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ROMAN G. NOVIKOV

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TRANSPARENT POTENTIALS AND HAMILTONIAN SYSTEMS

R.G. NOVIKOV

*This report is based on the joint work of J.-P.FRANCOISE and R.G.NOVIKOV
(see [1]) .*

Theorems 1 and 2 of this report are variations of some results from [1].

Consider the equation

$$(i\partial_y + \partial_x^2 + v(x,y))\psi = 0 \quad (1)$$

where $v(x,y)$ is smooth rapidly decreasing potential. Consider also the scattering amplitude $f(k,\ell)$, $k \in \mathbb{R}$, $\ell \in \mathbb{R}$ for this equation. The scattering amplitude $f(k,\ell)$ can be defined using the solutions of (1) with the asymptotic behavior

$$\psi(x,y,k) = e^{i(kx - k^2 y)} + \theta(y) \frac{e^{ix^2/y}}{\sqrt{y}} f(k, \frac{x}{y}) + \bar{O}\left(\frac{1}{\sqrt{y}}\right),$$

$$x^2 + y^2 \rightarrow \infty; \quad x/y = \ell$$

(where $\theta(y)$ is the Heaviside function).

Consider the following problem. To describe and to construct all transparent potentials (i.e. potentials with zero scattering amplitude) for eq. (1).

In order to present some results about this problem let us introduce into consideration the hamiltonian Calogero-Moser system (see [2,3])

$$\dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x} \quad (2)$$

$$H_g(x,p) = \sum_{n=1}^N p_n^2 + g^2 \sum_{n \neq m} (x_n - x_m)^{-2}, \quad \text{where } n,m = 1 \dots N$$

For this report it is important to remind that

$$H_g(x,p) = \text{tr } L^2, \quad (3)$$

where

$$L_{nm} = p_n \delta_{nm} + (x_n - x_m)^{-1} ig(1 - \delta_{nm}) \quad (4)$$

Let $x_n(t)$ ($n = 1 \dots N$) be a solution of the Calogero-Moser system ($g^2 > 0$).

Write down its asymptotics for $t \rightarrow -\infty$ in the following form

$$x_n(t) \approx \xi_n t + \eta_n \quad \text{for } t \rightarrow -\infty \quad (5)$$

We shall consider solutions of (2) with additional symmetry

$$\xi_{2j} = \overline{\xi_{2j-1}}, \quad \eta_{2j} = \overline{\eta_{2j-1}}, \quad (6)$$

where $j = 1 \dots M$, $N = 2M$. We assume also that

$$\xi_n \neq \xi_m \quad \text{if } n \neq m_0 \quad (7)$$

we are sure that the following result is valid.

Hypothesis 1. If $x_n(t)$ ($n = 1 \dots 2M$) is a solution of the Calogero-Moser system ($g^2 > 0$) with the properties (5,6,7), then

$$\text{Im}(x_n(t)) \neq 0 \text{ for any } n = 1 \dots 2M \text{ and any } t \in \mathbb{R} \quad (8)$$

[This hypothesis can be also formulated in the following way. If $x(t)$ is a solution of the Calogero-Moser system ($g^2 > 0$) with the symmetric with respect real axis Cauchy data, i.e. $x_{2j}(0) = \overline{x_{2j-1}(0)}$, $\dot{x}_{2j}(0) = \overline{\dot{x}_{2j-1}(0)}$, and $x_n(0) \neq x_m(0)$ if $n \neq m$, then $\text{Im}(x_n(t)) \neq 0$ for any $n = 1 \dots 2M$ and any $t \in \mathbb{R}$].

It is clear (for us) how to prove the hypothesis 1 if $\text{Re}(\xi_n) = \text{Re}(\eta_n) = 0$ for $n = 1 \dots 2M$. It is quite possible, however, that somebody knows how to prove this result in the general case.

Following result gives unexpected relation between transparent potentials for eq. (1) and the Calogero-Moser system.

Theorem 1. Let $x(t)$ is a solution of the Calogero-Moser system ($g^2 = 1$) with the properties (5,6,7) then

$$v(x,y) = - \sum_{n=1}^{2M} (x - x_n(y))^{-2}, \text{ where } x,y \in \mathbb{R} \quad (9)$$

is a transparent potential for eq. (1); this potential is real, rational, decreasing at infinity function. Besides, if (8) is valid, then this potential is bounded. Using numbers ξ_n and η_n from (5), this potential can be written in the following form

$$v(x,y) = 2\partial_x^2 \ln \det \Theta, \quad (10)$$

where

$$\Theta_{nn} = -x + \xi_n y + \eta_n, \quad \Theta_{nm} = 2i / (\xi_n - \xi_m) \text{ for } n \neq m.$$

Theorem 1 arises from the Krichever and Fokas-Ablovitz results [4,5].

