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RADIATION CONDITIONS AND SCATTERING THEORY FOR N-PARTICLE HAMILTONIANS (MAIN IDEAS OF THE APPROACH)

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Abstract

The correct form of radiation conditions is found in scattering problem for N-particle quantum systems. The estimates obtained allow us to give an elementary proof of asymptotic completeness for such systems in the framework of the theory of smooth perturbations. Here we outline main ideas of this proof.

1. One of the main problems of scattering theory is a description of asymptotic behaviour of N interacting quantum particles for large times. The complete classification of all possible asymptotics (channels of scattering) is called asymptotic completeness. The final result can easily be formulated in physics terms. Two particles can either form a bound state or are asymptotically free. In the case $N \geq 3$ a system of N particles can additionally be decomposed for large times into non-trivial subsystems (clusters). Particles of the same cluster form a bound state and different clusters do not interact with each other.

There are two essentially different approaches to a proof of asymptotic completeness for multiparticle ($N \geq 3$) quantum systems. The first of them, suggested by L. D. Faddeev [1], relies on the detailed study of a set of equations derived by him for the resolvent of the corresponding Hamiltonian. This approach was developped in [1] for the case of three particles and was further elaborated by J. Ginibre and M. Moulin [2] and L. Thomas [3]. The attempts [4, 5] towards a straightforward generalization of Faddeev's method to an arbitrary number of particles meet with numerous difficulties. However, the results of [6] for weak interactions are quite elementary.

Another approach relies on the commutator method [7] of T. Kato. In the theory of N-particle scattering it was introduced by R. Lavine [8, 9] for

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repulsive potentials. The proof of asymptotic completeness in the general case is much more complicated and is due to I. Sigal and A. Soffer [10] (see also the article [11] by J. Derezinski for the proof of intermediary analytical results and the article [12] by H. Tamura for more careful presentation of the method of [10]). In the recent paper [13] G. M. Graf gave an accurate proof of asymptotic completeness in the time-dependent framework. The distinguishing feature of [13] is that all intermediary results are also purely time-dependent and most of them have a direct classical interpretation. Papers [10, 13] were to a large extent inspired by V. Enss (see e.g. [14]) who was the first to apply a time-dependent technique for the proof of asymptotic completeness.

The aim of the present paper is to give an elementary proof of asymptotic completeness for N-particle Hamiltonians with short-range potentials which fits into the theory of smooth perturbations [7, 15]. This proof is quite similar to the one [16] suggested by the author for three-particle Hamiltonians. One of the advantages of the theory of smooth perturbations is that it admits two equivalent formulations. The first of them, time-dependent, is given in terms of unitary groups of the Hamiltonians considered. Another, the stationary one, is based on their resolvents. In particular, the stationary version automatically gives (see e.g. [17]) formulas for basic objects of scattering theory: wave operators, scattering matrix etc. Properties of these representations, specific for N-particle systems, will hopefully be discussed elsewhere.

Our proof of asymptotic completeness relies on new estimates which establish some kind of radiation conditions for N-particle systems. Compared to the limiting absorption principle (see below) radiation conditions-estimates give us an additional information on the asymptotic behaviour of a quantum system for large distances and large times. The limiting absorption principle suffices for a proof of asymptotic completeness in the case of two-particle Hamiltonians with short-range potentials. However, radiation conditions-estimates are crucial in scattering for long-range potentials (see e.g. [18]), in scattering by unbounded obstacles [19, 20] and in scattering for anisotropically decreasing potentials [21]. In the latter paper the role of radiation conditions was also advocated for three-particle Hamiltonians.

2. Our interpretation of radiation conditions is, of course, different from the two-particle case. Before discussing their precise form let us introduce the generalized N-particle Hamiltonians. We consider the self-adjoint Schrödinger operator $H = -\Delta + V(x)$ in the Hilbert space $\mathcal{H} = L_2(\mathbf{R}^d)$. Suppose that some finite number α_0 of subspaces X^α of $X := \mathbf{R}^d$ is given and let x^α, x_α be the orthogonal projections of $x \in X$ on X^α and $X_\alpha = X \ominus X^\alpha$ respectively. We assume that

