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A Generalization of the Radiation Condition of Sommerfeld for N-body Schrödinger Operators

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1 Radiation conditions and uniqueness theorems

It was in 1912 that Sommerfeld introduced the radiation condition [10] to derive the uniqueness of the solution of the reduced wave equation in $\mathbf{R}^3$:

$$(-\Delta - \lambda)u = f, \qquad \lambda > 0. \tag{1}$$

If f decays sufficiently rapidly, this equation has two solutions $u_\pm$ defined by

$$u_\pm = \frac{1}{4\pi}\int_{\mathbf{R}^3} \frac{e^{\pm i\sqrt{\lambda}|x-y|}}{|x-y|} f(y)dy.$$

They behave like

$$u_\pm \sim r^{-1}e^{\pm i\sqrt{\lambda}r}a_\pm(\omega), \quad \omega = x/r, \quad r = |x| \to \infty, \tag{2}$$

and satisfy the following conditions at infinity:

$$u_\pm = O(r^{-1}), \qquad (\frac{\partial}{\partial r} \mp i\sqrt{\lambda})u_\pm = o(r^{-1}). \tag{3}$$

The important fact is that the solution of (1) is unique if it satisfies the condition (3) for u_+ or u_-. The former is usually called the outgoing radiation condition, since $r^{-1}e^{i\sqrt{\lambda}(r-t)}$ represents an outgoing wave for the wave equation $\partial_t^2 v = \Delta v$, and by the similar reason the latter is called the incoming radiation condition. This condition is closely related with the asymptotic behavior at infinity of the resolvent of $-\Delta$. In fact, we have

$$u_\pm = (-\Delta - \lambda \mp i0)^{-1}f.$$

The theorem of Sommerfeld was made mathematically rigorous by Rellich [8], Vekua [12] and extended to general elliptic operators, 1st order systems and also to potential scattering by Vainverg [11], Grushin [4], Eidus [2], Shulenberger-Wilcox [9] and Agmon-Hörmander [1]. Let us recall some well-known results for 2-body Schrödinger operators. First we introduce some notations. For $x \in \mathbf{R}^n$, let $<x> = (1+|x|^2)^{1/2}$. For $s \in \mathbf{R}$, let $L^{2,s} = L^{2,s}(\mathbf{R}^n)$ be the Hilbert space defined by

$$u \in L^{2,s} \Longleftrightarrow \|u\|_s = \| <x>^s u(x)\|_{L^2} < \infty.$$

For two Banach spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, let $\mathbf{B}(\mathcal{H}_1;\mathcal{H}_2)$ denote the totality of bounded operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. We consider the 2-body Schrödinger operator $H = -\Delta + V(x)$ in $\mathbf{R}^n$, where $V(x)$ is a real-valued function satisfying

$$|\partial_x^m V(x)| \leq C{<}x{>}^{-\rho-m}, \quad m = 0, 1,$$

for some $\rho > 0$. Here and in the sequel, ∂_x^m denotes an arbitrary derivative of order m with respect to x. One can also allow certain local singularities for V. Let $R(z) = (H-z)^{-1}$. Then for any $\lambda > 0$ and small $\epsilon > 0$ we have

$$R(\lambda \pm i0) \in \mathbf{B}(L^{2,1/2+\epsilon}; L^{2,-1/2-\epsilon}), \tag{4}$$

$$(\nabla \mp i\sqrt{\lambda}\hat{x})R(\lambda \pm i0) \in \mathbf{B}(L^{2,1/2+\epsilon}; L^{2,-1/2+\epsilon}), \quad \hat{x} = x/|x|. \tag{5}$$

Moreover, the solution of the equation $(H-\lambda)u = f$ satisfying

$$u_\pm \in L^{2,-1/2-\epsilon}, \quad (\nabla \mp i\sqrt{\lambda}\hat{x})u_\pm \in L^{2,-1/2+\epsilon} \tag{6}$$

is unique ([5]). It was also proved that the resolvent has the following asymptotic expansion at infinity :

$$R(\lambda \pm i0)f \sim r^{-(n-1)/2} e^{\pm i\varphi(x,\lambda)} a_\pm(\omega), \quad r = |x| \to \infty,$$

where $\varphi(x,\lambda)$ behaves like $\sqrt{\lambda}r$ and solves the eikonal equation $|\nabla\varphi|^2 + V = \lambda$ for large $|x|$. This fact was used to obtain the generalized Fourier transformation associated with H.

