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CHARACTERISTIC PROPERTIES OF DISTRIBUTIONS ASSOCIATED WITH THE WAVE GROUP

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Let us consider the spectral problem

$$\textbf{(A)} \qquad Av = \lambda^{2m} v\,,$$

$$\textbf{(B)} \qquad (B^{(j)}v)\Big|_{\partial M} = 0\,, \qquad j = 1, 2, \ldots, m,$$

where $\lambda > 0$ is the spectral parameter, A is a positive self-adjoint elliptic linear differential operator of order $2m$ ($m \in \mathbb{N}$) acting on half-densities on a compact n-dimensional ($n \geq 2$) manifold M with boundary ∂M or without boundary ($\partial M = \emptyset$). The $B^{(j)}$ are "boundary" linear differential operators describing regularly elliptic (see [1]) boundary conditions in the case $\partial M \neq \emptyset$.

Under our assumptions the differential operator A initially defined on

$$D(A) = \{v \in C^\infty(M) : (B^{(j)}v)\Big|_{\partial M} = 0\,, \quad j = 1, 2, \ldots, m\}$$

has a self-adjoint closure $\mathcal{A}$ in $L_2(M)$ with a domain of definition

$$D(\mathcal{A}) = \{v \in H^{2m}(M) : (B^{(j)}v)\Big|_{\partial M} = 0\,, \quad j = 1, 2, \ldots, m\}$$

where $H^{2m}(M)$ denotes the Sobolev space of half-densities belonging to $L_2(M)$ together with all their partial derivatives of order $\leq 2m$. It is well known [1] that the operator $\mathcal{A}$ has a positive discrete spectrum $0 < \nu_1 \leq \nu_2 \leq \ldots$ accumulating to $+\infty$ (we numerate the eigenvalues taking their multiplicities into account). The numbers $\lambda_k = \nu_k^{1/2m}$, $k = 1, 2, \ldots$ may be interpreted as the eigenvalues of the operator $\mathcal{A}^{(1/2m)}$. It is also well known that the respective eigenfunctions (more precisely, half-densities) v_k are infinitely smooth on M , satisfy (**A**), (**B**) and form an orthonormal basis in $L_2(M)$.

Note that whereas $\mathcal{A}$ is an operator in the full sense of the word A is not really an operator: it is more correct to call A a "differential expression". Nevertheless, for the sake of simplicity we speak in both cases of "operators", distinguishing $\mathcal{A}$ and A by different script.

This paper is devoted to the study of the operator $\exp(-it\mathcal{A}^{1/(2m)})$, $t \in (T_-, T_+)$. Information on this operator is essential for obtaining two-term spectral asymptotics for the eigenvalue problem (**A**), (**B**), see [2–8].

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX

1. Functional spaces and basic notation. In this section we define precisely the functional spaces of our distributions in order to prepare the ground for the formulation and subsequent proof of Theorem 1.

Local coordinates on M will be denoted by $x = (x_1, x_2, \ldots, x_n)$ or by $y = (y_1, y_2, \ldots, y_n)$. In a neighborhood of ∂M we will use only special coordinate systems of the type $x = (x', x_n)$, where $x' = (x_1, x_2, \ldots, x_{n-1})$ are coordinates on ∂M (the boundary coordinates) and $x_n \geq 0$ is the "normal" coordinate in the sense that $\partial M = \{x_n = 0\}$.

Following Schwartz [9, Sect. 3.7] and Hörmander [10, vol. 1, Sect. 2.3] we denote by $\mathcal{E}(M)$ the vector space of infinitely differentiable (up to the boundary!) complex-valued half-densities $v(x)$ on M equipped with the usual C^∞-topology defined by the semi-norms

$$v \to \sum_{p=1}^{q} \sum_{|\alpha| \leq k} \max_{x \in M^{(p)}} |\partial_x^\alpha (\chi_p v)|$$

where k ranges over all integers ≥ 0 and $1 = \sum_{p=1}^{q} \chi_p(x)$, $\chi_p \in C^\infty(M)$, $\operatorname{supp} \chi_p \subset M^{(p)}$, is some partition of unity on M with local coordinates x in coordinate maps $M^{(p)}$. We denote by $\mathcal{E}_B(M)$ the subspace of $\mathcal{E}(M)$ consisting of all the half-densities which satisfy the boundary conditions

$$(1) \qquad (B^{(j)} A^r \overline{v})\Big|_{\partial M} = 0\,, \qquad j = 1, 2, \ldots, m, \qquad r = 0, 1, 2, \ldots;$$

the topology on $\mathcal{E}_B(M)$ is taken to be the same as on $\mathcal{E}(M)$. By $\mathcal{E}'(M)$, $\mathcal{E}'_B(M)$ we denote the dual spaces of $\mathcal{E}(M)$, $\mathcal{E}_B(M)$ respectively (i.e. spaces of linear continuous functionals on $\mathcal{E}(M)$, $\mathcal{E}_B(M)$) equipped with the dual (strong) topology generated by the initial topology on $\mathcal{E}(M)$, $\mathcal{E}_B(M)$, see [9, Sect. 3.2, 3.3 and 3.7]. Obviously, $\mathcal{E}'(M) \subset \mathcal{E}'_B(M)$ because $\mathcal{E}_B(M) \subset \mathcal{E}(M)$. The value of the functional (distribution) u on the test half-density v will be denoted by $\langle u, v \rangle_x$ with the subscript x emphasizing the variable in which the distribution is acting.

