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# Some remarks on Weyl pseudodifferential operators

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**Abstract**

We study properties of the Weyl symbols of functions of pseudodifferential operators, especially of pseudodifferential operators with quadratic symbols.

## 1 Introduction.

The name "pseudodifferential operators" is usually used in two different (although related) meanings. First, it is used to denote operators on $L^2(\mathbb{R}^n)$ defined by certain integral formulas which stress the phase space properties of the operaror. The operators which we call $x$-$D$-pseudodifferential operators are defined (at least formally) as

$$(a_1(x,D)\psi)(x) = \int \frac{d\xi dy}{(2\pi)^n} a_1(x,\xi)\psi(y)e^{i(x-y)\xi}. \tag{1.1}$$

The Weyl pseudodifferential operators are defined (also at least formally) by

$$\left(a_2^W(x,D)\psi\right)(x) = \int \frac{d\xi dy}{(2\pi)^n} a_2\left(\frac{x+y}{2},\xi\right)\psi(y)e^{i(x-y)\xi}. \tag{1.2}$$

Essentially every operator on $L^2(\mathbb{R}^n)$ can be represented in the above form. The functions $a_1$ and $a_2$ are uniquely determined in the sense of distributions by the operators themselves. We will call them $x$-$D$- and Weyl symbols respectively.

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In its second meaning the word "pseudodifferential operators" is used to describe certain classes of operators whose definitions make use of the integral representations (1.1) and (1.2). For example, a big amount of literature is devoted to the classes $\Psi^m_{\rho,\delta}$ of pseudodifferential operators, which roughly speaking are defined as the operators whose symbols belong to a certain space $S^m_{\rho,\delta}$ (see eg. [Hö1,Ta,Tr]).

It is probably the second meaning that is more commonly atributed to the word "pseudodifferential operators" in the literature. In this paper though we will use the first meaning. Moreover, we will concentrate our attention on Weyl symbols, which have better "symplectic properties".

The literature about pseudodifferential operators is vast. They are used extensively in the theory of PDE-s (see eg. [Hö1,L,Ro,Sj,Shu,Ta,Tr]). They express in a natural way the principle of correspondence of the quantum and classical mechanics [We,FH,MaFe,Ro]. Nevertheless, in section 2 we give a short introduction to the subject. The introduction is somewhat different from what one can usually find in the literature, because we are not directly interested in standard classes of pseudodifferential operators. Instead, we try to present various properties of symbols of pseudodifferential operators under as general conditions as we can. We do not use asymptotic series, which are the usual tool in the calculus of pseudodifferential operators. We use only exact identities.

Some of these identities are contained in the literature, especially in [Hö1,BeSh] where one can find formulas which allow to go from the $x$-$D$-symbol to the Weyl symbol and formulas for the symbol of the product of two pseudodifferential operators. We try to give rather general conditions for these identities to be well defined.

The problem of computing the symbol of a function of a pseudodifferential operator was studied eg. in [Se,CV,HR1,2,Ro1,2,Sh]. The approaches used in these papers were based on the assumption that the pseudodifferential operator had a smooth symbol. Their formulas were usually expressed in terms of asymptotic series. We avoid asymptotic formulas and use just compact exact identities. We give examples when they can be given an exact meaning for rather general classes of pseudodifferential operators.

Special attention we devote to quadratic Hamiltonians (pseudodifferential operators whose symbols are quadratic polynomials). It turns out that the Weyl symbol of a function of a quadratic Hamiltonian can be computed from an especially simple expression.

Suppose that $F$ is a function of $(x,\xi)$ that depends just on $x^2+\xi^2$. Then there exists a function $f$ such that

$$F^W(x,D) = f(x^2+D^2).$$

The converse statement is also true: functions of the harmonic oscillator have rotationally symmetric Weyl symbols. It is possible to give quite simple formulas that make it possible to go from $f$ to $F$ and from $F$ to $f$.

The above properties of the harmonic oscillator can be generalized to some other quadratic Hamiltonians. For instance, functions of $x^2-D^2$ have Weyl symbols that depend just on $x^2-\xi^2$.

