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An Introduction to symplectic Topology

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The aim of this talk is to present some results and questions in symplectic topology obtained in the last years. We shall also sketch some applications to eigenvalues of differential operators, mostly from our very partial understanding of [**F**].

Symplectic topology is the qualitative study of symplectic manifolds, their submanifolds and the symplectic maps between them. The author wishes to thank the organizers of ther meeting for inviting him to such a pleasant and instructive meeting.

1. Some basic facts of symplectic geometry.

A symplectic manifold is a pair (M,ω) where ω is a 2 form on M, which is:
-closed : $d\omega = 0$
-nondegenerate i.e. $\forall x\ \omega(x,y) = 0 \Rightarrow y = 0$
This implies in particular that M is $2n$ - dimensional. Classical examples are:

(1) $(\mathbb{R}^{2n},\omega_0)$ where $\omega_0 = \sum_{i=1}^n dx^i \wedge dy_i$. Note that since ω_0 is invariant by the standard $\mathbb{Z}^{2n}$ action this induces a symplectic structure on T^{2n}.

(2) If M an n-dimensional manifold, $(T^*M, dp \wedge dq)$ where $dp \wedge dq$ is the two form given locally by $dp \wedge dq = \sum_{i=1}^n dp^i \wedge dq_i$ where (p^i) are local coorinates dual to the (q_i). It is easy to check that our definition is indipendent of the choice of the local coordinates.

It is a classical result that in contrast to Riemannian geometry there is no local symplectic invariant. Indeed, we have:

DARBOUX'S THEOREM. *Two symplectic manifolds of the same dimension are locally symplectomorphic. Thus a symplectic manifold is locally equivalent to* $(\mathbb{R}^{2n},\omega_0)$

The proof is essentially an application of Moser's lemma, which allows one to extend the above theorem as follows. First remember that a Lagrange submanifold is an n dimensional submanifold of (M,ω) on which the symplectic form induces the zero form. Then, we have:

WEINSTEIN'S THEOREM. *If L is a Lagrange sumbanifold of (M,ω), then, there is a symplectic diffeomorphism from a neighbourhood of the zero section in $(T^*L, dp \wedge dq)$ into (M,ω) which sends the zero section to L.*

Finally we are also interested in symplectic diffeomorphisms, these are diffeomorphisms ψ satisfying $\psi^*\omega = \omega$. We denote by $\mathrm{Diff}_s(M,\omega)$ the group of such diffeomorphisms. The group $\mathrm{Diff}_s(M,\omega)$ is quite large. Indeed if $H(t,x)$ is a function on $\mathbb{R} \times M$, and X_H the time dependent vector field on M defined by the relation

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$\omega(X_H,\xi) = dH(t,x)\xi \ \forall \xi \in T_xM$ (here dH is the space derivative), then provided the flow ψ_t of this vector field is defined (this is always the case if M is compact), it lies in the group $\mathrm{Diff}_h(M,\omega)$ of hamiltonian diffeomorphisms. This is a subgroup of $\mathrm{Diff}_s(M,\omega)$. On T^*M the flow is written in local coordinates as the familiar

$$\dot{q} = \frac{\partial H}{\partial p} \quad \dot{p} = -\frac{\partial H}{\partial q}$$

Note that in $\mathbf{R}^{2n}$ a symplectic diffeomorphisms coincides on an arbitrary large compact set with the time one maps of a Hamiltonian flow

The group $\mathrm{Diff}_s(M,\omega)$ is obviously a subgroup of $\mathrm{Diff}_v(M,\omega^n)$, the group of volume preserving diffeomorphisms. It has been a long standing question as to whether symplectic maps were fundamentally different from volume preserving ones. Mathematically this is a question about the closure of $Diff_s$ in $Diff_{vol}$ (the space of volume preserving diffeomorphisms) for the C^0 topology. That the second space is closed follows from the remark that volume preserving is equivalent to measure preserving, which is a C^0 notion (ψ is measure preserving is expressed by the fact that for any open set, U, we have $\mu(\psi(U)) = \mu(U)$, and for each U the map $\psi \to \mu(\psi(U))$ is continuous for the C^0 topology). A related question is what obstructions are there, other than the volume, to symplectically embed an open set in an other one. We shall see in **2.A** and **2.B** how this was answered by Gromov and Eliashberg.