$$V(x) = \sum_{\alpha=1}^{\alpha_0} V^\alpha(x^\alpha), \tag{1}$$

where V^α are decreasing real functions of variables x^α. Without loss of generality, one can suppose that the linear sum of all subspaces X^α exhausts X. The two-particle Hamiltonian H is recovered if (1) consists of only one term with $X^\alpha = X$. The three-particle problem is distinguished from the general situation by the condition $X_\alpha \cap X_\beta = \{0\}$ for $\alpha \neq \beta$. We prove asymptotic completeness under the assumption that V^α are short-range functions of x^α but many intermediary results (in particular, radiation conditions-estimates) are as well true for long-range potentials. Clearly, $V^\alpha(x^\alpha)$ tends to zero as $|x| \to \infty$ outside of any conical neighbourhood of X_α and $V^\alpha(x^\alpha)$ is constant on planes parallel to X_α. Due to this property the structure of the spectrum of H is much more complicated than in the two-particle case. Operators H considered here were introduced in [22] and are natural generalizations of N-particle Hamiltonians. Consideration of a more general class of operators allows to unravel better the geometry of the problem.

The spectral theory of the operator H starts with the following geometrical construction. Let us introduce the set $\mathcal{X}$ of linear sums

$$X^a = X^{\alpha_1} + X^{\alpha_2} + \ldots + X^{\alpha_k}$$

of subspaces X^{α_j}. The zero subspace $X^0 = \{0\}$ is included in the set $\mathcal{X}$ and X itself is excluded. The index a (or b) labels all subspaces $X^a \in \mathcal{X}$ and can be interpreted as the collection of all those α_j for which $X^{\alpha_j} \subset X^a$. Let x^a and x_a be the orthogonal projections of $x \in X$ on the subspaces X^a and

$$X_a := X \ominus X^a = X_{\alpha_1} \cap X_{\alpha_2} \cap \ldots \cap X_{\alpha_k},$$

respectively. Since $X = X_a \oplus X^a$, $\mathcal{H}$ splits into a tensor product

$$L_2(X) = L_2(X_a) \otimes L_2(X^a). \tag{2}$$

In the multiparticle terminology, index a parametrizes decompositions of an N-particle system into noninteracting clusters; x^a is the set of "internal" coordinates of all clusters, x_a describes the relative motion of clusters.

Let us introduce for each a an auxiliary operator $H_a = T + V^a$, $T = -\Delta$, with a potential

$$V^a = \sum_{X^\alpha \subset X^a} V^\alpha, \tag{3}$$

which does not depend on x_a. In the representation (2)

$$H_a = T_a \otimes I + I \otimes H^a, \tag{4}$$

where $T_a = -\Delta_{x_a}$ acts in the space $\mathcal{H}_a = L_2(X_a)$ and

$$H^a = T^a + V^a, \quad T^a = -\Delta_{x^a}$$

are the operators in the space $\mathcal{H}^a = L_2(X^a)$. Set $\mathcal{H}^0 = \mathbb{C}, V^0 = 0, H^0 = 0$. The operator H^a corresponds to the Hamiltonian of clusters with their centers-of-mass fixed at the origin, T_a is the kinetic energy of the center-of-mass motion of these clusters and H_a describes an N-particle system with interactions between different clusters neglected. Eigenvalues of the operators H^a are called thresholds for the Hamiltonian H.

3. Let us introduce the following notation: $E(\Lambda) = E(\Lambda; H)$ is the spectral projection of the operator H corresponding to a Borel set $\Lambda \subset \mathbb{R}$; $\mathcal{H}^{(ac)}(H)$ is the absolutely continuous subspace of H; $P^{(ac)}(H)$ is the orthogonal projection on $\mathcal{H}^{(ac)}(H)$; $\mathcal{H}^{(p)}(H)$ is the subspace spanned by all eigenvectors of the operator H; Q is the operator of multiplication by $(x^2+1)^{1/2}$.