It is not easy to generalize these facts to the N-body problem ($N \geq 3$). Because of the presence of many channels, the resolvent of the N-body Schrödinger operator does not behave like that of the Laplacian (see e.g. [6]). It means that the operator $\frac{\partial}{\partial r} \mp i\sqrt{\lambda}$ of Sommerfeld does not work for the N-body problem. We propose a new formulation of the radiation condition for the N-body problem.

We consider a system of N-particles moving in $\mathbf{R}^\nu$ with mass m_i and position x^i ($1 \leq i \leq N$). Let $\mathcal{X}$ be defined by

$$\mathcal{X} = \{(x^1, \cdots, x^N); \sum_{i=1}^N m_i x^i = 0\}, \tag{7}$$

and consider the Schrödinger operator

$$H = H_0 + \sum_{i<j} V_{ij}(x^i - x^j), \tag{8}$$

where $-H_0$ is the Laplace-Beltrami operator on $\mathcal{X}$ equipped with the Riemannian metric induced from $ds^2 = 2\sum_{i=1}^N m_i (dx^i)^2$ on $\mathbf{R}^{N\nu}$. We assume that each V_{ij} is a real-valued C^∞-function on $\mathbf{R}^\nu$ and satisfies for some constant $\rho > 0$

$$|\partial_y^m V_{ij}(y)| \leq C_m{<}y{>}^{-m-\rho}, \quad \forall m \geq 0. \tag{9}$$

Our radiation condition is defined in terms of pseudo-differential operators (Ps.D.Op.'s). For $k > 0$ and $a \in \mathbf{R}$, we introduce

DEFINITION 1.1 *$\mathcal{R}^k_\pm(a)$ is the set of C^∞-functions $p(x,\xi)$ on $\mathcal{X} \times \mathcal{X}^*$ such that*

$$|\partial_x^m \partial_\xi^n p(x,\xi)| \le C_{mn} <x>^{-m} <\xi>^{-k}, \tag{10}$$

for $0 \le m \le k,\ 0 \le n \le k$ and on supp $p(x,\xi)$

$$\inf_{x,\xi} \pm \frac{x \cdot \xi}{<x>} > \pm a, \tag{11}$$

where the sign $+$ corresponds to $\mathcal{R}^k_+(a)$ and $-$ to $\mathcal{R}^k_-(a)$.

Let Λ be the set of thresholds of H. For $\lambda \in \sigma_{ess}(H) \cap \Lambda^c$, we define

$$a(\lambda) = \inf\{\lambda - t; t \in \Lambda, t < \lambda\}. \tag{12}$$

Note that $a(\lambda) = \lambda$ if $\lambda > 0$, which follows from the absence of positive eigenvalues of Schrödinger operators.

We first recall the resolvent estimates. Let $R(z) = (H - z)^{-1}$. Then by [7] we have for any $\epsilon > 0$ and $\lambda \in \sigma_{ess}(H) \cap \sigma_p(H)^c \cap \Lambda^c$,

$$R(\lambda \pm i0) \in \mathbf{B}(L^{2,1/2+\epsilon}; L^{2,-1/2-\epsilon}). \tag{13}$$

As a result corresponding to (5), we proved the following theorem in [3].