Let $T_- < T_+$ be real numbers, finite or $\pm\infty$. In accordance with traditional notation we denote by $\mathcal{D}'(T_-, T_+)$ the vector space of linear continuous functionals on $C_0^\infty(T_-, T_+)$. The value of the distribution $f \in \mathcal{D}'(T_-, T_+)$ on the test function $g \in C_0^\infty(T_-, T_+)$ will be denoted by $\langle f, g \rangle_t$. For the sake of simplicity we, following Hörmander [10, vol. 1, Sect. 2.1], equip $\mathcal{D}'(T_-, T_+)$ with the weak* topology defined by the semi-norms

$$\mathcal{D}'(T_-, T_+) \ni f \to |\langle f, g \rangle_t|$$

where g is any fixed function from $C_0^\infty(T_-, T_+)$.

By $C^\infty((T_-, T_+) \times M_y; \mathcal{E}'(M_x))$, $C^\infty((T_-, T_+) \times M_y; \mathcal{E}'_B(M_x))$ we shall denote the class of distributions from $\mathcal{E}'(M_x)$, $\mathcal{E}'_B(M_x)$ respectively infinitely differentiably depending on $(t, y) \in (T_-, T_+) \times M_y$ as on a parameter (the subscripts x and y are used to distinguish the two copies of the manifold M). By $C^\infty(M_x \times M_y; \mathcal{D}'(T_-, T_+))$ we shall denote the class of distributions from $\mathcal{D}'(T_-, T_+)$ infinitely differentiably depending on $(x, y) \in M_x \times M_y$ as on a parameter. Here infinite differentiability is understood in the strong (Fréchet) sense with account of the respective topologies in $\mathcal{E}'(M_x)$, $\mathcal{E}'_B(M_x)$, $\mathcal{D}'(T_-, T_+)$.

We shall use the notation "$f = O(|\lambda|^{-\infty})$ as $\lambda \to -\infty$" to describe the fact that the function $f(\lambda) \in C^\infty(\mathbb{R})$ vanishes faster than any given negative power of $|\lambda|$ when $\lambda \to -\infty$. More generally, we shall use this notation for "functions" $f(\lambda, x, y)$ depending on additional parameters $x \in M_x$, $y \in M_y$ to describe the fact that f as well as any given derivative of f with respect to x, y vanishes faster than any given negative power of $|\lambda|$ as $\lambda \to -\infty$ uniformly over $M_x \times M_y$.

By $\mathcal{F}^{-1}_{t\to\lambda}(\cdot) \overset{\text{def}}{=} (2\pi)^{-1}\int_{-\infty}^{+\infty} \exp(it\lambda)(\cdot)\,dt$ we shall denote the inverse Fourier transform.

2. Main result. The main result of this paper is the following abstract theorem which plays a fundamental role in the asymptotic analysis of higher order ($m > 1$) spectral problems. It allows us to avoid the consideration of an ill-posed Cauchy problem (in the variable t) for the equation $D_t^{2m}u = A_x u$ ($D_t \overset{\text{def}}{=} -i\partial/\partial t$); see [6–8] for applications of this theorem.

Theorem 1. *Let $T_- < 0 < T_+$ be real numbers, finite or $\pm\infty$, and let*

$$(2) \qquad u(t,x,y) \in C^\infty((T_-, T_+) \times M_y; \mathcal{E}'_B(M_x)) \cap C^\infty(M_x \times M_y; \mathcal{D}'(T_-, T_+))$$

be a distribution which behaves as a function in the variable t and as a half-density in the variables x, y.

If

$$(3) \qquad u(t,x,y) - \exp(-itA_x^{1/(2m)})u(0,x,y) \in C^\infty((T_-, T_+) \times M_x \times M_y)$$

then

$$(4) \qquad D_t^{2m}u - A_x u \in C^\infty((T_-, T_+) \times M_x \times M_y),$$

$$(5) \qquad (B_x^{(j)}u)\Big|_{\partial M_x} \in C^\infty((T_-, T_+) \times \partial M_x \times M_y), \qquad j = 1, 2, \ldots, m,$$

$$(6) \qquad \mathcal{F}^{-1}_{t\to\lambda}[gu] = O(|\lambda|^{-\infty}) \quad \text{as} \quad \lambda \to -\infty$$

for any $g(t) \in C_0^\infty(T_-, T_+)$.

Inversly, if (4), (5) hold, (6) holds for some $g(t) \in C_0^\infty(T_-, T_+)$, $g \not\equiv 0$, and, in addition,

$$(7) \qquad (B_x^{(j)}A_x^r u)\Big|_{\partial M_x} = 0 \quad \text{at} \quad t = 0, \qquad j = 1, 2, \ldots, m, \quad r = 0, 1, 2, \ldots,$$

then (3) is fulfilled.

Before proceeeding to the proof of Theorem 1 let us explain in what sense formulas (2)–(7) should be understood (recall that we are dealing with distributions!).

Firstly, (2) means that for any $v(x) \in \mathcal{E}_B(M_x)$, $g(t) \in C_0^\infty(T_-, T_+)$ we have the equality

$$\int_{T_-}^{T_+} \langle u, v\rangle_x \, g\, dt = \int_{M_x} \langle u, g\rangle_t \, v\, dx$$

over all $y \in M_y$.