The limiting absorption principle asserts that the operator Q^{-r} is locally H-smooth (in the sense of T. Kato) for any $r > 1/2$. The term "locally" means that actually only the operator $Q^{-r}E(\Lambda)$ is H-smooth for an arbitrary bounded interval Λ which is separated from all thresholds and eigenvalues of H. A definition of H-smoothness of the operators $Q^{-r}E(\Lambda)$ can be given either in terms of the resolvent $(H-z)^{-1}$, $\operatorname{Im} z \neq 0$, of the operator H or of its unitary group $U(t) = \exp(-iHt)$. This is discussed e.g. in [17] or [23]. We recall here the definition in terms of $U(t)$: an H-bounded operator K is called H-smooth if for every $f \in \mathcal{D}(H)$

$$\int_{-\infty}^{\infty} \|K \exp(-iHt) f\|^2 \, dt \leq C \|f\|^2.$$

The limiting absorption principle ensures, in particular, that the singular continuous spectrum of H is empty,that is $\mathcal{H} = \mathcal{H}^{(p)}(H) \oplus \mathcal{H}^{(ac)}(H)$. Furthermore, in the case $N = 2$ (but not $N > 2$) it suffices for construction of scattering theory. There are many different proofs of the limiting absorption principle for $N = 2$ but the only one applicable for $N > 2$ relies on the Mourre estimate (see articles [24, 25, 26]).

The fundamental result of N-particle scattering theory called asymptotic completeness is the assertion that the evolution governed by the Hamiltonian H is decomposed as $t \to \pm\infty$ into a sum of simpler evolutions governed by the Hamiltonians H_a. This means that for every $f \in H^{(ac)}$ there exist $f_a^{\pm}$ such that

$$U(t)f \sim \sum_a U_a(t) f_a^{\pm}, \quad U_a(t) = \exp(-iH_a t), \quad t \to \pm\infty, \tag{5}$$

where " $\sim$ " denotes that the difference between left and right sides tends to zero. This relation is also called sometimes asymptotic clustering. Using separation of variables (4) and applying (5) to the Hamiltonians H^a (in place of H) one can desribe the asymptotics of $U(t)f$ in terms of the free operators T_a and of eigenvalues λ_n^a and eigenvectors ψ_n^a of the operators H^a. Actually, by inductive procedure, (5) yields

$$U(t)f \sim \sum_a \sum_n \exp(-i(T_a + \lambda_n^a)t) f_{a,n}^{\pm} \otimes \psi_n^a, \quad f_{a,n}^{\pm} \in \mathcal{H}_a. \tag{6}$$

In particular, in the two-particle case the right side of (6) consists of the single term $\exp(-iTt)f^{\pm}$ where $f^{\pm} \in \mathcal{H}$.

More detailed formulation of the scattering problem for N-particle Hamiltonians is given in terms of wave operators. Recall that for a couple of self-adjoint operators H_j, $j = 1, 2$, in a Hilbert space $\mathcal{H}$ and a bounded operator (identification) $J : \mathcal{H} \to \mathcal{H}$ the wave operator is defined by the relation

$$W^{\pm}(H_2, H_1; J) = s - \lim_{t \to \pm\infty} \exp(iH_2 t) J \exp(-iH_1 t) P^{(ac)}(H_1) \tag{7}$$

under the assumption that this limit exists. In this case the intertwining property

$$E(\Omega; H_2) W^{\pm}(H_2, H_1; J) = W^{\pm}(H_2, H_1; J) E(\Omega; H_1)$$

($\Omega \subset \mathbf{R}$ is any Borel set) holds. It follows that the range $R(W^{\pm}(H_2, H_1; J))$ of the operator (7) is contained in $\mathcal{H}^{(ac)}(H_2)$ and its closure is an unvariant subspace of H_2. Moreover, if the wave operator is isometric on some subspace $\mathcal{H}_1$, then the restrictions of H_1 and H_2 on the subspaces $\mathcal{H}_1$ and $\mathcal{H}_2 = W^{\pm}(H_2, H_1; J)\mathcal{H}_1$ respectively are unitarily equivalent. This equivalence is realized by the wave operator. Clearly, for every $f_2 = W^{\pm}(H_2, H_1; J) f_1$

$$\exp(-iH_2 t) f_2 \sim J \exp(-iH_1 t) f_1, \quad t \to \pm\infty.$$

In the case $J = I$ we omit dependence of wave operators on J. The operator $W^{\pm}(H_2, H_1)$ is obviously isometric on $\mathcal{H}^{(ac)}(H_1)$. The operator $W^{\pm}(H_2, H_1)$ is called complete if $R(W^{\pm}(H_2, H_1)) = \mathcal{H}^{(ac)}(H_2)$. This is equivalent to existence of the wave operator $W^{\pm}(H_1, H_2)$.