We give a simple algebraic condition for a quadratic form $p(x,\xi)$ on the phase space to have the property that, at least formaly, for every $f$ there exists $\tilde{f}$ such that

$$f\left(p^W(x,D)\right) = (\tilde{f}\circ p)^W(x,D). \tag{1.3}$$

We give a classification of all the real quadratic forms $p(x,\xi)$ that behave in a similar way.

The above properties of $x^2+D^2$ and $x^2-D^2$ were known before [U1,2], but it seems that the full classification of quadratic Hamiltonians with this property is a new result. Note also that in [HS1,2] B.Helffer and J.Sjöstrand use similar ideas on the level of asymptotic series.

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# 2 Basic pseudodifferential operators.

## 2.1 Symbols of an operator.

Throughout the paper $X$ will stand for an $n$-dimensional vector space and $X'$ for its dual. Instead of $X\times X'$ we will often write $T^*X$. We will denote the natural symplectic form on $T^*X$ by $\omega$.

Generic elements of $X$, $X'$ and $T^*X$ will be denoted by $x$, $\xi$ and $z$ respectively.

The spectrum of an operator $B$ will be denoted $\sigma(B)$. If $B$ is self-adjoint then $E_\Omega(B)$ will denote the spectral projection of $B$ onto the set $\Omega$. $D$ will denote the operator $\frac{1}{i}\nabla$.

We will study properties of certain operators acting on $L^2(X)$. A large class of such operators can be described by functions on $T^*X$ called symbols. There are various possibilities of doing this: the most popular ones are what we call $x$-$D$-symbols and Weyl symbols.

Let $\bar{\mathcal{S}}'(X)\otimes\mathcal{S}'(X)$ denote the space of sesquilinear forms on the space of Schwartz test functions $\mathcal{S}(X)$. We will view $\bar{\mathcal{S}}'(X)\otimes\mathcal{S}'(X)$ as a kind of an extension of the set of linear operators on $L^2(X)$. We will treat all the elements of this space as "pseudodifferential operators" and we will define their symbols.

Let $A\in\bar{\mathcal{S}}'(X)\otimes\mathcal{S}'(X)$. Then we say that $a_1\in\mathcal{S}'(T^*X)$ is the $x$-$D$-symbol of $A$ if for any $\phi,\psi\in\mathcal{S}(X)$

$$(\phi|A\psi) = \int\frac{dxd\xi dy}{(2\pi)^n}a_1(x,\xi)\bar{\phi}(x)\psi(y)e^{i(x-y)\xi}. \tag{2.1}$$

We will write

$$A = a_1(x,D). \tag{2.2}$$

We say that $a_2\in\mathcal{S}'(T^*X)$ is the Weyl symbol of $A$ if for any $\phi,\psi\in\mathcal{S}(X)$

$$(\phi|A\psi) = \int\frac{dxd\xi dy}{(2\pi)^n}a_2\left(\frac{x+y}{2},\xi\right)\bar{\phi}(x)\psi(y)e^{i(x-y)\xi}. \tag{2.3}$$

We will write

$$A = a_2^W(x, D). \tag{2.4}$$

Using basic properties of the Fourier transform on $\mathcal{S}'(T^*X)$ and the Schwartz's kernel theorem (see eg. Appendix to chapter V3 of vol I of [RS]) we easily see that every element of $\bar{\mathcal{S}}'(X) \otimes \mathcal{S}'(X)$ possesses a unique $x$-$D$-symbol and a unique Weyl symbol. Conversely, with any symbol in $\mathcal{S}'(T^*X)$ we can associate a unique $x$-$D$-pseudodifferential operator and a unique Weyl pseudodifferential operator. Note also that the maps from $\bar{\mathcal{S}}'(X) \otimes \mathcal{S}'(X)$ to $\mathcal{S}'(T^*X)$ that to a form assigne its $x$-$D$- and its Weyl symbols are homeomorphisms.

The following well known identity [Hö1] allows one to go from the $x$-$D$-symbol to the Weyl symbol:

$$e^{\frac{i}{2}D_xD_\xi} a_1 = a_2. \tag{2.5}$$

Note that 2.5 makes sense for symbols in $\mathcal{S}'(T^*X)$.

