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Solvability and Asymptotic Behaviour of Solutions of Ordinary Differential Equations with Variable Operator Coefficients

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The present talk is based upon our recent and mostly unpublished work on the solvability and asymptotic properties of solutions of ordinary differential equations with variable unbounded operator coefficients. The results obtained have direct applications to partial differential equations, which are described in the second part of the talk. A part of our results appeared in the preprints [1], [2].

Let $H_0, H_1, \ldots, H_\ell$ be Hibert spaces with the norms $||\cdot||_0$, $||\cdot||_1, \ldots, ||\cdot||_\ell$. We suppose that H_ℓ is dense in H_0, $H_\ell \subset H_{\ell-1} \subset \ldots \subset H_0$ and $||u||_j \leq ||u||_{j+1}$ for $j = 0, 1, \ldots, \ell-1$.

Introduce the operator pencil

$$\mathcal{A}(\lambda) = \sum_{0 \leq q \leq \ell} A_{\ell-q} \lambda^q : H_\ell \to H_0 \tag{1}$$

where A_q is a linear bounded operator from H_q into H_0. We assume that the following two conditions on $\mathcal{A}(\lambda)$ are fulfilled.

Condition I. The operator $\mathcal{A}(\lambda)$ is Fredholm for every $\lambda \in C$ and it is invertible at least for one value of λ.

Condition II. *There exist numbers* $\Theta \in (0,\pi/2)$ *and* $\rho > 0$ *such that for all*

$$\lambda \in S_{\rho,\Theta} = \{z \in C: \ |\arg(\pm z)| \leq \Theta \ |z| \geq \rho\}$$

the following inequality holds

$$\sum_{0 \leq q \leq \ell} |\lambda|^q ||\phi||_{\ell-q} \leq c||\mathcal{A}(\lambda)\phi||_0 \qquad \forall \Phi \in H_\ell. \tag{2}$$

Then $\mathcal{A}(\lambda)$ is the Fredholm operator pencil and its spectrum consists of isolated eigenvalues $\{\lambda_\nu\}_{\nu \in Z}$ having finite algebraic multiplicities with the only possible limit point at infinity. Clearly, the set $S_{\rho,\Theta}$ contains no eigenvalues of $\mathcal{A}(\lambda)$.

The items [3-7] in our list of references partially reflect the development of the asymptotic theory of linear differential equations with operator coefficients. In particular in [2] it is shown that under Conditions I, II for solutions of the equation

$$\mathcal{A}(D_t)u = f \text{ on } \mathbb{R}^1, \tag{3}$$

where $D_t = i^{-1}\,\partial/\partial t$, the asymptotic formula

$$u(t) = \sum_{\{\nu:\mathrm{Im}\lambda_\nu = k\}} P_\nu(t)e^{i\lambda_\nu t} + o(e^{-kt}) \text{as } t \to +\infty, \tag{4}$$

holds. Here P_ν are polynomials in t with coefficients which are elements of Jordan chains generated by λ_ν.

We characterize the behaviour of the solution u and of the right-hand side f by the functions

$$||u||_{W^{\ell}(t,t+1)} = \Big(\int_t^{t+1} \sum_{q=0}^{\ell} ||D_t^q u(\tau)||_{\ell-q}^2 d\tau\Big)^{1/2},$$

$$||f||_{L_2(t,t+1;H_0)} = \Big(\int_t^{t+1} ||f(\tau)||_0^2 d\tau\Big)^{1/2}$$

and suppose everywhere that both functions are finite for each $t \in \mathbb{R}^1$.

With any strip $k_- < \mathrm{Im}\,\lambda < k_+$ free of eigenvalues of $\mathcal{A}(\lambda)$ we connect a class of solutions of the equation (3) with

$$\underline{\lim}_{t\to\pm\infty} e^{k_{\mp}t} ||u||_{W^{\ell}(t,t+1)} = 0. \tag{5}$$

For such a solution we obtain the pointwise estimate

$$||u||_{W^{\ell}(t,t+1)} \leq c\{\int_t^{\infty} e^{k_-(\tau-t)} (1+\tau-t)^{m_- -1} ||f||_{L_2(\tau,\tau+1;H_0)} d\tau +$$

$$+ \int_{-\infty}^{t} e^{k_+(\tau-t)} (1+t-\tau)^{m_+ -1} ||f||_{L_2(\tau,\tau+1;H_0)} d\tau,$$

where $m_\pm$ are the maximal lengths of the Jordan chains corresponding to eigenvalues of $\mathcal{A}(\lambda)$ on the lines $\mathrm{Im}\lambda = k_\pm$ (if there are no eigenvalues on the line $\mathrm{Im}\lambda = k_\pm$ we put $m_\pm = 1$). This estimate is equivalent to the following comparison principle for $u(t)$ and for the solution $w(t)$ of the ordinary differential equation

$$(-1)^{m_-}(\partial_t+k_+)^{m_+}(\partial_t+k_-)^{m_-} w(t) = ||f||_{L_2(t,t+1;H_0)} \tag{6}$$

satisfying $w(t) = o(e^{-k_{\mp}t})$ as $t \to \pm\infty$:

$$||u||_{W^{\ell}(t,t+1)} \leq b w(t),\ b = \text{const}. \tag{7}$$

The operator

$$\mathcal{M}(\partial_t) = (\partial_t + k_+)^{m_+}(-\partial_t - k_-)^{m_-}$$

is the simplest model for $\mathcal{A}(D_t)$.