By $\exp(-it\mathcal{A}^{1/(2m)})$, $t \in (T_-\,,\,T_+)$, we denote the bounded operator $L_2(M) \to L_2(M)$ (called the ***unitary exponent***) defined by the series

$$\exp(-it\mathcal{A}^{1/(2m)}) \;=\; \sum_{k=1}^{+\infty} \exp(-it\lambda_k)\, v_k(x) \int\limits_{M_y} (\,\cdot\,)\, \overline{v_k(y)}\, dy \tag{8}$$

where λ_k are the eigenvalues and v_k are the orthonormalized eigenfunctions of the problem (**A**), (**B**). In accordance with (8), by $\exp(-it\mathcal{A}_x^{1/(2m)})u(0,x,y)$ we denote the formal expression

$$\exp(-it\mathcal{A}_x^{1/(2m)})u(0,x,y) \;=\; \sum_{k=1}^{+\infty} \exp(-it\lambda_k)\, v_k(x) \langle\, u(0,x,y)\,,\, \overline{v_k(x)}\,\rangle_x\,. \tag{9}$$

Note that if $u(0,x,y) = \delta(x-y)$ then (9) is the Schwartz kernel of the unitary exponent. The formal expression (9) can be understood in the sense of $C^\infty((T_-\,,\,T_+) \times M_y;\, \mathcal{E}'_B(M_x))$- distributions as well as $C^\infty(M_x \times M_y;\, \mathcal{D}'(T_-\,,\,T_+))$-distributions. In the first case the value of the distribution $\exp(-it\mathcal{A}_x^{1/(2m)})u(0,x,y)$ on the test half-density $v(x) \in \mathcal{E}_B(M_x)$ is defined as

$$\begin{aligned}(10)\quad &\langle\, \exp(-it\mathcal{A}_x^{1/(2m)})u(0,x,y)\,,\, v\,\rangle_x \\ &\qquad = \sum_{k=1}^{+\infty} \exp(-it\lambda_k) \int\limits_{M_x} v_k(x)\, v(x)\, dx\, \langle\, u(0,x,y)\,,\, v_k(x)\,\rangle_x\,.\end{aligned}$$

In the second case the value of the distribution $\exp(-it\mathcal{A}_x^{1/(2m)})u(0,x,y)$ on the test function $g(t) \in C_0^\infty(T_-\,,\,T_+)$ is defined as

$$\begin{aligned}(11)\quad &\langle\, \exp(-it\mathcal{A}_x^{1/(2m)})u(0,x,y)\,,\, g\,\rangle_t \\ &\qquad = \sum_{k=1}^{+\infty} v_k(x) \int\limits_{T_-}^{T_+} \exp(-it\lambda_k)\, g(t)\, dt\, \langle\, u(0,x,y)\,,\, v_k(x)\,\rangle_x\,.\end{aligned}$$

Elementary integration by parts

$$\int\limits_{M_x} v_k(x)\, v(x)\, dx \;=\; \lambda_k^{-s} \int\limits_{M_x} v_k(x)\, (\overline{A_x^s}\, v(x))\, dx\,, \tag{12}$$

$$\int\limits_{T_-}^{T_+} \exp(-it\lambda_k)\, g(t)\, dt \;=\; (i\lambda_k)^{-s} \int\limits_{T_-}^{T_+} \exp(-it\lambda_k)\, (\partial_t^s g(t))\, dt \tag{13}$$

(obviously we used (1) in deriving (12)) with arbitrary $s \in \mathbb{N}$ proves that the quantities $\int_{M_x} v_k(x)\, v(x)\, dx$, $\int_{T_-}^{T_+} \exp(-it\lambda_k)\, g(t)\, dt$ vanish faster than any given negative power of λ_k as $k \to +\infty$, and, consequently (since we *a priori* know from classical works on one-term spectral asymptotics the rough estimate $\lambda_k \geq ck^{2m/n}$, $c > 0$), faster than any given negative power of k. On the other hand, in view of standard embedding theorems and of the fact that a distribution from $\mathcal{E}'_B(M_x)$ always has finite order (due to the definition of the topology on $\mathcal{E}_B(M_x)$), the quantities $v_k(x)$, $\langle u(0,x,y), v_k(x)\rangle_x$ grow not faster than some fixed positive power of k. This argument shows that the series (10), (11) are absolutely convergent. Moreover, this convergence is uniform over $(T_-, T_+) \times M_y$, $M_x \times M_y$, and does not suffer as a result of differentiation with respect to (t,y), (x,y); see a more detailed discussion after Lemma 8.

Thus, the left-hand side of (3) can be understood in the sense of $C^\infty((T_-, T_+)\times M_y; \mathcal{E}'_B(M_x))$- distributions as well as $C^\infty(M_x \times M_y; \mathcal{D}'(T_-, T_+))$-distributions. The C^∞-inclusion here means that there exists a $w(t,x,y) \in C^\infty((T_-, T_+) \times M_x \times M_y)$ such that

$$\langle u(t,x,y) - \exp(-it\mathcal{A}_x^{1/(2m)})u(0,x,y), v\rangle_x = \int_{M_x} w(t,x,y)\, v(x)\, dx ,$$

$$\langle u(t,x,y) - \exp(-it\mathcal{A}_x^{1/(2m)})u(0,x,y), g\rangle_t = \int_{T_-}^{T_+} w(t,x,y)\, g(t)\, dt$$

for $v(x) \in \mathcal{E}_B(M_x)$, $g(t) \in C_0^\infty(T_-, T_+)$.

The expressions $(B_x^{(j)}u)\Big|_{\partial M_x}$, and, consequently, the inclusions (5), are understood in the sense of $C^\infty(\partial M_x \times M_y; \mathcal{D}'(T_-, T_+))$-distributions:

$$\langle (B_x^{(j)}u)\Big|_{\partial M_x}, g\rangle_t \overset{\text{def}}{=} (B_x^{(j)}\langle u, g\rangle_t)\Big|_{\partial M_x} \tag{14}$$

for $g(t) \in C_0^\infty(T_-, T_+)$. When (6) holds we shall denote the infinitely differentiable "functions" $(B_x^{(j)}u)\Big|_{\partial M_x}$ by $b_j(t,x',y)$, $j = 1, 2, \ldots, m$.