Hypothesis 2. Potentials (9, 10) are dense in the space of all continuous, real, decreasing at infinity, transparent potentials for eq. (1).

Consider now a problem about transparent potentials for the equation

$$(-\partial_x^2 - \partial_y^2 + v(x,y)) \psi = \psi, \quad (11)$$

where $v(x,y)$ is smooth rapidly decreasing potential. It is convenient for us

introduce new variables potential. It is convenient for us introduce new variables $z = x + iy$, $\bar{z} = x - iy$ and rewrite eq. (11) as

$$(-4\partial_z \partial_{\bar{z}} + v(z, \bar{z})) \psi = \psi .$$

The scattering amplitude $f(\lambda, \lambda')$, $\lambda \in \mathbb{C}$, $\lambda' \in \mathbb{C}$, $|\lambda| = |\lambda'| = 1$ for eq. (12) can be defined using the solutions of eq. (12) with the asymptotic behavior

$$\psi(z, \bar{z}, \lambda) = \exp \left[\frac{i}{2} (\lambda \bar{z} + z/\lambda) \right] + \frac{e^{i|z|}}{\sqrt{|z|}} f(\lambda, \frac{z}{|z|}) + \bar{O}\left(\frac{1}{\sqrt{|z|}}\right)$$

$|z| \rightarrow \infty$.

In order to give an analogy of the theorem 1 for eq. (12) , we have to introduce the lower analogy of the Calogero-Moser system (see [1])

$$\dot{x} = \frac{\partial \tilde{H}}{\partial p} , \quad \dot{p} = - \frac{\partial \tilde{H}}{\partial x} , \quad \tilde{H}_g = \text{tr } L^{-1} , \quad (13)$$

where L is defined by (4) .

Let $\chi_n(t)$ ($n = 1 \dots N$) be a solution of the lower Calogero-Moser system ($g^2 > 0$) . Write down its asymptotics for $t \rightarrow -\infty$ in the following form

$$\chi_n(t) \approx \zeta_n^{-2} t - \eta_n \quad \text{for } t \rightarrow -\infty . \quad (14)$$

We shall consider solutions of (13) ($g^2 > 0$) with additional symmetries

$$\zeta_{2j} = - \zeta_{2j-1} , \quad \eta_{2j} - \eta_{2j-1} = - 2i/\zeta_{2j-1} ; \quad (15)$$

$$\zeta_{4j-1} = - 1/\bar{\zeta}_{4j-3} , \quad \zeta_{4j} = - 1/\bar{\zeta}_{4j-2} ,$$

$$\bar{\eta}_{4j-3} = 2i\zeta_{4j-1} - \eta_{4j-1} \zeta_{4j-1}^2 , \quad \bar{\eta}_{4j-2} = 2i\zeta_{4j} - \eta_{4j} \zeta_{4j}^2 ,$$

where $j = 1 \dots M$, $N = 4M$. We assume also that

$$\zeta_n \neq \zeta_m \quad \text{if } n \neq m \quad (16)$$

Let us reproduce one Grinevich result from [6] .

If numbers ζ_j , η_j , $j = 1 \dots 4M$ have the properties (15,16) , then

$$v(z, \bar{z}) = - 8 \partial_z \partial_{\bar{z}} \ln \det \Theta , \quad (17)$$

where $\Theta_{nn} = \bar{z} - \zeta_n^2 z + \eta_n$, $\Theta_{nm} = 2/i(\zeta_n - \zeta_m)$ for $n \neq m$, is a real, decreasing at infinity, transparent potential eq. (12). Besides, it is stated in [6], that this potential is bounded.

Now it is time to present analogies of the theorem 1 and the hypothesis 2 for the case of eq. (12).

Theorem 2. If $\chi(t)$ is a solution to the hamiltonian system (13) ($g = 2$) with the properties (14,15,16) then

$$v(z, \bar{z}) = - 8 \sum_{n=1}^{4M} \frac{(d/dz) \chi_n(z)}{(\bar{z} + \chi_n(z))^2} \quad (18)$$

is a transparent potential for eq. (12) .

Using numbers ζ_n and η_n from (14), this potential can be written in the form (17) .

Hypothesis 3. Potentials (17,18) are dense in the space of all continuous, real, decreasing at infinity, transparent potentials for eq. (12) .

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UNIVERSITÉ PARIS VI
ANALYSE COMPLEXE
ET GÉOMÉTRIE
Unité Associée au C.N.R.S. (U.A. 213)
AILE 45-46 - Boîte 172 - 5^e ÉTAGE
4, PLACE JUSSIEU
75252 PARIS - CEDEX 05
TÉL. (1) 44.27.53.44 - FAX (1) 44.27.53.45