2. Three theorems in symplectic topology.

A. Fixed point theorems

The first result in modern symplectic topology is the Conley-Zehnder fixed point theorem:

THEOREM 2.1. *Let ψ_t be a Hamiltonian flow on (T^{2n},ω_0), then ψ_1 has at least $2n+1$ fixed points.*

The above theorem can be seen as a generalization of the Poincaré -Birkhoff theorem on diffeomorphisms of the annulus (it is an exercise for the reader to show that for $n=2$ Poincaré -Birkhoff's theorem follows from the above result). The Conley-Zehnder theorem has been further generalized to other symplectic manifolds, mostly by M. Gromov and A.Floer (cf. [**G 1**],[**G 2**], [**Fl 1-4**]). We see at once that such a theorem implies that $\mathrm{Diff}_h(T^{2n},\omega_0)$ is not dense for the C^0 topology in $\mathrm{Diff}_v(M,\omega^n)$, since there are volume preserving maps of T^{2n} without fixed points (this remark is due to M. Herman). Other results assert the existence of periodic orbits for Hamiltonian systems. Let us remind the reader that if H is time independent, then the Hamiltonian flow preserves the levels of H. Thus it makes sense to look for periodic orbits on a given level set of H. We have ([**V 6**]:

THEOREM 2.2. *Let $c \in \mathbf{R}$, and $\epsilon > 0$. Suppose that $H^{-1}([c-\varepsilon, c+\varepsilon])$ is compact. Then it contains at least one periodic orbit of the flow.*

B. Non embeddability and rigidity properties

In [G 1] Gromov proved that obstructions to symplectic embeddings are stronger than the volume restrictions:

THEOREM 2.3. *If $B^{2n}(r)$ is a ball of radius r, then there is no embedding of $B^{2n}(r)$ into $B^2(r') \times \mathbf{R}^{2n-2}$ unless $r \leq r'$*

Eliashberg proved that this implies:

THEOREM 2.4. *$Diff_s$ is closed for the C^0 topology.*

REMARK. *This becomes false if we replace the C^0 topology, by the L^p topology. One may approximate any volume preserving map by a symplectic map in the L^p topology. Also, it is possible to embed symplectically $B^{2n}(r)$ into $B^2(r') \times \mathbf{R}^{2n-2}$ up to a set of arbitrary small measure. The idea is to replace the ball by a cube with sides one, that we cut in smaller squares plus "some fat" (see figure 1). Then each of the smaller squares may be embedded symplectically into $B^2(r') \times \mathbf{R}^{2n-2}$, with $r' < r$ and it is easy to extend the map defined on the small cubes to a symplectic map defined on the large cube. A refinement of this ([**H**]), following an idea by Katok , may be used to approximate any volume preserving map in the L^p topology. For instance one may try to approximate a volume preserving diffeomorphism of the large cube which restricts to some permutation of the small cubes. Any volume preserving map may in fact be approximated by such a map (provided the small cubes are small enough, and the "fat part" has small measure). It is thus enough to realize a transposition of two cubes inside a parallepiped. But this may be reduced to switching the two halves of a ball, which can be realized by a unitary rotation, obviously symplectic.*

Gromov proved also some "packing inequalities": If one tries to squeeze symplectically two balls of radius r in a ball of radius R, then we see from volume considerations that we must have that $2r^n \leq R^n$. Another theorem by Gromov tells us that this is not enough:

THEOREM 2.5. *If $B^{2n}(R)$ contains two disjoint symplectically embeded balls of radius r_1 and r_2, then we must have $r_1^2 + r_2^2 \leq R^2$. If $n = 2$, and $B^4(R)$ contains $d(d+3)/2$ embbedded balls of radius r_i, then we have $\sum_{i=1}^{d(d+3)/2} r_i^2 \leq d \cdot R^2$.*

C. Lagrange submanifolds

We finally give some properties of Lagrange submanifolds. Lagrange submanifolds have half the dimension of the ambient manifold . So it is not clear whether they should still exhibit some "rigidity" properties. We shall see that it is indeed the case.

