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Asymptotic completeness in classical N-body systems

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Abstract

We describe some properties of classical trajectories which are related to the concept of the asymptotic completeness.

1 Introduction

The scattering theory of quantum nonrelativistic systems is quite well understood. In particular, it is known that N-body quantum systems are asymptotically complete under rather general assumptions [E,SigSof1,2,Graf,Ya,De1]. The asymptotic completeness roughly means that one can fully classify all states in the Hilbert space according to the asymptotic properties of their evolution as $t \to \infty$.

It is natural to ask if there is an analog of this property in the classical case. In this exposé we would like to describe a number of such analogs. We will indicate the difficulties which one encounters if one wants to introduce the notion of the asymptotic completeness in the classical case. For simplicity, we will restrict our attention to the short range case.

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2 2-body systems

A 2-body system is described by a Hamiltonian of the form $H(x,\xi) := \frac{1}{2}\xi^2 + V(x)$ defined on the phase space $X \times X'$ where X is a finite dimensional vector space. We want to study its trajectories, that is functions $y(t)$ that solve the equations

$$\frac{d^2}{dt^2}y(t) = -\nabla V(y(t)).$$

The scattering theory of 2-body classical systems is quite well understood [Sim,RS]. The most basic asymptotic property of classical trajectories is the existence of the asymptotic velocity. This property is actually true also for long range potentials. We describe it in the following theorem.

Theorem 2.1 *Suppose that*

$$\|\nabla V(x)\| \leq c\langle x\rangle^{-1-\mu}$$

with $\mu > 0$. Then for any trajectory $y(t)$ there exists

$$\lim_{t\to\infty} \frac{y(t)}{t}. \tag{2.1}$$

If 2.1 is not equal to 0 then it is equal to

$$\lim_{t\to\infty} \dot{y}(t). \tag{2.2}$$

If (y,η) are the initial conditions for the trajectory $y(t), \dot{y}(t)$ then we set $\xi^+(y,\eta)$ to be equal to 2.1. In this way we define a function on the phase space $X \times X'$ with values in the momentum space X'. We will call this function the asymptotic velocity.

Let us mention that the set of all trajectories can be naturally divided into following three categories

a) bounded trajectories;

b) "almost-bounded trajectories", that is unbounded trajectories for which the asymptotic velocity is zero; and

c) scattering trajectories, that is trajectories with a non-zero asymptotic velocity.

It is possible to give a rather complete classification of scattering trajectories in the short range 2-body case. This will be described in the following two theorems.

Theorem 2.2 *Suppose that*

$$\|\nabla V(x)\| \leq c\langle x\rangle^{-1-\mu}$$

with $\mu > 1$. Then for any scattering trajectory $y(t)$ with asymptotic velocity ξ^+ there exists

$$\lim_{t\to\infty} y(t) - t\xi^+. \tag{2.3}$$

Theorem 2.3 *Assume in addition that*

$$\|\nabla^2 V(x)\| \leq c\langle x\rangle^{-1-\mu}$$

with $\mu > 1$. *Then for any* $x^+, \xi^+ \in X$ *with* $\xi^+ \neq 0$ *there exists a unique trajectory* $y(t)$ *such that its asymptotic velocity equals* ξ^+ *and 2.3 equals* x^+.

One can restate the above theorems as follows. Suppose that $\phi(t)$ and $\phi_0(t)$ denote the flows generated by the Hamiltonians H and $H_0 := \frac{1}{2}\xi^2$ respectively. Then theorem 2.2 says that there exists $\lim_{t\to\infty}\phi_0(-t)\phi(t)$ on $(\xi^+)^{-1}(X'\backslash\{0\})$ and theorem 2.3 says that there exists $\lim_{t\to\infty}\phi(-t)\phi_0(t)$ on $X \times (X'\backslash\{0\})$. Thus, theorem 2.2 describes what can be called the asymptotic completeness of classical 2-body systems and theorem 2.3 describes the existence of "wave transformations". One may remark that the proof of theorem 2.2 is easier than the proof of theorem 2.3; besides it needs weaker assumptions on the potential. Note that in the quantum case it is usually easier to show the existence of wave operators then their completeness.

3 N-body systems

Next we would like to discuss the N-body case. The results described below are taken from [De2].

A system of N classical particles interacting with pair potentials can be described with a Hamiltonian of the form

$$H = \sum_{i=1}^{N}\frac{1}{2m_i}\xi_i^2 + \sum_{i>j=1}^{N} V_{ij}(x_i - x_j) \tag{3.1}$$

defined on the phase space $X \times X'$ where $X = \mathbf{R}^{3N}$ and X' is its conjugate space. Following Agmon [A] it has become almost standard in the mathematically oriented literature to replace 3.1 with an essentially more general class of Hamiltonians, sometimes called generalized N-body Hamiltonians. They are functions on $X \times X'$ of the form

$$H = \frac{1}{2}\xi^2 + \sum_{a\in\mathcal{A}} V^a(x^a) \tag{3.2}$$

where X is a Euclidean space, $\{X^a : a \in \mathcal{A}\}$ is a family of subspaces closed wrt the algebraic sum and containing $\{0\}$, and x^a denotes the orthogonal projection of x onto X^a. It is easy to see that after a change of coordinates any Hamiltonian of the form 3.1 belongs to the class 3.2.