## 2.2 Special classes of operators

Let us describe some classes of operators whose symbols have special properties.

The operators whose $x$-$D$-symbols are polynomials (both in $x$ and in $\xi$) form an algebra. This algebra is equal to the class of operators whose Weyl symbols are polynomials.

The set of Hilbert-Schmidt operators is exactly equal to the set of operators whose symbols (both Weyl and $x$-$D$-) are square integrable. This follows from the following well known identity:

$$\begin{aligned} \mathrm{Tr}A^*A &= \int \frac{dxd\xi}{(2\pi)^{2n}} |a_1(x,\xi)|^2 \\ &= \int \frac{dxd\xi}{(2\pi)^{2n}} |a_2(x,\xi)|^2 \end{aligned} \tag{2.6}$$

Let $C_\infty(Z)$ denote the space of continuous functions on $Z$ that go to zero as $|z| \to \infty$. For $1 \le p < \infty$ we define

$$J^p(L^2(Z)) := \{B \in B(L^2(Z)) \mid \|B\|_p := (\mathrm{Tr}|B|^p)^{1/p} < \infty\}.$$

$J^\infty(L^2(Z))$ will stand for the set of compact operators on $L^2(Z)$ and we will write $\|B\|_\infty$ to denote the usual norm of $B$.

The following nice properties of Weyl symbols follow easily from [G].

**Proposition 2.1** *Suppose that the numbers $1 \le p, q \le \infty$ satisfy $p^{-1} + q^{-1} = 1$.*
*a) If $a^W(x, D) \in J^1(L^2(Z))$, then $a \in C_\infty(Z)$. If $a^W(x, D) \in J^p(L^2(Z))$ for $1 \le p \le 2$, then $a \in L^q(Z)$ and*

$$\|a\|_q \le (2\pi)^{2n/p} \|a^W(x, D)\|_p. \tag{2.7}$$

*b) If* $1 \leq q \leq 2$ *and* $a \in L^q(Z)$*, then* $a^{\mathrm{W}}(x,D) \in J^p(L^2(Z))$ *and*

$$\|a\|_q \geq (2\pi)^{2n/p}\|a^W(x,D)\|_p. \tag{2.8}$$

**Proof.** Let $\delta_{y,\eta}$ denote the deltafunction concentrated in $(y,\eta) \in T^*X$. Then $\delta^W_{y,\eta}(x,D)$ is a unitary operator. (In fact, it is equal to $e^{ix\eta}P_y e^{-ix\eta}$ where $P_y$ is the inversion centered at $y$). The family $\delta^W_{y,\eta}(x,D)$ depends weakly continuously on the parameters $y$, $\eta$ and goes weakly to zero as the parameters go to infinity. Hence the fact that $a \in C_\infty(Z)$ and (2.7) with $p = 1$ follow from the following identity [G]:

$$a(y,\eta) = (2\pi)^{2n}\mathrm{Tr}\left(a^{\mathrm{W}}(x,D)\delta^{W*}_{y,\eta}(x,D)\right). \tag{2.9}$$

(2.7) with $p = 2$ follows immediately from (2.6). Now to obtain (2.7) we apply the complex interpolation between $p = 1$ and $p = 2$ (see eg. vol. II of [RS]).

The estimate (2.8) for $q = 1$ follows from the identity [G]:

$$a^{\mathrm{W}}(x,D) = \int dy d\eta a(y,\eta)\delta^{\mathrm{W}}_{y,\eta}(x,D). \tag{2.10}$$

Therefore to obtain (2.8) it is enough to apply the complex interpolation between $q = 1$ and $q = 2$. □