The expression $D_t^{2m}u - A_x u$, and, consequently, the inclusion (4), can be understood in the sense of $C^\infty((T_-, T_+) \times M_y; \mathcal{E}'_B(M_x))$-distributions (if we already know that (5) holds) or $C^\infty(M_x \times M_y; \mathcal{D}'(T_-, T_+))$-distributions. In the first case

$$\langle D_t^{2m}u - A_x u, v\rangle_x \overset{\text{def}}{=} D_t^{2m}\langle u, v\rangle_x - \langle u, \overline{A_x} v\rangle_x - \sum_{j=1}^{m} \int_{\partial M_x} b_j\, (C_x^{(j)}v)\Big|_{\partial M_x} dx'$$

for $v(x) \in \mathcal{E}_B(M_x)$; here the $C^{(j)}$ are "boundary" differential operators defined uniquely (for a given set of $B^{(j)}$) by Green's formula

$$\int_M (Au(x))\, v(x)\, dx - \int_M u(x)\, (\overline{A}v(x))\, dx = \sum_{j=1}^{m} \int_{\partial M} (B^{(j)}u)\Big|_{\partial M} (C^{(j)}v)\Big|_{\partial M} dx',$$

$\forall u \in \mathcal{E}(M)$, $\forall v \in \mathcal{E}_B(M)$. In the second case

$$\langle D_t^{2m} u - A_x u , g \rangle_t \stackrel{\text{def}}{=} \langle u , D_t^{2m} g\mathcal{A}, \rangle_t - A_x \langle u , g \rangle_t$$

for $g(t) \in C_0^\infty(T_- , T_+)$. In both cases the subscript x is used to emphasize the variable in which the respective differential operators are acting.

The expressions $(B_x^{(j)} A_x^r u)\Big|_{\partial M_x}$ appearing in the left-hand side of (7) are understood in the sense of $C^\infty(\partial M_x \times M_y; \mathcal{D}'(T_- , T_+))$-distributions similarly to (14):

$$\langle (B_x^{(j)} A_x^r u)\Big|_{\partial M_x} , g \rangle_t \stackrel{\text{def}}{=} (B_x^{(j)} A_x^r \langle u , g \rangle_t)\Big|_{\partial M_x}$$

for $g(t) \in C_0^\infty(T_- , T_+)$. It is easy to see that if (4), (5) hold then

$$(15) \qquad (B_x^{(j)} A_x^r u)\Big|_{\partial M_x} \in C^\infty((T_- , T_+) \times \partial M_x \times M_y), \quad j = 1,2,\ldots,m, \qquad r = 0,1,2,\ldots,$$

so the equalities (7) can be understood in the classical sense. We shall denote the infinitely differentiable "functions" $(B_x^{(j)} A_x^r u)\Big|_{\partial M_x}$ by $b_{jr}(t,x',y)$.

Finally, the expression $\mathcal{F}_{t\to\lambda}^{-1}[gu]$ is the $C^\infty(\mathbb{R} \times M_x \times M_y)$-"function" (more precisely, function in the variable λ and half-density in the variables x, y) defined as $\mathcal{F}_{t\to\lambda}^{-1}[gu] = (2\pi)^{-1} \langle u(t,x,y), \exp(it\lambda) g(t) \rangle_t$.

3. Proof of Theorem 1. If (2), (3) hold then (4)–(6) are obviously fulfilled. So we have to prove only the inverse statement of Theorem 1. It is convenient to split this proof into several parts which we shall consider as separate lemmas.

Lemma 2. *Let* $b_{jr}(t,x') \in C^\infty((T_- , T_+) \times \partial M)$, $j = 1,2,\ldots,m$, $r = 0,1,2,\ldots$, *be a set of "functions" which satisfy* $b_{jr}(0,x') = 0$. *Then there exists* $w(t,x) \in C^\infty((T_- , T_+) \times M)$ *such that*

$$(16) \qquad (B^{(j)} A^r w)\Big|_{\partial M} = b_{jr}, \qquad j = 1,2,\ldots,m, \qquad r = 0,1,2,\ldots,$$

$$(17) \qquad w|_{t=0} = 0.$$

Proof of Lemma 2. In order to simplify notation let us renumerate our boundary "functions", boundary operators and their orders with one index $k = j + mr$:

$$b_{j+mr} \stackrel{\text{def}}{=} b_{jr}, \qquad B^{(j+mr)} \stackrel{\text{def}}{=} B^{(j)} A^r, \qquad m_{j+mr} \stackrel{\text{def}}{=} m_j + 2mr,$$

$j = 1,2,\ldots,m$, $r = 0,1,2,\ldots$ Here $0 \le m_1 < m_2 < \ldots < m_m \le 2m-1$ are the orders of $B^{(1)}, B^{(2)}, \ldots, B^{(m)}$.

Let us consider some local coordinate system $x = (x', x_n)$ and some compact $K \subset \partial M$ which lies inside the chosen coordinate map. Without loss of generality we shall assume that $\operatorname{supp}_{x'} b_k \subseteqq K$ for all $t \in (T_-, T_+)$ and $k = 1, 2, \ldots$; the general case is reduced to this one by a partition of unity.

