JOURNÉES ÉQUATIONS AUX DÉRIVÉES PARTIELLES

ANTONIO SA BARRETO

Evolution of semilinear conormal waves

Journées Équations aux dérivées partielles (1991), p. 1-18
<http://www.numdam.org/item?id=JEDP_1991____A12_0>

Evolution Of Semilinear Conormal Waves

ANTÔNIO SÁ BARRETO

1 Introduction

Let $\Omega \subset \mathbf{R}^3$ be an open subset and let P be a second order strictly hyperbolic differential operator in Ω with smooth coefficients. Let $t \in C^\infty(\Omega)$ be a time function for P and define

$$\Omega^\pm = \Omega \cap \{\pm t > 0\}. \tag{1.1}$$

Assume that Ω is a domain of dependence of Ω^-. Let f be a smooth function of its arguments and suppose $u, Du \in L^\infty_{loc}(\Omega)$ satisfies

$$Pu = f(z, u, Du); \quad z \in \Omega. \tag{1.2}$$

The general question on propagation of singularities of solutions of (1.1) is how do singularities of u in Ω^- influence singularities of u in Ω. We shall concentrate in the study of some geometric singularities called conormal and the first example is conormality to a smooth hypersurface. Thus let $S \subset \Omega$ be a smooth hypersurface which is characteristic for P, let $\mathcal{V}_S$ be the Lie algebra of smooth vector fields tangent to S and denote

$$I_k L^2_{loc}(\Omega, \mathcal{V}_S) = \{u \in L^2_{loc}(\Omega) : \mathcal{V}^j_S u \subset L^2_{loc}(\Omega), \;\; j \leq k\}. \tag{1.3}$$

Observe that if $u \in I_\infty L^2_c(\Omega, \mathcal{V}_S)$, then u is smooth away from S. In fact one can easily show that in this case the wavefront set of u is contained in the conormal bundle to S.

Theorem 1.1 (Bony, [4]) *Let $u, Du \in H^s_{loc}(\Omega)$, $s > \frac{3}{2}$, satisfy* (1.2). *If $u, Du \in I_k L^2_{loc}(\Omega^-, \mathcal{V}_S)$, then $u, Du \in I_k L^2_{loc}(\Omega, \mathcal{V}_S)$.*

This result shows that as long as S is smooth u remains conormal to it, but in general characteristic hypersurfaces of P can have rather complicated singulariries. In this talk we shall describe the results of [16] and [17] concerning the propagation of conormal singularities for solutions of (1.2) along a hypersurface Σ with either a cusp or a swallowtail singularity. These are in some sense, see [2], the only cases where the singularities are stable under small pertubations. These problems have been also studied by M. Beals [3] and R. Melrose [9], in the case of the cusp and G. Lebeau, [6], [7] and J-M.

Delort [5] in the case of the swallowtail with the hypotheses that P has real analytic coefficients and the regular part of Σ is real analytic.

Before stating our results we have to introduce some notation. Let $\mathcal{W}$ be a Lie algebra and C^∞ module of smooth vector fields on a manifold with corners X and let μ be a smooth measure on X. The space of iteratively regular distributions with respect to $\mathcal{W}$ is then defined as

$$I_k L^2_{\mu,c}(X,\mathcal{W}) = \{u \subset L^2_{\mu,c}(X); \mathcal{W}^j u \in L^2_{\mu,c}(X), \ \ j \leq k\}. \tag{1.4}$$

2 The Cusp

Let G be a hypersurface with a cusp singularity at a line L, i.e there are local coordinates near $q \in L$ such that

$$G = \{(x,y,z) \in \Omega : y^3 = x^2\}, \ \ L = \{(x,y,z) : x = y = 0\}. \tag{2.1}$$

Assume that the smooth part of G is characteristic for P. Let $\mathcal{V}_G$ be Lie algebra of smooth vector fields tangent to G. It is easy to show that the Lie algebra $\mathcal{V}_G$ is characteristic complete, i.e

$$[P,\mathcal{V}_G] \subset \Psi^0(\Omega)\cdot P + \Psi^1(\Omega)\cdot \mathcal{V}_G + \Psi^1(\Omega). \tag{2.2}$$

Where $\Psi^j(\Omega)$ denotes the space of properly supported pseudodifferential operators of order j in Ω. Then by commutator methods, see [4], one obtains