## 2.3 Symbol of the product and the symplectic invariance of Weyl pseudodifferential operators

There exists a number of apparently different formulas that can be used to compute the product of two pseudodifferential operators. Namely, if

$$a^W(x,D)b^W(x,D) = c^W(x,D), \tag{2.11}$$

then (at least formally)

$$c(x,\xi) = e^{\frac{i}{2}(D_xD_\eta - D_yD_\xi)}a(x,\xi)b(y,\eta)_{|x=y,\xi=\eta} \tag{2.12}$$

$$= a\left(x - \frac{1}{2}D_\xi, \xi + \frac{1}{2}D_x\right) b(x,\xi) \tag{2.13}$$

$$= a^W\left(x - \frac{1}{2}D_\xi, \xi + \frac{1}{2}D_x\right) b(x,\xi) \tag{2.14}$$

$$= b\left(x + \frac{1}{2}D_\xi, \xi - \frac{1}{2}D_x\right) a(x,\xi) \tag{2.15}$$

$$= b^W\left(x + \frac{1}{2}D_\xi, \xi - \frac{1}{2}D_x\right) a(x,\xi). \tag{2.16}$$

(2.12 can be found e.g. in [Höl]; other expressions follow immediately from 2.12).

Of course, one needs some assumptions on $a$ and $b$ to make sense of above expressions. For example, 2.13 and 2.14 are well defined if $a$ is a polynomial and $b$ belongs to $\mathcal{S}(T^*X)'$. Later on we will give more general conditions that guarantee their well-definedness.

Now we would like to make some comments on the symplectic invariance of the Weyl calculus. The facts that we will recall below are well known and can be found eg. in [Höl,Le].

First of all note that the formulas 2.12-2.16 can be written in a manifestly symplectically invariant way. Let us do it for instance for 2.14 and 2.16:

$$c(z) = a^W(z - \frac{1}{2}\omega D_z)b(z) \tag{2.17}$$

$$= b^W(z + \frac{1}{2}\omega D_z)a(z). \tag{2.18}$$

Recall that $z = (x, \xi)$ and $\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Let $\mathrm{Sp}(T^*X)$ denote the group of linear symplectic transformations of $T^*X$. If $a \in \mathcal{S}'(T^*X)$ and $\gamma \in \mathrm{Sp}(T^*X)$ then we set

$$\gamma^* a(z) = a(\gamma^{-1} z).$$

It is easy to see eg. from 2.17 that if $a$, $b$ and $c$ satisfy 2.11 and $\gamma \in \mathrm{Sp}(T^*X)$ then

$$\gamma^* a^W(x, D) \gamma^* b^W(x, D) = \gamma^* c^W(x, D).$$

Thus the transformation

$$a^W(x, D) \mapsto \gamma^* a^W(x, D),$$

preserves the multiplication and restricted to $B(L^2(X))$ is a homomorphism.

It is well known [Höl,Le] that this transformation can be implemented by a unitary operator. To see this consider $\gamma \in \mathrm{Sp}(T^*X)$ for which there exists a quadratic form $p$ on $T^*X$ such that

$$\gamma^* a = \exp\{p, \cdot\} a, \tag{2.19}$$

where $\{\cdot, \cdot\}$ denotes the Poisson bracket. (Such $\gamma$ generate the whole $Sp(T^*X)$). The identity 2.19 can be "quantized" as described in the following proposition [Le]:

**Proposition 2.2** *Let $\gamma \in \mathrm{Sp}(T^*X)$ and $p$ be a quadratic form $p$ on $T^*X$ such that 2.19 holds. Then we have for any $a \in \mathcal{S}'(T^*X)$*

$$(\exp\{p, \cdot\} a)^W(x, D) = e^{ip^W(x,D)} a^W(x, D) e^{-ip^W(x,D)}. \tag{2.20}$$

**Proof.** To see 2.20 it is enough to check that

$$\frac{d}{d\tau}(\exp\{\tau p,\cdot\}a)^W(x,D)_{|\tau=0} = \frac{d}{d\tau}e^{i\tau p^W(x,D)}a^W(x,D)e^{-i\tau p^W(x,D)}{}_{|\tau=0}. \tag{2.21}$$

In fact, by 2.17 the Weyl symbol of $\left[p^W(x,D), a^W(x,D)\right]$ equals

$$p^W\left(z-\frac{1}{2}\omega D_z\right)a(z) - p^W\left(z+\frac{1}{2}\omega D_z\right)a(z) = i\{p,a\}(z).$$

Therefore 2.21 is true. □

Let us come back to the question when the formal identities 2.12-2.16 for the product of two operators are correct. It turns out that each of the following two hypotheses guarantees that 2.14 and 2.17 have a rigorous meaning.