Let P^a be the orthogonal projection in $\mathcal{H}^a$ on the subspace spanned by all eigenvectors of H^a. Set $P_a = I \otimes P^a$ where the tensor product is defined by (2). According to (4) the orthogonal projection P_a commutes with the operator $H_a = T + V_a$ and its functions. Set also $V^0 = 0, H_0 = T, P_0 = I$. The basic result of the scattering theory for N-particle Schrödinger operators is the following

Theorem. *Suppose that operators $V^{\alpha}(T^{\alpha} + I)^{-1}$ are compact in the space $\mathcal{H}^{\alpha}$ and*

$$(|x^{\alpha}| + 1)^{\rho} V^{\alpha} (T^{\alpha} + I)^{-1}$$

are bounded in $\mathcal{H}^{\alpha}$ for some $\rho > 1$. Then the wave operators

$$W_a^{\pm} = W^{\pm}(H, H_a; P_a)$$

exist and are isometric on the ranges $R(P_a)$ of projections P_a. The ranges $R(W_a^{\pm})$ of $W_a^{\pm}$ are mutually orthogonal and the asymptotic completeness holds:

$$\sum_a \oplus R(W_a^{\pm}) = \mathcal{H}^{(ac)}(H).$$

4. Our proof of this assertion relies on new estimates which we call radiation conditions-estimates. Actually, there is only one estimate which looks differently in different regions of the configuration space X. Denote by $\langle \cdot, \cdot \rangle$ the scalar product in the space $\mathbb{C}^d$. Let $\nabla_a = \nabla_{x_a}$ be the gradient in the variable x_a (i.e. $\nabla_a u$ is the orthogonal projection of ∇u on X_a) and let $\nabla_a^{(s)}$,

$$(\nabla_a^{(s)} u)(x) = (\nabla_a u)(x) - |x_a|^{-2} \langle (\nabla_a u)(x), x_a \rangle x_a,$$

be its orthogonal projection in X_a on the plane orthogonal to the vector x_a. Let $\mathbf{\Gamma}_a$ be a cone in $\mathbf{R}^d$ such that $\overline{\mathbf{\Gamma}}_a \cap X_b = \{0\}$ if $X_a \not\subset X_b$ and let $\mathbf{Y}_a$ be an intersection of $\mathbf{\Gamma}_a$ with some conical neighbourhood of X_a. In other words, $\mathbf{Y}_a$ is a neighbourhood of X_a with some neighbourhoods of all X_b, $X_a \not\subset X_b$, removed from it. Denote by $\chi(\cdot)$ the characteristic function of a corresponding set. Our main analytical result is that for every a the operator

$$\mathcal{G}_a = \chi(\mathbf{\Gamma}_a) Q^{-1/2} \nabla_a^{(s)}$$

is locally H-smooth. This result is formulated as a certain estimate (expressed either in terms of the resolvent of H or of its unitary group) which, by analogy with the two-particle problem, we call the radiation conditions-estimate. Actually, it suffices to verify local H-smoothness of operators

$$G_a = \chi(\mathbf{Y}_a) Q^{-1/2} \nabla_a^{(s)}.$$

Considering the collection of these operators for all a we obtain H-smoothness of the operators $\mathcal{G}_a E(\Lambda)$.

Let us compare the limiting absorption principle with the radiation conditions-estimates. Note that the operator $Q^{-1/2}$ is definitely not H-smooth even in the free case $H = -\Delta$. Thus the radiation conditions-estimates show that the differential operators $\nabla_a^{(s)}$ improve the fall-off of functions $(U(t)f)(x)$ for large t and $x \in \mathbf{\Gamma}_a$. In particular, in the free region $\mathbf{\Gamma}_0$, where all potentials V^α are vanishing, we have that the operator $Q^{-1/2} \nabla^{(s)}$ is H-smooth. This result is not very astonishing from the viewpoint of analogy with the classical mechanics. Indeed, for the free motion the vector $x(t)$ of the position of a particle is directed asymptotically as its momentum p (corresponding to the operator $-i\nabla$). So the projection of p on the plane orthogonal to $x(t)$ tends to zero. According to the conjecture (6) in the region $\mathbf{\Gamma}_a$ (for arbitrary a) the evolution in the variable x_a (corresponding to the relative motion of clusters of particles) is also asymptotically free. Therefore one can expect that the operator $\chi(\mathbf{\Gamma}_a)\nabla_a^{(s)}$ is "improving" for all a.