The first result we will consider is the folowing

THEOREM 2.6. *Let $j : T^n \to \mathbf{R}^{2n}$ be a Lagrange embedding. Then there is a loop $\gamma : S^1 \to T^n$ such that:*

(i) $\int_{S^1} \gamma^*(pdq) > 0$

(ii) $\langle \mu(j), \gamma \rangle \in [2, n+1]$, *where $\mu(j)$ denotes the Maslov class of the embedding.*

The number $\langle \mu(j), \gamma \rangle$ is essentially a measure of how much the tangent space of $j(T^n)$ winds as we move along γ.

This theorem tells us about a different type of symplectic particularity: some algebraic invariants of a Lagrange submanifold may not be arbitrarily prescribed. This has several consequences.

For instance we saw before that a ball may not be squeezed in an arbitrarily small cylinder. Now this is also true for a Lagrange torus, even though it is a very thin subset of $\mathbf{R}^{2n}$. Indeed we have:

PROPOSITION 2.7. *Let $j : T^n \to \mathbf{R}^{2n}$ be a Lagrange embedding. Then there is a number $r(j)$ such if for some $\psi \in Diff_s(\mathbf{R}^{2n})$, we have $\psi(j(T^n)) \in B^2(r) \times \mathbf{R}^{2n-2}$, then $r > r(j)$*

Note that any set of dimension at most $n-1$ may be squeezed in an arbitrarily small cylinder, thus n is really the critical dimension. We also point out that being lagrangian plays a crucial role in the above argument (for instance a symplectic subset of codimension 4 may be arbitrarily squezed).

3. Two methods of proof.

Gromov's original proofs are based on the study of pseudo holomorphic curves in symplectic manifolds and rest on the following principles. First we define in a symplectic manifold (M,ω) an almost complex structure as an automorphism J of TM such that J defines a complex structure on each fiber (i.e. $J^2 = -I$) and $\omega(\xi, J\xi) = g(\xi, xi)$ where g is some Riemannian metric on M. Such a J always exists and the set of all J is contractible. Then pseudo holomorphic curves are maps u from a Riemann surface (Σ, J_0) to (M, J) commuting with the complex structures; that is $Du \circ J_0 = J \circ Du$. This is equivalent a Cauchy-Riemann equation, is thus elliptic, and not too difficult to study. (In fact it is very close to the harmonic map equation studied by Sacks and Uhlenbeck, and exhibits the same blowing-up phenomenon). We denote by $\mathcal{H}_x(U)$ the set of pseudo holomorphic curves through x, which are closed in U. Then Gromov defines the width of an open set as

$$w(U) = \sup_J \inf_{\Sigma\in\mathcal{H}_x(U)} \int_{\Sigma\cap U} \omega$$

It is easy to check that the above definition is independent of the choice of x, and that it is a monotone symplectic invariant. Then the use of Lelong's isoperimetric inequality (see [**Le**]) allows one to see that using the standard complex structure in $\mathbf{R}^{2n} \simeq \mathbf{C}^n$ that $w(B^2(r) \times \mathbf{R}^{2n-2}) \geq w(B^{2n}(r)) \geq \pi r^2$. In fact a subtle existence theorem for holomorphic curves shows that the above inequality is in fact an equality. Because the width is monotone, (i.e. if $U \subset V$ then $w(U) \leq w(U)$), this clearly implies theorem 2.3.

The other approach is through the concept of generating functions. It was motivated by the approach of Ekeland and Hofer in [**E -H 1**], [**E -H 2**], and by results in [**Si 1**], as well as earlier work by the author [**V 1**].