Each operator $B^{(k)}$, $k = 1, 2, \ldots$, can be represented in the form $B^{(k)} = c_k(x')\partial_{x_n}^{m_k} - \tilde{B}^{(k)}$, $c_k(x') \neq 0$, where $\tilde{B}^{(k)}$ is a "boundary" differential operator of order m_k without the leading conormal derivative:

$$\tilde{B}^{(k)} = \sum_{p=0}^{m_k - 1} \tilde{B}^{(kp)} \partial_{x_n}^p .$$

Here the $B^{(kp)}$ are differential operators in x' of order $\leq m_k - p$. Without loss of generality we shall assume that $c_k(x') = 1$; the general case is reduced to this one by an obvious renormalization of the operators $B^{(k)}$ and the "functions" $b_k(t, x')$.

Let us construct w as a formal Taylor expansion in x_n

$$w(t,x) \sim \sum_{k=1}^{+\infty} w_k(t,x') \frac{x_n^{m_k}}{m_k!} . \tag{18}$$

Substituting (18) into (16) we obtain an infinite system of differential equations:

$$w_1 = b_1 , \tag{19}$$

$$w_k = \sum_{l=1}^{k-1} \left(\tilde{B}^{(km_l)} w_l\right) + b_k , \qquad k = 2, 3, \ldots \tag{20}$$

Due to its triangular structure this system is solved explicitly: (19) gives w_1 and (20) gives a recurrent procedure for the determination of w_k, $k = 2, 3, \ldots$ Note that we have $w_k|_{t=0} = 0$ and $\operatorname{supp}_{x'} w_k \subseteqq K$ for all $t \in (T_-, T_+)$ because the b_k possess these properties.

It is known that given an arbitrary formal Taylor expansion (18) one can construct an infinitely smooth "function" w with such Taylor coefficients and with support lying in our coordinate map. Moreover, as in our case the Taylor coefficients vanish at $t = 0$, w can be chosen to vanish identically over M at $t = 0$. (These simple statements are proved analogously to [11, Proposition 3.5].)

Lemma 2 is proved.

Lemma 3. *Let $t_- < 0 < t_+$ be finite real numbers and let $a(t,x) \in C^\infty([t_-, t_+] \times M)$ be a "function" which satisfies*

$$\left.(B^{(j)} A^r a)\right|_{\partial M} = 0, \qquad j = 1, 2, \ldots, m, \qquad r = 0, 1, 2, \ldots \tag{21}$$

Then there exists $w(t,x) \in C^\infty([t_-, t_+] \times M)$ such that equalities

$$D_t^{2m} w = Aw + a , \tag{22}$$

$$(23) \qquad (B^{(j)}A^r w)\Big|_{\partial M} = 0\,, \qquad j = 1,2,\dots,m, \qquad r = 0,1,2,\dots,$$

and (17) hold.

Proof of Lemma 3. Set

$$(24) \qquad w(t,x) \;=\; \sum_{k=1}^{+\infty} u_k(x)\,\big(\,w_k(t) \;-\; w_k(0)\,\cos\lambda_k t\,\big)\;,$$

$$(25) \qquad w_k(t) \;=\; \sum_{l=1}^{m} w_{kl}^{+}(t) \;+\; \sum_{l=m+1}^{2m} w_{kl}^{-}(t)\;,$$

$$(26) \qquad w_{kl}^{\pm}(t) \;=\; -\frac{i\exp(-it\lambda_{kl})}{2m\lambda_{kl}^{2m-1}} \int_{t_\pm}^{t} \exp(i\tau\lambda_{kl})\,a_k(\tau)\,d\tau\,,$$

$$(27) \qquad a_k(t) \;=\; \int_M a(t,x)\,\overline{v_k(x)}\,dx\,,$$

$$(28) \qquad \lambda_{kl} = \lambda_k \exp(i\pi(l-1)m^{-1})\,, \qquad l = 1,2,\dots,2m\,.$$

Recall that by λ_k and v_k, $k = 1,2,\dots$, we denote the eigenvalues and the orthonormalized eigenfunctions of the problem (**A**), (**B**). Due to the boundary conditions (21) the quantity $a_k(t)$ defined by (27) vanishes faster than any given negative power of k as $k \to +\infty$ uniformly over $[t_-\,,\,t_+]$; the same is true for any given derivative of $a_k(t)$ with respect to t. An elementary analysis of formulas (25), (26), (28) shows that this rapid decay property is inherited by the terms of the series (24): this series converges absolutely, uniformly over $[t_-\,,\,t_+] \times M$, as well as the series of any given derivatives with respect to t, x. Thus (24) defines an infinitely smooth "function" which can be differentiated under the $\sum_{k=1}^{+\infty}$ sign.

Straightforward substitution shows that the constructed w satisfies (22), (23) and (17).

Remark 4. Lemmas 2, 3 and their proofs remain true if the "functions" b_{jr} and a depend smoothly on the additional parameter $y \in M_y$. In this case the resulting w will also smoothly depend on y.

Lemma 5. *It is sufficient to prove the inverse statement of Theorem 1 with*

$$(29) \qquad D_t^{2m} u \;=\; A_x u\,,$$

$$(30) \qquad (B_x^{(j)} u)\Big|_{\partial M_x} \;=\; 0\,, \qquad j = 1,2,\dots,m,$$

instead of (4), (5), (7).

Proof of Lemma 5. It follows immediately from (13), Lemma 2 and Remark 4 that we can turn (5), (7) into (30) by adding to u an infinitely smooth with respect to all the variables "function" which vanishes at $t = 0$. Obviously such an operation does not spoil (2), (4), (6) and it also has no infiuence on the formula (3) which we are proving.