We turn to the ordinary differential equataion with variable operator coefficients

$$L(t,D_t)u = f \text{ on } \mathbb{R}^1. \tag{8}$$

We confine ourselves to the following three topics connected with (8)

(i) Uniqueness and solvability theorems.

(ii) Estimates for solutions.

(iii) Conditions ensuring the asymptotics (4).

The operator $L(t,D_t)$ will be considered as a perturbation of $\mathcal{A}(D_t)$. Therefore we introduce the function

$$\omega(t) = b||L(t,D_t)-\mathcal{A}(D_t)||_{W^\ell(t,t+1)\to L_2(t,t+1;H_0)}, \tag{9}$$

where b is the constsnat in (7).

In the variable coefficients case the role of the equation (6) is played by

$$(M(\partial_t) - \omega)w = h \text{ on } \mathbb{R}^1. \tag{10}$$

We shall suppose everywhere that either

$$\sup \omega(t) < m_+^{m_+} m_-^{m_-}\left(\frac{k_+ - k_-}{m_+ + m_-}\right)^{m_+ + m_-} \tag{11}$$

or

$$\int_{\mathbb{R}^1} \omega(\tau)d\tau < \frac{(m_+-1)!(m_--1)!}{(m_+ + m_- - 2)!} (k_+ - k_-)^{m_++m_--1} . \tag{12}$$

Each of these conditions guarantee the existence of the Green function $g_\omega(t,\tau)$ of the equation (10) and the convergence of the Neumann series

$$g_\omega(t,\tau) = \sum_{k=0}^{\infty} \int_{\mathbb{R}^k} g(t-\tau_1)\omega(\tau_1)g(\tau_1-\tau_2)\ldots\omega(\tau_k)g(\tau_k-\tau)d\tau_1,\ldots d\tau_k,$$

where g is the Green function of the unperturbed equation (6). The constants on the right in (11) and (12) are best possible.

It can be shown that g is positive, which implies the positivity of g_ω.

We state a theorem on the unique solvability of the equation (10).

Theorem 1. (i). *If*

$$\int_{\mathbb{R}^1} g_\omega(0,\tau) \; |h(\tau)|d\tau < \infty \tag{13}$$

then the integral

$$w(t) = \int_{\mathbb{R}^1} g_\omega(t,\tau)h(\tau)d\tau \tag{14}$$

satisfies the equation (10) *and*

$$w(t) = o(\sup_{\tau \in \mathbb{R}^1} \frac{g_\omega(t,\tau)}{g_\omega(0,\tau)}) \text{ as } t \to \pm \infty. \tag{15}$$

(ii) *If a solution* w *of* (10) *with* h = 0 *satisfies*

$$w(t) = o(\underline{\lim}_{\tau\to\pm\infty} \frac{g_\omega(t,\tau)}{g_\omega(0,\tau)}) \text{ as } t \to \pm \infty \tag{16}$$

then w = 0.

By introducing special assumptions on the equation (8) we can obtain an explicit information about its solutions. We show it by three examples of estimates for the Green function g_ω.

Theorem 2. *Let* $m_+ = m_- = 1$ *and let*

$$\sup \omega(t) < (k_+ - k_-)^2/4.$$

Then

$$g_\omega(t,\tau) \leq \exp\{- \int_\tau^t \alpha_\pm(\Omega(s))ds\} \quad \text{for } t \gtrless \tau,$$

where

$$\Omega(\tau) = \begin{cases} \sup\limits_{x\geq\tau} \omega(x) & \text{if } \tau \geq 0, \\ \sup\limits_{x\leq\tau} \omega(x) & \text{if } \tau < 0, \end{cases}$$

and $\alpha_\pm(\sigma)$ *are the roots of the equation*

$$(\alpha - k_+)(\alpha - k_-) + \sigma = 0, \ k < \alpha_-(\sigma) < \alpha_+(\sigma) < k_+.$$