We shall shortly review this and refer for more details to [**V 5**]. We are first going to sketch the proof of the Conley-Zehnder theorem. Let L be a Lagrange submanifold in T^*N and $S : N \times \mathbf{R}^k \to \mathbf{R}$ be a C^∞ function. We shall say that S is a generating function for L if $L = \{(x, \frac{\partial S}{\partial x}(x,\xi) \mid \frac{\partial S}{\partial \xi}(x,\xi) = 0\}$. Let us remark that if S is a function on $N \times \mathbf{R}^k$, such that the map $(x,\xi) \to, \frac{\partial S}{\partial \xi}(x,\xi)$ has 0 as a regular value , then the submanifold L given by the above equations is an immersed Lagrange submanifold. The simplest case is $k = 0$, and then L is the graph of dS. We are in fact interested in a special class of generating functions, those which coincide with a nondegenerate quadratic form at infinity. This is because we are looking for points in $L \cap N$, (we identify N to the zero section in T^*N), and a critical point of S corresponds to such a point. The first result we need is:

PROPOSITION. *If $\phi \in Diff_h(T^*N)$ then $\phi(N)$ has a generating function quadratic at infinity (we shall write g.f.q.i. for short)* .

Now a special feature of a g.f.q.i. is that, like functions on N, they must have critical points. In fact if $\alpha_1, \dots, \alpha_l$ are closed forms of nonzero degree on N, such that $\alpha_1 \wedge \cdots \wedge \alpha_l$ is not exact, then any g.f.q.i. on $N \times \mathbf{R}^k$ has at least $l+1$ critical points. The largest l such that forms $\alpha_1, \dots, \alpha_l$ with the above property exists is called the cup lenght of N, (abbreviated as c.l.). For instance $\mathrm{c.l.}(T^n) = n+1$, as we see by taking for α_j, the coordinate forms $d\theta_j = p_j^*(d\theta)$ ($p_j : T^n = (S^1)^n \to S^1$ is the projection on the j-th factor).

We thus have:

PROPOSITION. *If $\phi \in Diff_h(T^*N)$ then $\phi(N) \cap N$ contains at least* $\mathrm{c.l.}(N)$ *points.*

We now show how this implies the Conley-Zehnder theorem. Let $\overline{\mathbf{R}^{2n}}$ be the symplectic manifold $(\mathbf{R}^{2n}, -\omega_0)$. Then if $\phi \in Diff_h(T^*T^n)$ and $\psi \in Diff_h(\mathbf{R}^{2n})$ is the lift of ϕ to $\mathbf{R}^{2n}$ ($(\mathbf{R}^{2n}, \omega_0)$ is the universal cover of $(T^*T^n, dp \wedge dq)$), then $\Gamma_\psi = \{(x, \phi(x)) \mid x \in \mathbf{R}^{2n}$, the graph of ψ, is a Lagrange submanifold of $\overline{\mathbf{R}^{2n}} \times \mathbf{R}^{2n}$ (compute the induced symplectic form!). Now, $\overline{\mathbf{R}^{2n}} \times \mathbf{R}^{2n}$ is symplectically isomorphic to $T^*\Delta_{\mathbf{R}^{2n}}$, where $\Delta_{\mathbf{R}^{2n}}$ is the diagonal $\overline{\mathbf{R}^{2n}} \times \mathbf{R}^{2n}$, the isomorphism being given by