Theorem 2.1 *Let $u, Du \in H^s_{loc}(\Omega)$, $s > \frac{3}{2}$, satisfy equation* (1.2). *If $u, Du \in I_k L^2_{loc}(\Omega^-, \mathcal{V}_G)$, then $u, Du \in I_k L^2_{loc}(\Omega, \mathcal{V}_G)$.*

Next we recall the spaces of marked Lagrangian distributions introduced by R. Melrose in [9]. Let $\Lambda_G = \operatorname{clos}[N^*(G\setminus L)]$, Λ_G is a smooth conic Lagrangian submanifold of $T^*\mathbf{R}^3$. Let $\Lambda_L = N^*L$ and

$$\mathcal{M}_1(G) = \{A \in \Psi^1(\Omega) : a = \sigma_1(A) = 0 \ \text{ at } \ \Lambda_G, \\ H_a \ \text{ is tangent to } \ \Lambda_G \cap \Lambda_L\}. \tag{2.3}$$

$$\mathcal{M}_1(L) = \{A \in \Psi^1(\Omega) : a = \sigma_1(A) = 0 \ \text{ at } \ \Lambda_L, \\ H_a \ \text{ is tangent to } \ \Lambda_G \cap \Lambda_L\}. \tag{2.4}$$

Let

$$J^{G,m}_k(\Omega) = I_k L^2_{loc}(\Omega, \mathcal{M}_1(G)) + I_k L^2_{loc}(\Omega, \mathcal{M}_1(L)). \tag{2.5}$$

In [9] Melrose proves that

$$J^{G,m}_k \subsetneq I_k L^2_{loc}(\Omega, \mathcal{V}_G) \tag{2.6}$$

and

Theorem 2.2 (Melrose, [9]) *Let $u, Du \in H^s_{loc}(\Omega)$, $s > \frac{3}{2}$, satisfy equation (1.2). If $u, Du \in J^{G,m}_k(\Omega^-)$, then $u, Du \in J^{G,m}_k(\Omega)$.*

Finally we introduce a third space of distributions associated to the cusp. Observe that in local coordinates where (2.1) holds one finds that G is invariant under the $\mathbf{R}^+$ action

$$m_s^{3-2}(x,y) = (s^3 x, s^2 y). \tag{2.7}$$

This leads to the definition quasi-homogeneous polar coordinates, thus consider the non-round circle

$$S^1_{3-2} = \{(\omega_1, \omega_2) \in \mathbf{R}^2 : \omega_1^4 + \omega_2^6 = 1\} \tag{2.8}$$

and the manifold with boundary

$$X_{3-2} = S^1_{3-2} \times [0,\infty) \times \mathbf{R}. \tag{2.9}$$

Then define the blow-down map

$$\beta_{3-2} : X_{3-2} \longrightarrow \mathbf{R}^3, \quad \beta_{3-2}(\omega, r, z) = (r^3\omega_1, r^2\omega_2, z). \tag{2.10}$$

Let $\mathcal{W}_G$ be the Lie algebra of smooth vector fields in X_{3-2} which are tangent to ∂X_{3-2} and to $G^{(1)} = \mathrm{clos}\beta^{-1}_{3-2}[G \setminus L]$. Let μ be the pull back of the Lebesgue measure by the map β_{3-2}. Then one defines

$$J^G_k(\Omega) = \{u \in L^2_{loc}(\Omega) : \beta^*_{3-2}u \in I_k L^2_c(X_{3-2}, \mathcal{W}_G)\}. \tag{2.11}$$

One can easily show that the space $J^G_k(\Omega)$ does not depend on the choice of coordinates such that (2.1) holds. Then see [16], one can show that if $\mathcal{W}^1_G$ is the Lie algebra of smooth vector fields in X_{3-2} that are tangent to ∂X_{3-2} to $G^{(1)}$ and to the lines $\{\omega_1 = 0, r = 0\}$, $\{\omega_2 = 0, r = 0\}$, then the blow down map β_{3-2} induces an isomorphism

$$\beta^*_{3-2} : J^{G,m}_k(\Omega) \leftrightarrow I_k L^2_c(X_{3-2}, \mathcal{W}^1_G). \tag{2.12}$$