Theorem 3. *Let* $\omega(t) \leq \gamma t^{-\alpha}$ *for large positive* t, *where* $\gamma > 0$ *and* $0 < \alpha \leq \min\{m_-,m_+\}$. *Then for positive* t,τ

$$g_\omega(t,\tau) \leq c\left(\frac{t+1}{\tau+1}\right)^{q_\pm} \exp\{-\int_\tau^t \alpha_\pm(\gamma s^{-\alpha})ds\}, \ t \gtrless \tau.$$

Here $\alpha_\pm(\sigma)$ *are the roots of the equation*

$$(-\alpha+k_+)^{m_+}(\alpha-k_-)^{m_-}-\sigma = 0, \ k_-<\alpha_-(\sigma)<\alpha_+(\sigma)<k_+$$

and

$$q_\pm = \pm\frac{1 - m_\pm}{2}(k_+ - k_-)^{m_\mp/m_\pm} \gamma^{-1/m_\pm}.$$

Theorem 4. *Let*

$$\int^{\infty} \omega^{1/m_{\pm}}(\tau)d\tau = \infty.$$

Then for postive t,τ

$$g_{\omega}(t,\tau) \leq C \exp\{-k_{\pm}(t-\tau) + c|\int_{\tau}^{t} \Omega^{1/m_{\pm}}(s)ds|\}, \quad t \gtrless \tau.$$

By analogy with the constant coefficients case we obtain the following comparison principle for solution of the equation (8).

Theorem 5. *Let*

$$\int_{\mathbb{R}^1} g_{\omega}(0,\tau)||f||_{L_2(\tau,\tau+1;H_0)}d\tau < \infty. \tag{17}$$

Then there exists a solution of (8) *satisfying*

$$||u||_{W^{\ell}(t,t+1)} \leq b\, w(t), \tag{18}$$

where w *is the solution of* (10) *with*

$$h(t) = ||f||_{L_2(t,t+1;H_0)} \tag{19}$$

mentioned in Theorem 1.

A direct consequence of (14), (18), (19) is the inequality

$$||u||_{W^{\ell}(t,t+1)} \leq b \int_{\mathbb{R}^1} g_{\omega}(t,\tau)||f||_{L_2(\tau,\tau+1;H_0)}d\tau, \tag{20}$$

which along with Theorems 2-4 lead to explicit estimates for $||u||_{W^{\ell}(t,t+1)}$.

Moreover from (20) and Theorems 2-4 one obtains two-weighted estimates for solutions of (8), i.e. the estimates of the form

$$||u||_{W^{\ell}(\mathbb{R}^1;\gamma)} \leq c||f||_{L_2(\mathbb{R}^1;H_0;\Gamma)}, \tag{21}$$

where $W^{\ell}(\mathbb{R}^1;\gamma)$ and $L_2(\mathbb{R}^1;H_0;\Gamma)$ are the spaces with the norms

$$||u||_{W^{\ell}(\mathbb{R}^1;\gamma)} = (\int_{\mathbb{R}^1} \gamma^2(\tau) \sum_{0\leq q\leq \ell} ||D_\tau^q u(\tau)||^2_{\ell-q}\, d\tau)^{1/2},$$

$$||u||_{L_2(\mathbb{R}^1;H_0;\Gamma)} = (\int_{\mathbb{R}^1} \Gamma^2(\tau)||u(\tau)||_0^2 d\tau)^{1/2}.$$

We restrict ourselves to the following example.

Theorem 6. *Let* $m_+ = m_- = 1$ *and*

$$\omega(t) \leq a|t|^{-1} \text{ for large } t.$$

Then for every f $L_2(\mathbb{R}^1;H_0;\Gamma)$ *there exists one and only one solution* u $W^{\ell}(\mathbb{R}^1;\gamma)$, *where*

$$\gamma(t) = e^{k_\pm t}|t|^{\alpha_\pm},\ \Gamma(t) = e^{k_\pm t}|t|^{\alpha_\pm+1},$$

$\alpha_\pm > -1/2$. *This solution satisfies* (21).

This theorem is precise, i.e. if Γ is prescribed as above then the exponent $\alpha_\pm$ in the definition on γ can not be enlarged even in the case $\omega = 0$.

Now we formulate a uniqueness theorem for the equation (8) similar to Theorem 1 (ii).

Theorem 7. *If a solution* u *of* (8) *with* f = 0 *satisfies*

$$||u||_{W^{\ell}(t,t+1)} = o(\lim_{\tau\to\pm\infty} \frac{g_\omega(t,\tau)}{g_\omega(0,\tau)}) \text{ as } t \to \pm\infty, \tag{22}$$

then u = 0.

Under additional requirements to the function ω the condition (22) can be made more explicit and some specific corollaries can be deduced from it. For example, the following variant of the Phragmèn-Lindelöf principle holds. (The notations are the same as in Theorem 2).