$$(q,p,Q,P) \to (x,y,\xi,\eta) = (\frac{q+Q}{2}, \frac{p+P}{2}, P-p, q-Q)$$

Now Γ_ψ is a Lagrange submanifold hamiltonianly isotopic to the zero section (the isotopy is given by $id \times \psi_t$, and the points of $\Gamma_\psi \cap \Delta_{\mathbf{R}^{2n}}$ correspond to fixed points of ψ, hence to fixed points of ϕ. Moreover Γ_ψ is invariant by the $\mathbf{Z}^{2n}$ symmetry of $\overline{\mathbf{R}^{2n}} \times \mathbf{R}^{2n}$ given by $\nu * (z, Z) = (z+\nu, Z+\nu)$ (where $z = (q,p)$, $Z = (Q,P)$), that is $\nu * (x,y,\xi,\eta) = (x+\nu, y+\nu, \xi, \eta)$. It is easy to see that the quotient space of $T^*\Delta_{\mathbf{R}^{2n}}$ by this action is T^*T^{2n},and Γ_ψ descends to a Lagrange submanifold $\tilde{\Gamma}_\psi$ of T^*T^{2n}, hamiltonianly isotopic to the zero section. Again the points of $\tilde{\Gamma}_\psi \cap T^{2n}$ correspond to fixed points of ϕ. According to our proposition, there are at least $2n+1$ such points, this proves the theorem of Conley and Zehnder.

Now, the same idea may be applied to construct symplectic invariants associated to ϕ an element in $Diff_h^0(\mathbf{R}^{2n})$ the set of compact supported maps of $Diff_h(\mathbf{R}^{2n})$. We see again that the graph $G(\phi)$ of ϕ is a Lagrange submanifold in $\overline{\mathbf{R}^{2n}} \times \mathbf{R}^{2n} = T^*\Delta_{\mathbf{R}^{2n}}$. Moreover, $G(\phi)$ coincides with Δ outside a compact set. Thus we may simultaneously compactify $G(\phi)$ and the base of $T^*\Delta_{\mathbf{R}^{2n}}$ to get $\bar{G}(\phi)$ a Lagrange submanifold of T^*S^{2n}. As before, $\bar{G}(\phi)$ has a g.f.q.i. which has certain critical points. In fact a g.f.q.i. S on

S^{2n} has at least two critical levels, that we shall denote by $c_+(S)$ and $c_-(S)$. It looks as if these critical values really depend on S, and not on L (L does not uniquely define S). But one can exactly describe the set of all g.f.q.i. of L, and check that, provided $c_+(S)$ and $c_-(S)$ are properly defined, they really only depend on L, that is here on ϕ. We write them as $c_+(\phi)$ and $c_-(\phi)$. These invariants have several properties, summarized by the following:

PROPOSITION.
(i) $c_(\phi) \leq 0 \leq c_+(\phi)$
(ii) $c_(\phi) = c_+(\phi) = 0$ *if and only if* $\phi = Id$
(iii) *There exists x_+, a fixed point of ϕ such that* $c_+(\phi) = \int_{\phi_t(x_+)} pdq - H\,dt$, *and there is a point x_- such that a similar statement holds for $c_-(\phi)$.*
(iv) $c_+(\psi\phi\psi^{-1}) = c_+(\phi)$, *and the same holds for c_-.*

We may now construct a symplectic invariant by setting:

DEFINITION. *Let U be an open set in $\mathbf{R}^{2n}$, we define $c(U)$ as* $c(U) = \sup\{c_+(\phi) \mid \phi = \phi_1$, *where ϕ_t is the flow of a Hamiltonian supported in* $U\}$

Now, because of *(iv)* of the preceding proposition, $c(U)$ is a symplectic invariant. By its definition we see right away that it is monotone . Of course the difficult fact is to prove that $c(U)$ is bounded for U bounded and to compute it explicitly, for instance if U is a ball or a cylinder. In fact Gromov's theorem follows from the fact that $c(B^{2n}(r)) = c(B^2(r)) \times \mathbf{R}^{2n-2}$.