**Hypothesis 2.3.I**

*1.* $b \in \mathcal{S}'(T^*X)$

*2.* $a^W(x,D)$ *maps* $\mathcal{S}'(X)$ *continuously into itself.*

**Hypothesis 2.3.II**

*1.* $b^W(x,D)$ *is Hilbert-Schmidt.*

*2.* $a^W(x,D)$ *is bounded on* $L^2(X)$.

Assume first I. Note that the following identity is true:

$$a^W\left(x-\frac{1}{2}D_\xi, \xi+\frac{1}{2}D_x\right) = e^{\frac{i}{2}D_xD_\xi}e^{ix\xi}a^W(x,D_x)e^{-ix\xi}e^{-\frac{i}{2}D_xD_\xi}. \tag{2.22}$$

(This identity can be easily deduced from 2.20). By 2.22, if $a^W(x,D)$ maps continuously $\mathcal{S}'(X)$ into itself then $a^W\left(z-\frac{1}{2}\omega D_z\right)$ maps $\mathcal{S}'(T^*X)$ continuously into itself. Hence $c$ is well defined as an element of $\mathcal{S}'(T^*X)$. We leave to the reader the proof that $c$ actually satisfies 2.11.

Consider now II. Note first that $b \in L^2(T^*X)$. By the identity 2.22 the boundedness of $a^W(x,D)$ on $L^2(X)$ implies the boundedness of $a^W\left(z-\frac{1}{2}\omega D_z\right)$ on $L^2(T^*X)$. Hence $c$ is well defined as an element of $L^2(T^*X)$. (This reflects the obvious fact that the product of a bounded operator and a Hilbert-Schmidt operator is Hilbert-Schmidt.)

## 2.4 Symbol of functions of pseudodifferential operators.

Next we would like to study the Weyl symbols of functions of pseudodifferential operators. Formally we have:

$$f(p^W(x,D)) = F^W(x,D)$$

where

$$F(x,\xi) = f\left(p\left(x-\frac{1}{2}D_\xi, \xi+\frac{1}{2}D_x\right)\right)1, \tag{2.23}$$

where 1 is the function on $Z$ equal to 1.

The identity 2.23 can be written in a symplectically invariant way

$$F(z) = f\left(p\left(z - \frac{1}{2}\omega D_z\right)\right) 1. \tag{2.24}$$

Of course, in practice one needs to specify various assumptions both for $f$ and $p$ for 2.23 or 2.24 to make sense.

For instance, it is correct if $f$ is a polynomial and $p^W(x, D)$ maps continuously $\mathcal{S}'(X)$ into itself. In this case it follows immediately from 2.17 if we take into account that the Weyl symbol of the identity is 1.

Below we list various hypotheses under which the identities 2.23 and 2.24 are well defined and correct.

**Hypothesis 2.4.I**

*1. $p^W(x, D)$ is Hilbert-Schmidt.*

*2. $f$ is analytic on a neighborhood of $\sigma(p^W(x, D))$.*

**Hypothesis 2.4.II**

*1. $p^W(x, D)$ is Hilbert-Schmidt and normal.*

*2. $f$ is defined on $\sigma(p^W(x, D))$ and*

$$|f(\lambda) - f(0)| \leq C|\lambda|.$$

**Hypothesis 2.4.III**

*1. $p^W(x, D)$ maps continuously $\mathcal{S}'(X)$ into itself and for some $N \in \mathbb{N}$ the operator $\left(p^W(x, D)\right)^{-N}$ is Hilbert-Schmidt.*

*2. $f$ is analytic on a neighborhood of $\sigma(p^W(x, D))$ and for some $M \in \mathbb{N}$*

$$u \mapsto \frac{f(u^{\frac{1}{N}})}{u^{\frac{M}{N}}}$$

*is analytic at infinity.*

**Hypothesis 2.4.IV**

*1. $p^W(x, D)$ maps continuously $\mathcal{S}'(X)$ into itself, is normal and $\left(p^W(x, D)\right)^{-N}$ is Hilbert-Schmidt.*