Our proof of H-smoothness of the operators G_a hinges on the commutator method rather than the integration-by-parts machinery which is used (see e.g. [18]) to derive the radiation conditions-estimates in the two-particle case. Actually, we construct such an H-bounded operator M that the commutator $[H, M] := HM - MH$ satisfies locally the estimate

$$i[H, M] \geq c(G_a^* G_a - Q^{-\rho}), \quad \rho > 1, \quad \forall\, a. \tag{8}$$

The arguments of [7] show that H-smoothness of the operator G_a is a direct consequence of this estimate and of the limiting absorption principle. We look for an operator M in a form of a first-order differential operator

$$M = \sum_{j=1}^{d}(m_j D_j + D_j m_j), \quad m_j = \partial m / \partial x_j, \tag{9}$$

with a suitably chosen real function m which we call generating for M. Note that m is a homogeneous function of degree 1 so that coefficients m_j of the operator M are bounded. The leading term $G_a^* G_a$ in the right side of (8) comes from $i[H_0, M]$. We emphasize that due to the operator $Q^{-\rho}$ values of m in a compact domain are inessential. The operator $Q^{-\rho}$ controls in (8) also the commutator

$$i[V^\alpha, M] = -2\langle \nabla m(x), \nabla V^\alpha(x^\alpha)\rangle. \tag{10}$$

To give an idea of the choice of m suppose for a moment that $m(x) = |x|$. Then there is the identity

$$i[H_0, M] = 4\nabla^{(s)}|x|^{-1}\nabla^{(s)}, \quad H_0 = T = -\Delta.$$

Furthermore, by (10),

$$i[V^\alpha, M] = -2|x|^{-1}\langle x^\alpha, \nabla V^\alpha(x^\alpha)\rangle. \tag{11}$$

Thus, under proper assumptions on V^α, we have that in the case $X^\alpha = X$

$$[V^\alpha, M] = O(|x|^{-\rho}), \quad |x| \to \infty, \tag{12}$$

for some $\rho > 1$. This yields the estimate (8) and hence smoothness of the operator $Q^{-1/2}\nabla^{(s)}$ with respect to the two-particle Schrödinger operator H.

However, if $X^\alpha \neq X$, then functions (11) decrease only as $|x|^{-1}$ at infinity. Actually, one can not expect that the operator $Q^{-1/2}\nabla^{(s)}$ is smooth with respect to the N-particle Hamiltonian H. To prove a weaker result about H-smoothness of the operators G_a the function $m(x)$ should be modified in such a way that the estimate (12) with $\rho > 1$ holds for all α. According to (10), this is true if $m(x)$ depends only on the variable x_α in some cone where $V^\alpha(x^\alpha)$ is concentrated. A similar idea was applied by G. M. Graf [13] in the time-dependent context. We emphasize that our requirement on the function $m(x)$ ensures that $m(x) = m(x_a)$ in some conical neighbourhood of every X_a. In other words, a level surface $m(x) = const$ (which is a sphere for $m(x) = |x|$) should be flattened in a neighbourhood of each X_a. Another restriction on $m(x)$ is that the commutator $i[H_0, M]$ should be positive (up to an error $O(|x|^{-\rho})$, $\rho > 1$). This demands that $m(x)$ be a convex function. In this case we can neglect the region $X \setminus \mathbf{Y}_a$ by the derivation of the estimate (8). It turns out that flattening and convexity are compatible. However, the commutator $i[H_0, M]$ gets smaller compared to the case $m(x) = |x|$ so that radiation conditions-estimates in the N-particle case are weaker for $N > 2$ than for $N = 2$. Note also that due to localization in energy in this estimate we can easily dispense with derivatives of V^α and prove H-smoothness of the operators G_a both in short-range and long-range cases.

5. Our approach to the N-particle scattering theory starts with consideration of the wave operators

$$W^\pm(H, H_a; M^{(a)} E_a(\Lambda)), \quad W^\pm(H_a, H; M^{(a)} E(\Lambda)) \tag{13}$$

with "identifications"

$$M^{(a)} = \sum_{j=1}^{d} (m_j^{(a)} D_j + D_j m_j^{(a)}), \quad m_j^{(a)} = \partial m^{(a)} / \partial x_j, \tag{14}$$

which are first-order differential operators with suitably chosen "generating" functions $m^{(a)}$. The "effective perturbation" equals