THEOREM 1.2 *(Resolvent estimates). Let $\lambda \in \sigma_{ess}(H) \cap \sigma_p(H)^c \cap \Lambda^c$ and $a(\lambda)$ be as in (12). Then for for any $s > -1/2$ and $t > 1$, there exists $k = k(s) > 0$ such that*

$$P_\mp R(\lambda \pm i0) \in \mathbf{B}(L^{2,s+t}; L^{2,s}),$$

for any $P_\mp \in \mathcal{R}^k_\mp(\pm\sqrt{a(\lambda)})$.

This result as well as some related estimates have been also obtained by X.P.Wang [13] independently by a different method.

It may thus be inferred from Theorem 1.2 that in the N-body problem the operators in $\mathcal{R}^k_\mp(\pm\sqrt{a(\lambda)})$ play the role of $\frac{\partial}{\partial r} \mp i\sqrt{\lambda}$. This turns out to be true by the following

THEOREM 1.3 *(Uniqueness). Let $0 \le \alpha < 1/2 < s \le 1$ and $\lambda \in \sigma_{ess}(H) \cap \sigma_p(H)^c \cap \Lambda^c$. Suppose that $u \in L^{2,-s}$ satisfies $(H - \lambda)u = 0$ and that there exist $k_0 > 0$ and $\epsilon > 0$ such that $Pu \in L^{2,-\alpha}$ either for any $P \in \mathcal{R}^{k_0}_+(-\epsilon)$ or for any $P \in \mathcal{R}^{k_0}_-(\epsilon)$. Then $u = 0$.*

In other words, for $f \in L^{2,s}$, $u_\pm = R(\lambda \pm i0)f$ satisfies

$$P_\mp u_\pm \in L^{2,-\alpha}, \quad \forall P_\mp \in \mathcal{R}^{k_0}_\mp(\pm\epsilon), \tag{14}$$

and the solution $u \in L^{2,-s}$ of the equation $(H - \lambda)u = f$ satisfying (14) is unique. Therefore (14) is worth calling the radiation condition. Note that $\mathcal{R}^{k_1}_\pm(a) \subset \mathcal{R}^{k_2}_\pm(a)$ if $k_1 > k_2$ so that one can take k_0 in Theorems 1.2 and 1.3 large enough if necessary.

Theorem 1.3 includes the uniqueness theorem for the 2-body problem, since we have

LEMMA 1.4 *Let $0 \le \alpha < 1/2 < s \le 1$, $\lambda > 0$ and $k_0 > 0$ be large enough. Suppose that $u \in L^{2,-s}$ satisfies $(\frac{\partial}{\partial r} \mp i\sqrt{\lambda})u \in L^{2,-\alpha}$. Then $P_\mp u \in L^{2,-\alpha}$ for any $P_\mp \in \mathcal{R}^{k_0}_\mp(\pm\epsilon)$, $0 < \epsilon < \sqrt{\lambda}$.*

It is worthwhile to note that in Theorem 1.3 one can replace the family of operators $\mathcal{R}_{\pm}^{k_0}(\mp\epsilon)$ by one operator. Take $\rho_{\pm}(t) \in C^{\infty}(\mathbf{R})$ such that $\rho_{\pm}(t) = 1$ if $\pm t > -\epsilon/2$, $\rho_{\pm}(t) = 0$ if $\pm t < -\epsilon$. Let $P_{\pm}$ be the Ps.D.Op. with symbol

$$\rho_{\pm}(\frac{x\cdot\xi}{<x>}) <\xi>^{-2k} \in \mathcal{R}_{\pm}^{k}(\mp\epsilon),$$

k being chosen large enough. Suppose that $u \in L^{2,-s}$ satisfies $P_{\pm}u \in L^{2,-\alpha}$, where $0 \le \alpha < 1/2 < s \le 1$. Then $Q_{\pm}u \in L^{2,-\alpha}$ for any $Q_{\pm} \in \mathcal{R}_{\pm}^{3k}(\mp\epsilon/2)$.