*2. $f$ is defined on $\sigma(p^W(x, D))$ and satisfies*

$$|f(\lambda)| \leq C\langle\lambda\rangle^M.$$

In fact, let us assume I. Then we can write

$$f(\lambda) = f(0) + \lambda\tilde{f}(\lambda),$$

where $\tilde{f}$ is analytic. Thus

$$F(z) = f(0) + \tilde{f}\left(p^W\left(z - \frac{1}{2}\omega D_z\right)\right) p(z). \tag{2.25}$$

By 2.22 we have

$$\tilde{f}\left(p^W\left(z - \frac{1}{2}\omega D_z\right)\right) = e^{\frac{i}{2}D_x D_\xi} e^{ix\xi} \tilde{f}\left(p^W(x, D_x)\right) e^{-ix\xi} e^{-\frac{i}{2}D_x D_\xi}. \tag{2.26}$$

But $\tilde{f}\left(p^W(x, D)\right)$ is clearly bounded on $L^2(X)$. Hence 2.26 is a bounded operator on $L^2(T^*X)$. Note also that $p \in L^2(T^*X)$. Hence $F$ makes sense as a constant plus a square integrable symbol.

The case II is very similar. We also use 2.25 and 2.26 and note that $\tilde{f}\left(p^W(x, D)\right)$ is bounded.

To deal with III it suffices to write

$$f(\lambda) = \lambda^{M+N} \tilde{f}\left(\lambda^{-N}\right) \lambda^{-N},$$

where $\tilde{f}$ is analytic. Let $\tilde{p}$ denote the symbol of $\left(p^W(x, D)\right)^{-N}$, which belongs to $L^2(T^*X)$. Clearly,

$$\tilde{f}\left(\tilde{p}^W\left(z - \frac{1}{2}\omega D_z\right)\right) \tilde{p}$$

also belongs to $L^2(T^*X)$. Hence

$$F(z) = p^W\left(z - \frac{1}{2}D_z\right)^{N+M} \tilde{f}\left(\tilde{p}^W\left(z - \frac{1}{2}\omega D_z\right)\right) \tilde{p}$$

is in $\mathcal{S}'(T^*X)$.

The proof of the well-definedness of IV is similar.

# 3 Functions of quadratic Hamiltonians.

It is well known that quadratic Hamiltonians (pseudodifferential operators whose symbols are quadratic polynomials) have special properties. For example, if $p$ is a quadratic polynomial on $T^*X$ and then 2.24 can be simplified:

$$F(z) = f\left(p(z) + \frac{1}{4}D_z\omega\nabla^2 p\omega D_z\right) 1. \tag{3.1}$$

To see this note that by 2.24

$$F(z) = f\left(p(z) - \frac{1}{2}\nabla p(z)\omega D_z + \frac{1}{4}D_z\omega\nabla^2 p\omega D_z\right) 1. \tag{3.2}$$

We easily compute that the following commutators vanish:

$$[p(z), \nabla p(z)\omega D_z] = \nabla p(z)\omega\nabla p(z) = 0$$

and

$$[D_z\omega\nabla^2 p\omega D_z, \nabla p(z)\omega D_z] = 2D_z\omega\nabla^2 p\omega\nabla^2 p\omega D_z = 0.$$

We commute $\nabla p(z)\omega D_z$ in 2.25 to the right and use $\nabla p(z)\omega D_z 1 = 0$. This yields 3.1.

Let us consider more closely the case of the harmonic oscillator. To simplify we will assume that we have just one degree of freedom (the generalization to $n$ degrees of freedom is obvious).

Let us denote

$$h := x^2 + D^2,$$

$$H := x^2 + \xi^2 + \frac{1}{4}\left(D_x^2 + D_\xi^2\right)$$

and

$$L := xD_\xi - \xi D_x.$$

Let us remark that $H$ commutes with $L$. Moreover, $h$ is unitarily equivalent to $H$ restricted to the kernel of $L$ (the rotationally symmetric subspace). The spectrum of $h$, and also of $H$, is equal to $\{1, 3, 5, 7, \ldots\}$. We will denote by $\phi_n$ a (unique up to a phase) normalized rotationally symmetric eigenvector of $H$ with eigenvalue $2n + 1$.