We write $a \subset b$ iff $X^a \subset X^b$. The orthogonal complement of X^a in X is denoted X_a. We will write x_a for the orthogonal projection of x onto X_a.

There will be a special symbol for the sets

$$Z_a := X_a \setminus \bigcup_{b \not\subset a} X_b. \tag{3.3}$$

Let us introduce also the so-called cluster Hamiltonians

$$H_a := \frac{1}{2}\xi^2 + \sum_{b \subset a} V^b(x^b).$$

Note that $H_a = \frac{1}{2}\xi_a^2 + H^a$ where

$$H^a := \frac{1}{2}(\xi^a)^2 + \sum_{b \subset a} V^b(x^b).$$

It turns out that also in the N-body case every trajectory possesses an asymptotic velocity. This result has a quantum analog [De1] and is inspired by [Graf]. Curiously enough, its quantum analog was discovered first.

Theorem 3.1 *Assume that for every $a \in \mathcal{A}$ and some $\mu > 0$*

$$|\nabla V^a(x^a)| \leq c\langle x^a \rangle^{-1-\mu}. \tag{3.4}$$

Let $y(t)$ be a trajectory of H. Then there exists

$$\lim_{t \to \infty} t^{-1} y(t). \tag{3.5}$$

If this limit belongs to Z_a then it equals

$$\lim_{t \to \infty} \dot{y}_a(t). \tag{3.6}$$

The configuration space X is the disjoint union of sets Z_a. Hence the condition

$$\lim_{t \to \infty} t^{-1} y(t) \in Z_a \tag{3.7}$$

separates the set of all trajectories into distinct categories labelled with elements of $\mathcal{A}$.

It turns out that one can say a lot more about the "sub-a" coordinates then the "super-a" coordinates of such a trajectory. In particular, in the short range case the "sub-a" coordinates are asymptotic to the free motion.

Theorem 3.2 *Suppose that $y(t)$ is a trajectory of H that satisfies 3.7. Let ξ_a^+ denote 3.5. Suppose that for any $b \in \mathcal{A}$ $|\partial^\alpha V^b(x_b)| < c\langle x^b \rangle^{-|\alpha|-\mu}$ for $|\alpha| = 1, 2$. Let $\mu > 1$. Then there exists a unique $x_a^+ \in X_a$ such that*

$$\lim_{t \to \infty} (y_a(t) - x_a^+ - t\xi_a^+) = 0. \tag{3.8}$$

In the classical case N-body case it is not clear what should be the definition of the notion to be called the asymptotic completeness. One of candidates for this name is given in the above theorem. Unfortunately, it is not very satisfactory, because it yields a very incomplete classification of all trajectories. Our next theorem describes a much more complete classification of N-body trajectories. Its part a) can be viewed as an analog of the existence of wave operators and part b) as an analog of the asymptotic completeness. Unfortunately, it is true under very restrictive assumptions on the decay of the potentials.

Theorem 3.3 *Suppose that for every $b \in \mathcal{A}$ $\nabla^2 V^b(x^b)$ is bounded and for every $\theta > 0$ there exists σ such that $|\nabla V^a(x^a)| < \sigma e^{-\theta|x^a|}$. Then the following statements are true.*
a) For any trajectory $x(t)$ of H_a such that

$$\lim_{t\to\infty} t^{-1}x(t) \in Z_a \tag{3.9}$$

there exists a unique trajectory $y(t)$ of H such that for any $\theta > 0$

$$\lim_{t\to\infty} e^{\theta t}(x(t) - y(t)) = 0 \tag{3.10}$$

and

$$\lim_{t\to\infty} t(\dot{x}(t) - \dot{y}(t)) = 0. \tag{3.11}$$

b) For any trajectory $y(t)$ of H such that

$$\lim_{t\to\infty} t^{-1}y(t) \in Z_a \tag{3.12}$$

there exists a unique trajectory $x(t)$ of H_a such that for any $\theta > 0$

$$\lim_{t\to\infty} e^{\theta t}(x(t) - y(t)) = 0 \tag{3.13}$$

and

$$\lim_{t\to\infty} t(\dot{x}(t) - \dot{y}(t)) = 0. \tag{3.14}$$

Note that all the trajectories of H_a are of the form $x(t) = x^a(t) + x_a^+ + t\xi_a^+$ where $x^a(t)$ is a trajectory of H^a. If $x(t)$ satisfies 3.9 then $x^a(t)$ is a bounded or almost-bounded trajectory. If potentials are compactly supported then there are no almost-bounded trajectory. Thus in this case all the trajectories can be asymptotically decomposed into a bounded intracluster motion and a free intercluster motion—which is probably the most intuitive candidate for the definition of the asymptotic completeness. This is an old result which was obtained by Hunziker [Hu].

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