The crucial step to prove Theorem 1.3 is the following

THEOREM 1.5 *Let $0 \le \alpha < 1/2$ and $\lambda \in \sigma_{ess}(H) \cap \sigma_p(H)^c \cap \Lambda^c$. Suppose $u \in L^{2,-\alpha}$ satisfies $(H-\lambda)u = 0$. Then $u = 0$.*

Note that the condition $\alpha < 1/2$ is optimal. In fact let $u = \int_{S^{n-1}} e^{ik\omega\cdot x}\varphi(\omega)d\omega$ for $k > 0$ and $\varphi \in C^{\infty}(S^{n-1})$. Then u satisfies $(-\Delta - k^2)u = 0$, $u = O(r^{-(n-1)/2})$, hence $u \in L^{2,-s}$ for $s > 1/2$. The above theorem roughly means that the non-trivial solution of the equation $(H-\lambda)u = 0$ cannot decay faster than $O(r^{-(n-1)/2}), n = (N-1)\nu$.

An extensive literature has been devoted to derive this slowly decreasing property. Most of them studied the 2-body problem, and if the N-body problem was concerned, the homogeneity assumption was usually imposed on the potential and the energy was restricted to be positive. Our theorem covers the more general N-body case except that the local singularities are not allowed for the 2-body potentials.

We must point out that too much smoothness assumption is imposed on the potential. One can relax it considerably, but still needs rather high differentiability of the potential compared with the 2-body case. To extend the above results to singular potentials is our future problem.

2 Sketch of the proof

Although the above theorems are formulated by Ps.D.Op.'s, the main part of the proof is done in an algebra introduced in [3] consisting of operators which have some particular commutation relations with the operator

$$B = \frac{1}{2i}(\frac{x}{<x>}\cdot\nabla_x + \nabla_x\cdot\frac{x}{<x>}).$$

We first introduce some notations. For two operators P and A, we define

$$\begin{aligned} ad_0(P,A) &= P, \\ ad_n(P,A) &= [ad_{n-1}(P,A), A], \quad n \ge 1. \end{aligned}$$

For $m \in \mathbf{R}$, let $\mathcal{F}^m$ be the set of C^{∞}-functions on $\mathbf{R}$ such that

$$|f^{(k)}(x)| \le C_k(1+|x|)^{m-k}, \quad \forall k \ge 0. \tag{15}$$

Let $X = <x>$. For $m \in \mathbf{R}$, we introduce the following

DEFINITION 2.1 *$\mathcal{OP}^m(X)$ is the set of all operators P such that $X^{\alpha}ad_n(P,B)X^{\beta} \in \mathbf{B}$ for any $\alpha,\beta \in \mathbf{R}$ and $n \ge 0$ satisfying $\alpha+\beta = n-m$.*

The following lemma, which can be derived easily from the definition, shows that the set $\cup_m \mathcal{OP}^m(X)$ forms an algebra.

LEMMA 2.2 *(1) $P \in \mathcal{OP}^m(X) \iff$ There exists $P_0 \in \mathcal{OP}^0(X)$ such that $P = X^m P_0$.*
(2) $P \in \mathcal{OP}^m(X) \Longrightarrow [P, B] \in \mathcal{OP}^{m-1}(X)$.
(3) $P \in \mathcal{OP}^m(X) \Longrightarrow X^k P X^l \in \mathcal{OP}^{m+k+l}(X), \quad \forall k, l \in \mathbf{R}$.
(4) $P \in \mathcal{OP}^m(X) \Longrightarrow P^* \in \mathcal{OP}^m(X)$.
(5) $P \in \mathcal{OP}^m(X)$, $Q \in \mathcal{OP}^n(X) \Longrightarrow PQ \in \mathcal{OP}^{m+n}(X)$.