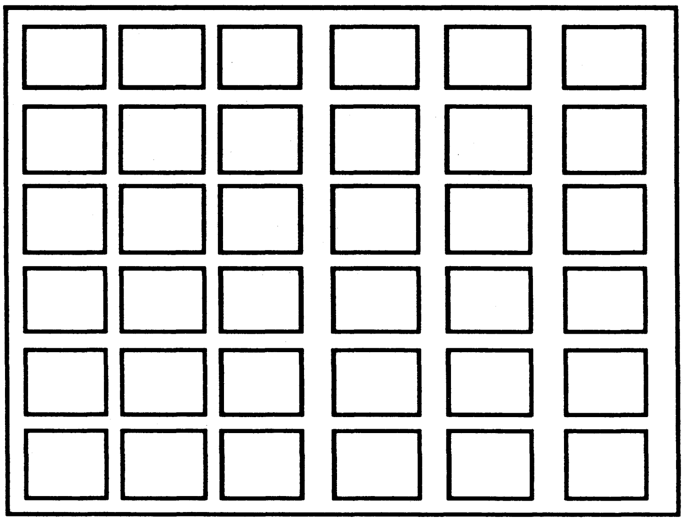

FIGURE 1

4. Small eigenvalues of elliptic operators.

Even though, there is definitely a relationship between symplectic topology, we failed to convince ourselves that any significantly new result in estimating eigenvalues of eliptic operators follows from the preceding sections. For the moment symplectic topology and the theory of elliptic operators are having an unsuccessfull love story. However we thought it worthwile to summarize one of the connections between these two fields.

We will mainly consider the operator $\mathrm{P}(x, D) : \psi \to (-\Delta + V(x))\psi$. We remind the reader that the number, $N(\mathrm{P}, \lambda)$ of eigenvalues of this operator less than λ is bounded from above by a constant times $V(\mathrm{P}, \lambda) =$ volume of the set$\{(x,\xi) \in T^*\mathbf{R}^n \mid (P(x,\xi) \leq \lambda\}$ i.e. we have $N(\mathrm{P}, \lambda) \leq CV(\mathrm{P}, \lambda)$ (C depends only on n). With reasonable assumptions, this inequality yields an accurate estimate as λ goes to infinity. On the other hand it is often very important to have an approximate value of $N(\mathrm{P}, \lambda)$ for small values of λ, and then, the value $V(P, \lambda)$ is often grossly inaccurate (we refer the reader to the example at the end of this section as well as to more examples in **[F1]**).

A more accurate estimate is provided by Fefferman (cf.**[F 1]**,**[F-P]**), it is based on the uncertainty principle. This tells us that if u is a function localized in $B(x_0, \rho)$ (i.e. $|u(x)| \leq 1/2\|u\|_{L^2}$ for x outside $B(x_0, \rho)$), and its Fourier transform, $\hat{u}$ is localized in $B(\xi_0, \hat{\rho})$, then we have $\hat{\rho} \times \rho \geq 2\pi$.

Moreover the equality is realized by $u^{\rho}_{q,p}(x) = e^{i(p,x)} e^{-\frac{|x-q|^2}{2\rho}}$.

Now the L^2 scalar product $< P(x,D)u, u >$ may be written as

$$\int_{\mathbf{R}^{2n}} P(x,\xi)u(x)\hat{u}(\xi)dxd\xi$$

Since $N(P, \lambda)$ is the maximal dimension of a subspace of $L^2(\mathbf{R}^n)$ on which $< P(x,D)u, u > \leq \lambda\|u\|^2$ and if we assume that $P(x,\xi) \leq \lambda$ on $\mathbf{B}(x_0, \xi_0, \rho) \overset{\text{def}}{=} B(x_0, \rho) \times B(\xi_0, \frac{2\pi}{\rho})$ we may write, for $u = u^{\rho}_{q,p}$:

$$\int_{\mathbf{B}(x_0,\xi_0,\rho)} P(x,\xi)u(x)\hat{u}(\xi)\,dxd\xi + \int_{\mathbf{R}^{2n}-\mathbf{B}(x_0,\xi_0,\rho)} P(x,\xi)u(x)\hat{u}(\xi)\,dxd\xi \leq$$
$$\frac{\lambda}{4}\|u\|^2 + \int_{\mathbf{R}^{2n}-\mathbf{B}(x_0,\xi_0,\rho)} P(x,\xi)u(x)\hat{u}(\xi)dxd\xi$$