Take now arbitrary finite real numbers t_-, t_+ such that

$$(31)) \qquad T_- < t_- < 0 < t_+ < T_+ .$$

Formulas (4), (30) imply that the "function" $a(t,x,y) \stackrel{\text{def}}{=} D_t^{2m} u - A_x u$ satisfies the conditions of Lemma 3. It follows from Lemma 3 and Remark 4 that we can turn (4) into (29) by adding to u an infinitely smooth with respect to all the variables "function" which satisfies (23) in the variable x and which vanishes at $t = 0$. Such an operation does not spoil (2), (6), (30) and it does not influence the formula (3) which we are proving, only the time interval becomes smaller ((t_-, t_+) instead of (T_-, T_+)). Assuming that the inverse statement of Theorem 1 with (4), (5), (7) replaced by (29), (30) is true, we have

$$(32) \qquad u(t,x,y) - \exp(-itA_x^{1/(2m)})u(0,x,y) \in C^\infty((t_-, t_+) \times M_x \times M_y) .$$

As t_-, t_+ are arbitrary numbers satisfying (31) formula (32) implies (3).

Lemma 5 is proved.

Thus we have reduced the proof of Theorem 1 to the proof of the following statement.

Let $u(t,x,y)$ be a distribution of the class (2) which behaves as a function in the variable t and as a half-density in the variables x, y and which satisfies (29), (30) and (6) for some $g(t) \in C_0^\infty(T_-, T_+)$, $g \not\equiv 0$. Then (3) is fulfilled.

Set

$$(33) \qquad u_k(t,y) = \langle u(t,x,y), \overline{v_k(x)} \rangle_x .$$

In view of (2) we have

$$(34) \qquad u_k(t,y) \in C^\infty((T_-, T_+) \times M_y) .$$

Moreover, formulas (29), (30) and (33) imply $D_t^{2m} u_k = \lambda_k u_k$, and consequently

$$(35) \qquad u_k(t,y) = \sum_{l=1}^{2m} u_{kl}(t,y) ,$$

$$(36) \qquad u_{kl}(t,y) = u_{kl}(0,y) \exp(-it\lambda_{kl}) = \sum_{p=1}^{2m} b_{lp} (-\lambda_k)^{1-p} D_t^{p-1} u_k(t,y) .$$

Here $\|a_{lp}\|$ is the symmetric $2m$ by $2m$ matrix with elements

$$a_{lp} = \exp(i\pi(l-1)(p-1)m^{-1}), \qquad l,p = 1,2,\ldots,2m,$$

and $\|b_{lp}\| = \|a_{lp}\|^{-1}$. The λ_{kl} are defined by (28). Note that formula (36) allows to express the functions $u_{kl}(t,y)$ in terms of derivatives of the function $u_k(t,y)$; with account of (34) this gives $u_{kl}(t,y) \in C^\infty((T_-, T_+) \times M_y)$.

Definition 6. Consider a sequence of "functions" (more precisely, functions in the variable t and half-densities in the variable y)

$$w_k(t,y) \in C^\infty((T_-\,,T_+)\times M_y)\,, \qquad k=1,2,\dots,$$

and let $1=\sum_{p=1}^q \chi_p(y)$, $\chi_p \in C^\infty(M_y)$, $\operatorname{supp}\chi_p \subset M_y^{(p)}$, be some partition of unity on M_y with local coordinates y in coordinate maps $M_y^{(p)}$. We will say that this sequence *increases slowly* if for any real numbers t_-, t_+ satisfying (31), any multiindex $\alpha \geq 0$ and any integer $r \geq 0$ there exists a natural s such that $k^{-s}\partial_t^r\partial_y^\alpha(\chi_p(y)\,w_k(t,y)) \to 0$ as $k\to+\infty$ uniformly over $t\in[t_-\,,t_+]$, $y\in M_y^{(p)}$, $p=1,2,\dots,q$. We will say that this sequence *decreases rapidly* if for any real numbers t_-, t_+ satisfying (31), any multiindex $\alpha\geq 0$, any integer $r\geq 0$ and any natural s $k^s\partial_t^r\partial_y^\alpha(\chi_p(y)\,w_k(t,y)) \to 0$ as $k\to+\infty$ uniformly over $t\in[t_-\,,t_+]$, $y\in M_y^{(p)}$, $p=1,2,\dots,q$.

Lemma 7. *The sequence $u_k(t,y)$, $k=1,2,\dots,$ increases slowly.*

Proof of Lemma 7. Suppose that the statement of Lemma 7 is false. Then there exist real numbers t_-, t_+ satisfying (31), a multiindex $\alpha\geq 0$, an integer $r\geq 0$, a coordinate map $M_y^{(p)}$ with local coordinates y, a cut-off function $\chi_p(y)\in C^\infty(M_y)$ with $\operatorname{supp}\chi_p \subset M_y^{(p)}$, and sequences $k_s\in\mathbb{N}$, $t_s\in[t_-\,,t_+]$, $y_s\in\operatorname{supp}\chi_p$, $s=1,2,\dots,$ such that $k_s\to+\infty$ and

$$(k_s)^{-s}\left(\partial_t^r\partial_y^\alpha(\chi_p(y)\,u_{k_s}(t,y))\right)\big|_{t=t_s,\ y=y_s} \not\to 0 \tag{37}$$