Note that if $P \in \mathcal{OP}^0(X)$, $X^s P X^{-s} \in \mathbf{B}$ for any $s \in \mathbf{R}$, hence $P \in \mathbf{B}(L^{2,s}; L^{2,s})$. The important fact is that this algebra contains functions of operators X, H and B.

LEMMA 2.3 *(1) $f(X) \in \mathcal{OP}^m(X)$ if $f \in \mathcal{F}^m$, $m \in \mathbf{R}$.*
(2) $f(H)$, $f(B) \in \mathcal{OP}^0(X)$ if $f \in \mathcal{F}^{-\epsilon}$, $\epsilon > 0$.

Using the formula of Helffer-Sjöstrand to represent the functions of self-adjoint operators, one can show the following asymptotic expansion of commutators.

LEMMA 2.4 *Let $P \in \mathcal{OP}^m(X)$, $f \in \mathcal{F}^0$, $m \in \mathbf{R}$. Then we have*

$$[P, f(B)] \sim \sum_{k \geq 1} (-1)^{k-1}/k!\ ad_k(P, B) f^{(k)}(B),$$

$$ad_k(P, B) \in \mathcal{OP}^{m-k}(X).$$

There are close connections between Ps.D.Op.'s in $\mathcal{R}^k_{\pm}(a)$ and the functions of the operator B.

DEFINITION 2.5 *For $a \in \mathbf{R}$, let $\mathcal{F}^0_{\pm}(a)$ be defined by*

$$\mathcal{F}^0_+(a) = \{f \in \mathcal{F}^0; \text{supp } f \subset (a, \infty)\},$$

$$\mathcal{F}^0_-(a) = \{f \in \mathcal{F}^0; \text{supp } f \subset (-\infty, a)\}.$$

The following lemma is intuitively obvious if one examines the supports of F and the symbol of P and recall that the symbol of B is equal to $\frac{x \cdot \xi}{\langle x \rangle}$.

LEMMA 2.6 *Let $F \in \mathcal{F}^0_-(a)$ and $P \in \mathcal{R}^k_+(b)$. Suppose $a < b$. Then for any $s \geq 0$, there exists $k = k(s) > 0$ such that $X^s P F(B) X^s \in \mathbf{B}$.*

Proof. We have only to show the following assertion : For any $s \geq 0$ there exists $k = k(s) > 0$ such that

$$X^s P F(B) \in \mathbf{B}. \tag{16}$$

We prove (16) by induction on s. It is true for $s = 0$. Suppose (16) is true for some $s \geq 0$. We put $B_1 = B - \alpha$, $\alpha = (a + b)/2$ and

$$P(t) = e^{tB_1} F(B)^* P^* X^{2s+1} P F(B) e^{tB_1}, \ t \geq 0.$$

Let $b(x, \xi) = \frac{x \cdot \xi}{\langle x \rangle} - \alpha$ be the symbol of B_1. Then on the support of the symbol of P, $b(x, \xi) \geq (b - a)/2 > 0$. Therefore, by Gårding's inequality, there exists a finite number of Ps.D.Op.'s $P_i \in \mathcal{R}^{l_i}_+(b), l_i = l_i(k)$ and $Q \in S^{-M}$ such that

$$-B_1 P^* X^{2s+1} P - P^* X^{2s+1} P B_1 \leq \sum_i P_i^* X^{2s} P_i + Q.$$

Since $l_i(k) \to \infty$ as $k \to \infty$, for sufficiently large $k > 0$, $X^s P_i F(B) \in \mathbf{B}$ by our induction hypothesis. Therefore,

$$\begin{aligned} -\frac{d}{dt}P(t) &\leq e^{tB_1}F(B)^*(\sum_i P^* X^{2s} P_i + Q)F(B)e^{tB_1} \\ &\leq Ce^{t(a-b)}, \end{aligned}$$

since $\lambda - \alpha \leq (a-b)/2$ on supp $F(\lambda)$. Noting that

$$F(B)P^*X^{2s+1}PF(B) = P(0) = -\int_0^\infty P'(t)dt,$$

we have $X^{s+1/2}PF(B) \in \mathbf{B}$. □

Using Lemma 2.6, one can rephrase the radiation condition (14) equivalently in terms of the functions of B.