**Theorem 3.1** *1) Let $f$ be a function on the spectrum of $h$ such that*

$$f(2n + 1) \leq c(2n + 1)^N \tag{3.3}$$

*for some $c$, $N$. Then the Weyl symbol of $f(h)$ is rotationally symmetric. More precisely, if we set*

$$F(x, \xi) = f(H)1 \tag{3.4}$$

*then $F$ is a rotationally symmetric distribution in $\mathcal{S}'(\mathbb{R}^2)$ and*

$$F^W(x, D) = f(h). \tag{3.5}$$

*2) Let $F \in \mathcal{S}'(\mathbb{R}^2)$ be rotationally symmetric. Then there exists a function $f$ on the spectrum of $h$ such that*

$$F^W(x, D) = f(h). \tag{3.6}$$

*Moreover,*

$$f(2n + 1) = \frac{\int \phi_n(x, \xi)F(x, \xi)dxd\xi}{\int \phi_n(x, \xi)dxd\xi}. \tag{3.7}$$

**Proof.** $1 \Rightarrow 2$ is a special case of 3.1 with hypothesis IV.

Let us show the implication $2 \Rightarrow 1$.

Any $F \in \mathcal{S}'(\mathbf{R}^2)$ is of the form $H^{-N}\tilde{F}$ for some $\tilde{F} \in L^2(\mathbf{R}^2)$. If $F$ is rotationally symmetric then so is $\tilde{F}$. The vectors $\phi_n$ span the rotationally symmetric subspace of $L^2(\mathbf{R}^2)$. Hence

$$\tilde{F}(x,\xi) = \sum_{n=1}^{\infty} \int \phi_n(y,\eta)\tilde{F}(y,\eta)dyd\eta\phi_n(x,\xi)$$

in the sense of the $L^2$ convergence. Consequently,

$$F(x,\xi) = \sum_{n=1}^{\infty} \int \phi_n(y,\eta)F(y,\eta)dyd\eta\phi_n(x,\xi) \tag{3.8}$$

in the sense of distributions. Clearly,

$$\phi_n(x,\xi) = \frac{E_{\{2n+1\}}(H)1}{\int \phi_n(y,\eta)dyd\eta}.$$

Hence by 1)

$$\phi_n^W(x,D) = \frac{E_{\{2n+1\}}(h)}{\int \phi_n(y,\eta)dyd\eta}. \tag{3.9}$$

Now 3.7, 3.8 and 3.9 imply 3.6. □

**Theorem 3.2** *Let $p(x,\xi)$ be a quadratic form on $T^*X$. Then the following conditions are eqivalent.*

*1) The set of operators of the form $f(p^W(x,D))$ where $f$ is a polynomial is equal to the set of operators of the form $F^W(x,D)$ where $F(x,\xi) = g(p(x,\xi))$ and $g$ is a polynomial.*

*2) There exists $c \in \mathbf{C}$ such that*

$$(\nabla^2 p\omega)^3 = c\nabla^2 p\omega. \tag{3.10}$$

**Proof.** Let us prove that 2) implies 1). Let $\mathcal{A}$ be the space of all polynomials on $T^*X$ of the form $g(p(z))$ for some polynomial $g$ and let $\mathcal{B}$ be the space of all polynomials $F$ on $T^*X$ such that there exists a polynomial $f$ and $f(p^W(x,D)) = F^W(x,D)$. Note that

$$\left(p(z) + \frac{1}{4}D_z\omega\nabla^2 p\omega D_z\right) g(p(z))$$

$$= p(z)g(p(z)) + \frac{1}{4}\mathrm{Tr}(\omega\nabla^2 p\omega\nabla^2 p)g'(p(z)) + \frac{1}{4}\nabla p(z)\omega\nabla^2 p\omega\nabla p(z)g''(p(z)). \tag{3.11}$$