as $s\to+\infty$. Without loss of generality we shall assume that $t_s\to\tilde t\in[t_-\,,t_+]$, $y_s\to\tilde y\in\operatorname{supp}\chi_p$ as $s\to+\infty$; this can always be achieved by extracting subsequences in view of the compactness of $[t_-\,,t_+]\times\operatorname{supp}\chi_p$. Let us introduce the set of half-densities $\mathcal{B}=\left\{(k_s)^{-s}\,\overline{v_{k_s}(x)}\ :\ s=1,2,\dots\right\}$. Obviously $\mathcal{B}$ is a *bounded set* (see [9, Sect. 3.2 and 3.7]) in $\mathcal{E}_B(M)$. Denote by $\tilde u(t,x,y)$ the distribution defined by the formula $\langle\tilde u, v\rangle_x=\partial_t^r\partial_y^\alpha(\chi_p(y)\,\langle u(t,x,y)\,,v(x)\rangle_x)$, $v(x)\in\mathcal{E}_B(M_x)$. Due to (2) we have $\tilde u(t,x,y)\in C^\infty((T_-\,,T_+)\times M_y;\mathcal{E}'_B(M_x))$. So, in particular, $\tilde u(t,x,y)$ is a continuous function of the parameters t, y at the point $t=\tilde t$, $y=\tilde y$ in the sense of the $\mathcal{E}'_B(M_x)$)-topology which means (see [9, Sect. 3.3 and 3.7]) that $\langle\tilde u(t,x,y)\,,v(x)\rangle_x\to\langle\tilde u(\tilde t,x,\tilde y)\,,v(x)\rangle_x$ as $t\to\tilde t$, $y\to\tilde y$ uniformly over all $v(x)\in\mathcal{B}$, where $\mathcal{B}$ is an arbitrary bounded set in $\mathcal{E}_B(M)$. But this contradicts (37).

Lemma 7 is proved.

Lemma 8. *For any $l=1,2,\dots,2m$ the sequence $u_{kl}(t,y)$, $k=1,2,\dots,$ increases slowly.*

Proof of Lemma 8. Lemma 8 follows immediately from Lemma 7 and formula (36).

Lemma 8 allows us to represent our distribution $u(t,x,y)$ in the form of a series of smooth "functions"

$$u(t,x,y)=\sum_{k=1}^{+\infty}\sum_{l=1}^{2m}u_{kl}(t,y)\,v_k(x)\,, \tag{38}$$

with the values of this distribution on the test "functions" $v(x) \in \mathcal{E}_B(M)$, $g(t) \in C_0^\infty(T_-, T_+)$ given by the natural formulas

$$(39) \qquad \langle u(t,x,y), v \rangle_x = \sum_{k=1}^{+\infty} \sum_{l=1}^{2m} u_{kl}(t,y) \int_{M_x} v_k(x)\, v(x)\, dx ,$$

$$(40) \qquad \langle u(t,x,y), g \rangle_t = \sum_{k=1}^{+\infty} \sum_{l=1}^{2m} v_k(x) \int_{T_-}^{T_+} u_{kl}(t,y)\, g(t)\, dt .$$

It follows from Lemma 8 and formulas (12),

$$(41) \qquad \int_{T_-}^{T_+} u_{kl}(t,y)\, g(t)\, dt = (i\lambda_{kl})^{-s} \int_{T_-}^{T_+} u_{kl}(t,y)\, (\partial_t^s g(t))\, dt$$

(cf. (13)) with arbitrary $s \in \mathbb{N}$ that the series (39), (40) converge absolutely, uniformly over $(T_-, T_+) \times M_y$, $M_x \times M_y$, as well as the series of any given derivatives with respect to (t,y), (x,y). This justifies the representation (38) in the sense of $C^\infty((T_-, T_+) \times M_y; \mathcal{E}'_B(M_x))$-distributions as well as $C^\infty(M_x \times M_y; \mathcal{D}'(T_-, T_+))$-distributions.

Lemma 9. *For any $l \neq 1, m+1$ the sequence $u_{kl}(t,y)$, $k = 1, 2, \ldots,$ decreases rapidly.*

Proof of Lemma 9. Assume $m \geq 2$ (otherwise none of the $l = 1, 2, \ldots, 2m$ satisfy the condition $l \neq 1, m+1$). Let us consider first the case $2 \leq l \leq m$. Having fixed l and arbitrary s, r, α, χ_p (in the notation of Definition 6) set

$$\mu_k(t_-, t_+) \stackrel{\text{def}}{=} \max_{t \in [t_-, t_+],\ y \in M_y^{(p)}} k^s \, |\partial_t^r \partial_y^\alpha (\chi_p(y)\, u_{kl}(t,y))|$$

where t_-, t_+ are arbitrary real numbers satisfying (31). Set $\varepsilon = (T_+ - t_+)/2$. Due to the exponential behavior of the "function" $u_{kl}(t,y)$ in the variable t (see (36)) we have

$$(42) \qquad \mu_k(t_-, t_+) = \exp(-\varepsilon\, \lambda_k \sin(\pi/m))\, \mu_k(t_-, t_+ + \varepsilon) .$$

According to Lemma 8 and Definition 6 there exists a natural $\tilde{s}$ such that

$$(43) \qquad k^{-\tilde{s}} \mu_k(t_-, t_+ + \varepsilon) \to 0 \qquad \text{as} \qquad k \to +\infty .$$

Combining (42) and (43) we get

$$(44) \qquad \mu_k(t_-, t_+) \leq c\, k^{s+\tilde{s}} \exp(-\varepsilon\, \lambda_k \sin(\pi/m))$$

where $c = \max_{k \in \mathbb{N}} (k^{-\tilde{s}} \mu_k(t_-, t_+ + \varepsilon))$. As the right-hand side of (44) contains an exponential term which is obviously stronger than the term $k^{s+\tilde{s}}$, it follows from (44) that $\mu_k(t_-, t_+)$ vanishes as $k \to +\infty$. This means rapid decrease in the sense of Definition 6.