LEMMA 2.7 *Let $0 \leq \alpha < 1/2 < s \leq 1$ and $a \in \mathbf{R}$. Suppose that $u \in L^{2,-s}$ satisfies $(H-\lambda)u = 0$. Then the following conditions (1) and (2) are equivalent.*
(1) There exists $k_0 > 0$ such that $Pu \in L^{2,-\alpha}$ for any $P \in \mathcal{R}_-^{k_0}(a)$.
(2) $F(B)u \in L^{2,-\alpha}$ for any $F \in \mathcal{F}_-^0(a)$.

LEMMA 2.8 *Let $0 \leq \alpha < 1/2 < s \leq 1$. Suppose that $u \in L^{2,-s}$ satisfies $(H-\lambda)u = 0$ for some $\lambda \in \mathbf{R}$ and that there exist $k_0 > 0$ and $\epsilon > 0$ such that $Pu \in L^{2,-\alpha}$ for any $P \in \mathcal{R}_-^{k_0}(\epsilon)$. Then $u \in L^{2,-\alpha}$.*

Sketch of the proof. The idea is classical. We take $\rho(t) \in C^\infty(\mathbf{R})$ such that $\rho(t) = 1$ if $t < 1$, $\rho(t) = 0$ if $t > 2$ and put $\chi_t(s) = \int_s^\infty \rho(s/t)^2 s^{-2\alpha} ds$. From the equation $Im((H-\lambda)u, \chi_t(X)u) = 0$, we obtain by integration by parts

$$(BX^{-\alpha}u_t, X^{-\alpha}u_t) = 0, \ \forall t > 0.$$

where $u_t = \rho(\frac{X}{t})u$. We next introduce $F_\pm(t) \in C^\infty(\mathbf{R})$ such that $F_+(t)^2 + F_-(t)^2 = 1$, $F_+(t) = 1$ if $t > 2\epsilon/3$, $F_+(t) = 0$ if $t < \epsilon/3$. Using the relation

$$(F_+(B)^2 BX^{-\alpha}u_t, X^{-\alpha}u_t) = -(F_-(B)^2 BX^{-\alpha}u_t, X^{-\alpha}u_t)$$

and the radation condition we have

$$\sup_{t>1} \|F_\pm(B)X^{-\alpha}u_t\| < \infty,$$

which proves that $\sup_{t>1} \|X^{-\alpha}u_t\| < \infty$, and hence $u \in L^{2,-\alpha}$. □

The above Lemma 2.8 reduces the proof of Theorem 1.3 to Theorem 1.5.

The idea of the proof of Theorem 1.5 is basically the classical one. We multiply a weight to u and appeal to the calculus of commutators. Let u be as in Theorem 1.5. For $\epsilon > 0$, we define

$$\Phi = \log(1+\epsilon X)^{-\alpha},$$
$$u_\epsilon = e^\Phi u = (1+\epsilon X)^{-\alpha}u \in L^2.$$

Then we get the following inequality

$$i([H,A]u_\epsilon,u_\epsilon) \le 4\alpha\|Bu_\epsilon\|^2 + C \tag{17}$$

C being independent of $\epsilon > 0$. For the 2-body problem, the proof is essntially finished since in (17) one can replace $\|Bu_\epsilon\|^2$ by $\lambda\|u_\epsilon\|^2$ because the potential vanishes at infinity, and by the Mourre inequality one can show that $\|u_\epsilon\|$ is uniformly bounded in $\epsilon > 0$. Letting $\epsilon \to 0$, one obtains $u \in L^2$, which shows $u = 0$ since $\lambda \notin \sigma_p(H)$.