Clearly, the first two terms on the right of 3.13 belong to $\mathcal{A}$. If 3.10 holds then

$$\nabla p(z)\omega\nabla^2 p\omega\nabla p(z) = z\nabla^2 p\omega\nabla^2 p\omega\nabla^2 pz = cz\nabla^2 pz = 2cp(z). \tag{3.12}$$

Hence 3.12 belongs to $\mathcal{A}$, and so does 3.11. Moreover, $1 \in \mathcal{A}$. Clearly, $\mathcal{B}$ is the smallest linear space containing 1 and invariant with respect to the operator $p(z) + \frac{1}{4} D_z \omega \nabla^2 p \omega D_z$. Thus $\mathcal{A} \supset \mathcal{B}$.

One can show the opposite inclusion with help of the induction. Suppose that we know that the elements of $\mathcal{A}$ of degree less than $2n$ belong to $\mathcal{B}$. Note that by 3.1

$$(p^n)^W(x,D) - (p^W(x,D)^n = q^W(x,D) \tag{3.13}$$

with $\deg q \leq 2n-2$. The polynomial $p^n$ belongs clearly to $\mathcal{A}$. The Weyl symbol of $(p^W(x,D)^n$ belongs to $\mathcal{B} \subset \mathcal{A}$. Therefore, $q \in \mathcal{A}$. Hence by the induction assumption, $q \in \mathcal{B}$. Consequently, $p^n \in \mathcal{B}$. Thus $\mathcal{A} \subset \mathcal{B}$.

This ends the proof of the implication $2 \Rightarrow 1$.

To see that $1 \Rightarrow 2$ we compute

$$\left(p^W(x,D)\right)^3 = F^W(x,D)$$

where

$$F(z) = p^3(z) + \frac{3}{4} p(z) \mathrm{Tr}\left(\nabla^2 p \omega \nabla^2 p \omega\right) + \frac{1}{2} \nabla p(z) \omega \nabla^2 p \omega \nabla p(z).$$

The first two terms of the right hand side belong clearly to $\mathcal{A}$. The third term belongs to $\mathcal{A}$ if and only if 3.10 is true. □

Note that any quadratic form $p$ on $T^*X$ can be associated with a unique bilinear symmetric form on $T^*X$ which we will denote $P$. $P$ can be identified with a linear map from $T^*X$ to $(T^*X)'$. Then we can write $p(z) = \langle z, Pz\rangle$ where $\langle\cdot,\cdot\rangle$ is the natural duality on $T^*X \times (T^*X)'$.

The symplectic form $\omega$ can be identified with a linear map from $T^*X$ to $(T^*X)'$. This map is invertible and its inverse will be denoted $\omega^{-1}$. The condition 3.10 is equivalent to

$$\left(\omega^{-1}P\right)^3 = c\omega^{-1}P. \tag{3.14}$$

Note that the operator $\omega^{-1}P$ is sometimes called the fundamental matrix or the Hamilton map [Hö1].

It is possible to classify symmetric forms in a symplectic space (see [Wi,A,Hö2]) Based on this classification we can give a list of all possible types of forms satisfying the condition 3.14.

**Theorem 3.3** *Let $P$ be a real symmetric form on $T^*X$ that satisfies 3.14. Set $\lambda := \sqrt{|c|}$.*
*1) If $c < 0$, then there exists a linear symplectic transformation and an integer $k \leq n$ such that $p$ is reduced to the form*

$$\lambda \sum_{i=1}^{k} (\xi_i^2 + x_i^2). \tag{3.15}$$

*2) If $c > 0$, then there exists a linear symplectic transformation and an integer $k \leq n$ such that $p$ is reduced to the form*

$$\lambda \sum_{i=1}^{k} (\xi_i^2 - x_i^2). \tag{3.16}$$

*3) If $c = 0$, then there exists a linear symplectic transformation and integers $k, m$ such that $p$ is reduced to the form*

$$\sum_{i=0}^{k} (\xi_{3i+1} x_{3i+2} + \xi_{3i+2} x_{3i+3}) + \sum_{j=3k+4}^{m} \xi_j^2. \tag{3.17}$$

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