueness for the Cauchy problem with $\dot{H}^{1/2} \times \dot{H}^{-1/2}$ data.

Theorem 6. *Let $n \geqslant 2$ and suppose that $V \in L^{\frac{n+1}{2}}([0,T] \times \mathbb{R}^n)$ and that $(f,g) \in \dot{H}^{1/2}(\mathbb{R}^n) \times \dot{H}^{-1/2}(\mathbb{R}^n)$. Then the equation*

$$(14) \qquad \begin{cases} \Box u = Vu \\ u(0,x) = f(x), \quad \partial_t u(0,x) = g(x) \end{cases}$$

has a unique (weak) solution $u \in L^\infty([0,T] \times \mathbb{R}^n; \dot{H}^{1/2}(\mathbb{R}^n)) \cap C^{0,1}([0,T] \times \mathbb{R}^n; \dot{H}^{-1/2}(\mathbb{R}^n))$. Moreover, if $0 < t < T$,

$$\|u(t,\cdot)\|_{\dot{\gamma}} \leqslant 2 \exp\Big(K \int_0^t \int_{\mathbb{R}^n} |V(s,x)|^{\frac{n+1}{2}}\, dx ds \Big) \cdot \|u(0,\cdot)\|_{\dot{\gamma}}, \quad \gamma = 1/2,$$

where K is a constant depending only on the dimension.

This follows from the above type of arguments using Strichartz's estimates. If one uses the more general results in Theorem 3, one can improve on this. For instance in $(3+1)$-dimensions one gets:

Theorem 7. *Suppose that $V \in L^2([0,T] \times \mathbb{R}^3)$ and that $(f,g) \in \dot{H}^\gamma(\mathbb{R}^3) \times \dot{H}^{\gamma-1}(\mathbb{R}^3)$, with $0 < \gamma < 1$. Then the equation (14) has a unique solution u belonging to $L^\infty([0,T] \times \mathbb{R}^3; \dot{H}^\gamma(\mathbb{R}^3)) \cap C^{0,1}([0,T] \times \mathbb{R}^3; \dot{H}^{\gamma-1}(\mathbb{R}^3))$. Moreover, if $0 < t < T$, there is a universal constant K_γ so that*

$$\|u(t,\cdot)\|_{\dot{\gamma}} \leqslant 2 \exp\Big(K_\gamma \int_0^t \int_{\mathbb{R}^3} |V(s,x)|^2\, dx ds \Big) \cdot \|u(0,\cdot)\|_{\dot{\gamma}}.$$

One can also improve Theorem 6 somewhat in higher dimensions. However, the range of γ is smaller, due to the fact that the range of γ in the favorable case (i) of Theorem 3 gets smaller with increasing dimensions.

These results greatly improve the regularity assumptions for the data of other uniqueness theorems for singular hyperbolic equations, for instance in [28], and this is of course important for nonlinear applications. The natural assumption for the potential is the same as in [28] and this also turns out to be important for applications. For instance in the $(3+1)$-dimensional case, if $0 < \gamma(\kappa) < 1$, one immediately gets the uniqueness part of Theorem 4, since if u and $\tilde{u}$ both solve both solve (8) with the same data, then $u - \tilde{u}$ must vanish identically since it has zero data and $\Box(u - \tilde{u}) = Vu$, where $V = (F_\kappa(u) - F_\kappa(\tilde{u}))/(u - \tilde{u})$ is in $L^2(S_{T_*})$, due to the fact that $u, \tilde{u} \in L^{2(\kappa-1)}(S_{T_*})$.

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