In (17) the term $4\alpha\|Bu_\epsilon\|^2$ arizes from the commutator of H and $(1+\epsilon X)^{-\alpha}$. Our idea to prove Theorem 1.5 is to use $<A>$ as a weight instead of X.

We first cut off the part of u near $B = 0$.

LEMMA 2.9 *Let $F(t) \in C^\infty(\mathbf{R})$ be such that $F(t) = 0$ if $|t| < \delta_0$,$F(t) = 1$ if $|t| > 2\delta_0$. Then for a sufficiently small $\delta_0 > 0$, there exists a constant $C > 0$ independent of $\epsilon > 0$ such that*

$$\|u_\epsilon\| \le C(\|F(B)u_\epsilon\| + 1).$$

As stated above our main device is to use $< A >$ as a weight instead of X. Since $A = XB + C(x)$, where $C(x)$ is a bounded function of x, it is intuitively natural to regard A as a weight if we cut off the part near $B = 0$.

We take $\varphi_1(t), \varphi_2(t) \in C_0^\infty(\mathbf{R})$ such that $\varphi(t)\varphi_1(t) = \varphi(t)$, $\varphi_1(t)\varphi_2(t) = \varphi_1(t)$ and put

$$w = \varphi_1(H)F(B)\varphi(H)u,$$

$$w_\epsilon = \varphi_2(H)(1+\epsilon <A>)^{-\alpha}w, \ \epsilon > 0.$$

$w_\epsilon \in L^2$ by virtue of Lemma 2.9 and $[\varphi_1(H), F(B)] \in \mathcal{OP}^{-1}(X)$. We put

$$A_\epsilon = 1 + \epsilon <A>,$$

$$Q = \varphi_1(H)[H, F(B)]\varphi(H),$$
$$T_1 = 2\mathrm{Im}(\varphi_2(H)[H, A_\epsilon^{-\alpha}]w, Aw_\epsilon),$$
$$T_2 = 2\mathrm{Im}(\varphi_2(H)A_\epsilon^{-\alpha}Qu, Aw_\epsilon).$$

Then we have

LEMMA 2.10

$$i([H,A]w_\epsilon, w_\epsilon) = T_1 + T_2.$$

Let $(*)$ denote a function $f(\epsilon)$ such that $\sup_{\epsilon>0}|f(\epsilon)| < \infty$. The key-fact is the following

LEMMA 2.11 *The following inequalities hold :*

$$T_1 \le 2\alpha i([H,A]w_\epsilon, w_\epsilon) + (*),$$

$$T_2 \le (*).$$

Since $2\alpha < 1$, one gets from Lemmas 2.10 and 2.11 that

$$i([H, A]w_\epsilon, w_\epsilon) \leq (*).$$

By the Mourre inequality one then proves that $\|w_\epsilon\|$ is uniformly bounded in $\epsilon > 0$. Letting $\epsilon \to 0$, one has $u \in L^2$ and the proof is complete.

To prove Lemma 2.11, we introduce another algebra of operators. We define

$$\mathcal{OP}^0(A) = \{P; ad_n(P, A) \in \mathbf{B}, \forall n \geq 0\},$$
$$\mathcal{OP}^m(A) = \{P; <A>^{-m} P \in \mathcal{OP}^0(A)\}, \quad m \in \mathbf{R}.$$

As in the case of $\mathcal{OP}^m(X)$, $\cup_m \mathcal{OP}^m(A)$ forms an algebra. We also have

LEMMA 2.12 *(1)* $f(X) \in \mathcal{OP}^0(A)$ *if* $f \in \mathcal{F}^0$.
(2) $f(H), f(B) \in \mathcal{OP}^0(A)$ *if* $f \in \mathcal{F}^{-\epsilon}$, $\epsilon > 0$.

In conclusion, the proof of our main theorem consists in introducing the algebraic framework which is deeply inspired by the classical idea of integration by parts and the algebra of pseudo-differential operators.

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