Theorem 8. *Let* $m_+ = m_- = 1$ *and*

$$\overline{\lim_{t\to+\infty}}\ \omega(t) < (k_+ - k_-)^2/4.$$

If u *is a solution of* $L(t,D_t)u = 0$ *for* $t > t_0$ *then either*

$$\underline{\lim_{t\to+\infty}}\ \exp\{\int_0^t \alpha_-(\Omega(s))ds\}\ ||u||_{W^\ell(t,t+1)} > 0$$

or

$$\overline{\lim_{t\to+\infty}}\ \exp\{\int_0^t \alpha_+(\Omega(s))ds\}\ ||u||_{W^\ell(t,t+1)} < \infty.$$

In the next theorem we give a condition which implies that the principal terms in the asymptotics are the same as in the constant coefficients case.

Theorem 9. *Let* m_0 *be an integer*, $m_0 \geq m_+$ *and*

$$\int^{+\infty} \omega(t)\, t^{m_0-1}\, dt < \infty. \tag{23}$$

Let also u *be a solution of* $L(t,D_t)u = 0$ *for large* t *and let*

$$||u||_{W^\ell(t,t+1)} \leq c e^{-(k_-+\varepsilon)t}$$

with some positive ε. *Then*

$$u(t) = \sum_{\{\nu:\operatorname{Im}\lambda_\nu = k_+\}} e^{i\lambda_\nu t} P_\nu(t) + o(e^{-k_+ t}\, t^{2m_+-1-m_0}) \tag{24}$$

where P_ν *are polynomials in* t *whose coefficients are elements of Jordan chains of the pencil* $\mathcal{A}(\lambda)$ *corresponding to the eigenvalue* λ_ν.

We remark that in the case of the equaiton $(M(\partial_t)-\omega)u = 0$ the condition (23) is also necessary for validity of the asymptotic formula (24). Therefore in general (23) is best possible.

We turn to some applications of previous results to elliptic partial differential equations.

First, consider two elliptic differential operators

$$A(x,D_x) = \sum_{|\alpha|=2m} A_\alpha(x)D_x^\alpha, \quad A(D_x) = \sum_{|\alpha|=2m} A_\alpha D_x^\alpha$$

with measurable and constant coefficients respectively.

We introduce the continuity modulus of the coefficients of $A(x,D_x)$ at a point $x_0 \in \mathbb{R}^n$

$$\rho(\tau) = \sum_{|\alpha|=2m} \sup_{\tau<|x-x_0|<2\rho} |A_\alpha(x)-A_\alpha|.$$

Theorem 9 yields the following description of the asymptotics of solutions of the equation

$$A(x,D_x)u = 0$$

in a neighbourhood $B_\varepsilon(x_0)$, $u \in H^{2m}(B_\varepsilon(x_0))$.

If

$$\int_0 \rho(\tau) \frac{d\tau}{\tau} < \infty$$

then there exists a polynomial P_{2m}, $\deg P_{2m}$)= 2m such that

$$u(x) = P_{2m}(x) + o(|x - x_0|^{2m}) \text{ as } x \to x_0.$$

More restrictive conditions of the Dini type appear for even n and $2m \geq n$, when one studies the asymptotic behaviour of a fundamental solution $\mathcal{G}$ of the operator $A(x,D_x)$ near x_0. Namely, if

$$\int_0 \rho(\tau)(\log\tau)^2\,\frac{d\tau}{\tau} < \infty$$

then

$$\mathcal{G}(x) = G(x - x_0) + o(|x - x_0|^{2m-n}),$$

where

$$G(x) = |x|^{2m-n}(\alpha(x/|x|)\log|x| + \beta(x/|x|))$$

is the fundamental solution of $A(x_0,D)$.

One more application of Theorem 9 concerns a variant of the Giraud theorem on the sign of the normal derivative. In $B_\varepsilon^+ = \{x: |x| < \varepsilon,\ x_n > 0\}$ we consider the uniformly elliptic equation

$$\partial_{x_i}(a_{ij}(x)\partial_{x_j}u) = 0 \tag{25}$$

with real measurable bounded coefficients. Let

$$\rho(\tau) = \sum_{i,j=1}^{n} \sup_{x\in B_\tau^+} |a_{ij}(x) - \delta_{ij}|.$$

By u we denote a function in $H^1(B_\varepsilon^+)$, satisfying (25) and the Dirichlet condition

$$u = \phi \text{ on } \{x: |x| < \varepsilon,\ x_n = 0\}, \tag{26}$$

where ϕ is a smooth function.

We prove that *under the Dini condition*

$$\int_0 \rho(\tau)\,\frac{d\tau}{\tau} < \infty$$

there exists a finite limit

$$c = \lim_{|x|\to 0} x_n^{-1}\{u(x) - \phi(0)\}.$$

Furthermore, *if* $u(x) \geq \phi(0)$ *on* B_ε *then* $c > 0$.

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