The case $m+2 \leq l \leq 2m$ is handled similarly by estimating $\mu_k(t_-, t_+)$ through $\mu_k(t_- - \varepsilon, t_+)$, $\varepsilon = (t_- - T_-)/2$.

Lemma 9 is proved.

Lemma 10. *The sequence* $u_{k(m+1)}(t,y)$, $k=1,2,\dots$, *decreases rapidly.*

Proof of Lemma 10. As in the proof of the previous lemma let us fix arbitrary s, r, α, χ_p. It follows from (6) that

$$k^s(i\lambda_k)^r\partial_y^\alpha\left(\chi_p(y)\int\limits_{M_x}\mathcal{F}^{-1}_{t\to\lambda}[g(t)\,u(t,x,y)]\,\overline{v_k(x)}\,dx\right)\to 0\quad\text{as}\quad\lambda\to-\infty \tag{45}$$

uniformly over $y\in M_y^{(p)}$, $k\in\mathbb{N}$; here uniformity over k is established by integration by parts in the variable x (similarly to (12)) with account of the boundary conditions (30). Substitution of (38), (36) for $u(t,x,y)$ turns (45) into

$$k^s(i\lambda_k)^r\partial_y^\alpha\left(\chi_p(y)\sum_{l=1}^{2m}\mathcal{F}^{-1}_{t\to(\lambda-\lambda_{kl})}[g(t)]\,u_{kl}(0,y)\right)\to 0\quad\text{as}\quad\lambda\to-\infty\,. \tag{46}$$

Let us choose and fix some $\mu\in\mathbb{R}$ such that $\mathcal{F}^{-1}_{t\to\mu}[g(t)]\neq 0$ (such a μ exists because $g\not\equiv 0$), and let us relate λ and k by the condition $\lambda=\mu-\lambda_k$. Then (46) takes the form

$$\sum_{l=1}^{2m}g_{kl}(y)\;\to\;0\qquad\text{as}\qquad k\to+\infty\,, \tag{47}$$

where

$$g_{kl}(y)\;=\;k^s(i\lambda_k)^r\,\mathcal{F}^{-1}_{t\to(\mu-\lambda_k-\lambda_{kl})}[g(t)]\,\left(\partial_y^\alpha(\chi_p(y)\,u_{kl}(0,y))\right). \tag{48}$$

Formula (48) can also be rewritten as

$$g_{kl}(y)\;=\;\frac{\exp(-i\pi r(l-1)m^{-1})}{2\pi}\int\limits_{T_-}^{T_+}\exp(it(\mu-\lambda_k))\,g(t)\,\left(\partial_t^r\partial_y^\alpha(\chi_p(y)\,u_{kl}(t,y))\right)dt\,;$$

in the case $l\neq 1,\,m+1$ the integral in this formula is easily estimated with the help of Lemma 9 which gives

$$g_{kl}(y)\to 0\quad\text{as}\quad k\to+\infty\,,\qquad l\neq 1,\,m+1\,. \tag{49}$$

Let us examine now formula (48) in the case $l=1$. As $\lambda_{k1}=\lambda_k$ the sequence $\mathcal{F}^{-1}_{t\to(\mu-\lambda_k-\lambda_{k1})}[g(t)]$ decreases rapidly (the usual property of the Fourier transform of a C_0^∞-function), and the presence of slowly increasing factors k^s, $(i\lambda_k)^r$, $\partial_y^\alpha(\chi_p(y)\,u_{kl}(0,y)))$ (see Lemma 8) in the right-hand side of (48) cannot spoil this rapid decrease. Thus

$$g_{k1}(y)\to 0\quad\text{as}\quad k\to+\infty\,. \tag{50}$$

Formulas (47)–(50) and $\lambda_{k(m+1)} = -\lambda_k$ imply that

$$g_{k(m+1)}(y) = k^s (i\lambda_k)^r \mathcal{F}^{-1}_{t\to\mu}[g(t)]\left(\partial_y^\alpha(\chi_p(y)\,u_{k(m+1)}(0,y))\right) \to 0 \quad \text{as} \quad k \to +\infty\,.$$

Dividing the latter formula by the non-zero constant $\mathcal{F}^{-1}_{t\to\mu}[g(t)]$ and multiplying it by the function $\exp(it\lambda_k)$ (which obviously has unit modulus) we arrive at

$$k^s\left(\partial_t^r\partial_y^\alpha(\chi_p(y)\,u_{k(m+1)}(t,y))\right) \to 0 \quad \text{as} \quad k \to +\infty\,. \tag{51}$$

Formula (51) is uniform over $y \in M_y^{(p)}$ because the initial formula (45) had this property and all our arguments leading from (45) to (51) preserved uniformity in y. Moreover, as the time dependence in $u_{k(m+1)}(t,y)$ is trivial (purely imaginary exponent) formula (51) is uniform over all $t \in \mathbb{R}$. So (51) means rapid decrease of the functional sequence $u_{k(m+1)}(t,y)$, $k = 1, 2, \ldots$, in the sense of Definition 6.

Lemma 10 is proved.

Now it only remains to rearrange (38) as

$$\begin{aligned} u(t,x,y) = {} & \exp(-it\mathcal{A}_x^{1/(2m)})u(0,x,y) \\ & + \sum_{k=1}^{+\infty}\sum_{l=2}^{2m}\left(u_{kl}(t,y) - u_{kl}(0,y)\exp(-it\lambda_k)\right)v_k(x)\,. \end{aligned} \tag{52}$$

Due to Lemmas 9, 10 the infinite sum in the right-hand side of (52) defines a $C^\infty((T_-\,,\,T_+)\times M_x\times M_y)$-"function". So the inclusion (3), and with it our Theorem 1, is proved.

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