Provided $P(x,\xi)$ does not grow too fast outside the box, the second term will be bounded by a constant times the first one, so we already see that if $\{(x,\xi) \mid P(x,\xi) \leq \lambda\}$ contains a box $\mathbf{B}(x_0, \xi_0, \rho)$ with, then $N(P,\lambda) \geq 1$ i.e. the smallest eigenvalue of $P(x,D)$ is greater or equal to $C\lambda$ where C only depends on the growth of the symbol. In general to show that $N(P,\lambda) \geq k$ it is not enough to find k disjoint testing boxes we must also show that the functions $u_{q,p}$ are linearly independent. We refer to **[F]** for a discussion of this point. The symplectic transformation come into play in the following manner. Let U be a unitary operator of L^2. Then the eigenvalues of $P(x,D)$ and those of UPU^{-1} are obviously the same. The idea is to apply Egorov's theorem to get from a symplectic map ψ (satisfying certain growth conditions) a unitary pseudodifferential operator U_ψ

such that the symbol of $U_\psi P U_\psi^{-1}$ is equal to $\tilde{P}(x,\xi) = P(\psi(x,\xi))$ + lower order terms. Then one may apply the above argument that is, if the set $\{(x,\xi) \mid \tilde{P}(x,\xi) \leq \lambda\} = \psi(\{(x,\xi) \mid P(x,\xi) \leq \lambda\})$ contains a box $\mathbf{B}(x_0,\xi_0,\rho)$, or else if $\{(x,\xi) \mid P(x,\xi) \leq \lambda\}$ contains $\psi^{-1}(\mathbf{B}(x_0,\xi_0,\rho))$, then $N(P,\lambda) \geq 1$. However, the set of admissible ψ is much smaller than $Diff_s(\mathbf{R}^{2n})$, because of all the growth conditions we had to impose. So the first eigenvalue of P may be estimated by the smallest λ such that $\{(x,\xi) \mid P(x,\xi) \leq \lambda\}$ contains the symplectic image of a box (note that any two boxes, $\mathbf{B}(x_0,\xi_0,\rho)$, are symplectomorphic), but this is still not too good. Taking into account the estimates that have to be satisfied by the symplectomorphisms usually gives a much better result.

EXAMPLE. *(from [**F 1**]) Let $P(x,D)u = -(\Delta + E\chi_I(x))u$, where $I = \{(x_1,\ldots,x_n) \mid |x_i| \leq \delta_i\}$ and χ_I is the characteristic function of I. We assume $\delta_1 \leq \cdots \leq \delta_n$. The volume of the set $\{(x,\xi) \mid \xi^2 - E\chi_I(x) \leq 0\}$ is $C_n\delta_1\cdots\delta_n \times E^{n/2}$ (C_n is the volume of the unit ball in $\mathbf{R}^n$). It is contained in a cylinder of sectional area $E^{1/2}\,\delta_1$ thus it cannot contain a symplectic box unless $E^{1/2}\,\delta_1 \geq 2\pi$, that is $E \geq \frac{4\pi^2}{\delta_1^2}$. On the other hand it contains a symplectic box as soon as $E \geq \frac{n^2}{\delta_1^2}$. Thus, if this relation is satisfied $-(\Delta + E\chi_I(x))$ should have negative eigenvalues. In fact according to [**F**], one may prove that a necessary and sufficient condition for this operator to have a negative eigenvalue is that $E \geq C(\delta_1,\delta_2,\delta_3)$ with $\frac{K_1}{\delta_1\delta_2(\delta_3/\delta_1)^\epsilon} \leq C(\delta_1,\delta_2,\delta_3) \leq \frac{K_2}{\delta_1\delta_2}$. Indeed we have $\frac{1}{(C_n\delta_1\cdots\delta_n^{n/2})} \leq C(\delta_1,\delta_2,\delta_3) \leq \frac{n^2}{\delta_1^2}$, the "volume estimate" is very bad, the "symplectic estimate" is better but still far from the exact value.*

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