$$H M^{(a)} - M^{(a)} H_a = [T, M^{(a)}] + [V^a, M^{(a)}] + V_a M^{(a)}, \tag{15}$$

where V^a is defined by (3) and

$$V_a = V - V^a = \sum_{X^\alpha \not\subset X^a} V^\alpha.$$

To prove existence of the wave operators (13) it suffices (see e.g. [17] or [23]) to verify that every term in the right side of (15) can be factorized into a product $K^* K_a$ where K is H-smooth and K_a is H_a-smooth (locally). Functions $m^{(a)}$ are chosen as homogeneous functions of order 1 (for $|x| \geq 1$). Therefore coefficients of the second-order differential operator $[T, M^{(a)}]$ decrease only as $|x|^{-1}$ at infinity. It turns out that this term can be considered with the help of the radiation conditions-estimates. Furthermore, similarly to $m(x)$, the function $m^{(a)}(x)$ depends only on x_α in some conical neighbourhood of X_α.

This ensures that $[V^{(a)}, M^{(a)}] = O(|x|^{-\rho})$ where $\rho > 1$. Finally, it is required that $m^{(a)}(x)$ equals zero in some conical neighbourhood of X_α such that $X_a \not\subset X_\alpha$. So coefficients of the operator $V_a M^{(a)}$ also vanish as $O(|x|^{-\rho})$, $\rho > 1$, at infinity. Thus the second and third terms in the right side of (15) can be taken into account by the limiting absorption principle. The obtained representation for the operator (15) ensures existence of both wave operators (13) (for all a).

Existence of the second wave operator (13) implies that for every vector $f^\pm \in E(\Lambda)\mathcal{H}$ and some $f_a^\pm$

$$M^{(a)}U(t)f^\pm \sim U_a(t)f_a^\pm, \quad t \to \pm\infty. \tag{16}$$

If the sum of $M^{(a)}$ over all a were equal the identity operator I, then summing up, as advised in [27], the relations (16) we would have obtained the asymptotic completeness (5). However, the equality $\sum_a M^{(a)} = I$ is incompatible with the definition (14). We choose functions $m^{(a)}$ in such a way that

$$\sum_a M^{(a)} = M,$$

where M is the same operator as in (9). Summing up the relations (16) we find only that

$$MU(t)f^\pm \sim \sum_a U_a(t)f_a^\pm, \quad t \to \pm\infty. \tag{17}$$

At the final step of the proof of the asymptotic completeness we get rid of the operator M in the left side of (17). To that end we introduce the observable

$$M^\pm(\Lambda) = W^\pm(H, H; ME(\Lambda))$$

and verify that the range of the operator $M^\pm(\Lambda)$ coincides with the subspace $E(\Lambda)\mathcal{H}$. Actually, we show that the operator $\pm M^\pm(\Lambda)$ is positively definite on $E(\Lambda)\mathcal{H}$. In virtue of the inequality $m(x) \geq |x|$ for $|x| \geq 1$, this can be derived from the Mourre estimate. Here we shall explain this result by analogy with classical mechanics. Let us consider a particle (of mass $1/2$) in an external field. In this case the observable $U^*(t)MU(t)$ corresponds, in the Heisenberg picture of motion, to the projection $\mathcal{M}(t) = |x(t)|^{-1}\langle \xi(t), x(t)\rangle$ of the momentum $\xi(t)$ of a particle on a vector $x(t)$ of its position. For positive energies λ and large t we have that $\xi(t) \sim \xi_\pm$, $\xi_\pm^2 = \lambda$, and $x(t) \sim 2\xi_\pm t + x_\pm$. Therefore $\mathcal{M}(t)$ tends to $\pm\lambda^{1/2}$ as $t \to \pm\infty$.

The asymptotic completeness in the form (5) can be easily deduced from these results. Actually, for every $f = M^\pm(\Lambda)f^\pm$ we have that

$$U(t)f \sim MU(t)f^\pm, \quad t \to \pm\infty. \tag{18}$$

Comparing (17) with (18) and taking into account that the range of $M^\pm(\Lambda)$ equals $E(\Lambda)\mathcal{H}$ we arrive at (5). It remains to establish existence of the wave operators $W^{\pm}(H, H_a)$. This is derived from existence of the first set of the wave operators (13). At this step we assume validity of the main Theorem for all operators H_a (in place of H). This additional assumption is, finally, removed by an inductive procedure.

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