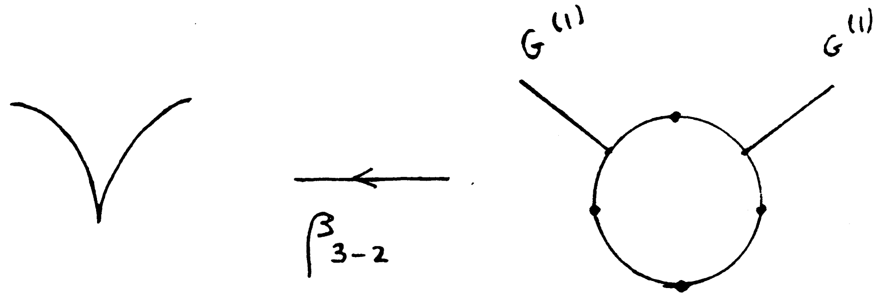
Similarly if $\mathcal{W}^0_G$ is the Lie algebra of smooth vector fields that are tangent to $G^{(1)}$ and vanish on ∂X_{3-2}, then

$$\beta^*_{3-2} : I_k L^2_c(\Omega, \mathcal{V}_G) \leftrightarrow I_k L^2_c(X_{3-2}, \mathcal{W}^0_G). \tag{2.13}$$

In particular one obtains from (2.12) and (2.13) that

$$J^G_k(\Omega) \subsetneq J^{G,m}_k \subsetneq I_k L^2_{loc}(\Omega, \mathcal{V}_G). \tag{2.14}$$

Figure 1:



The main difficulty in proving a propagation theorem for $J_k^G(\Omega)$ is that this space is not known to have a microlocal characterization. One of the main results of [16] is the following elliptic regularity type of theorem

Theorem 2.3 *If $u, Du \in H^s_{loc}(\Omega) \cap I_k L^2_{loc}(\Omega, G)$ satisfies equation* (1.2), *then $u, Du \in J_k^G(\Omega)$.*

Theorem 2.3 illustrates an important idea that will be used in the proof of Theorem 7.1. One first proves a propagation theorem for a bigger space which has a microlocal characterization and then uses the equation to show that the solution is actually in the smaller space.

3 The Swallowtail

Since the results we wish to prove are local we shall assume that $\Omega \subset \mathbf{R}^3$ is a sufficiently small neighborhood of $O = (0,0,0)$. Let $\Sigma \subset \Omega$ be a hypersurface with a swallowtail singularity at $O \in \Omega$, i.e there are smooth coordinates (x, y, z) in Ω such that

$$\Sigma = \{(x,y,z) : \delta(\lambda) = \lambda^4 + z\lambda^2 + y\lambda + x = 0, \quad \text{has a double real root}\}. \tag{3.1}$$

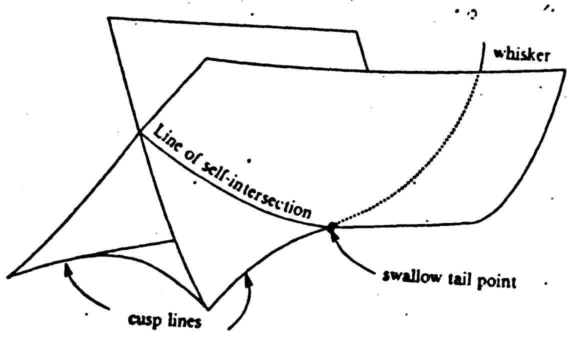
Σ has a cusp singularity at

$$L = \{(x,y,z) : \quad x = -\frac{z^2}{12}, \ y^2 = (-\frac{2}{3}z)^3\} \tag{3.2}$$

and a self-intersection at

$$H = \{(x,y,z) : \quad y = 0, \ x = -\frac{z^2}{4}, \ z \leq 0\}. \tag{3.3}$$

Fig 2:



The continuation of the line H to values of $z > 0$ corresponds to the set of (x, y, z) such that $\delta(\lambda)$ has two doulbe complex roots and therefore is not included in Σ. Let $\Sigma_{\text{reg}} = \Sigma \setminus [L \cup H]$ be the regular part of Σ.

The discriminant of the polynomial $\delta(\lambda)$ is given by

$$\Psi(x, y, z) = 16xz^4 - 4y^2z^3 - 128x^2z^2 + 144xzy^2 + 256x^3 - 27y^4. \quad (3.4)$$

Hence one deduces from (3.2) and (3.3) that

$$\Sigma_{\text{reg}} = \{(x, y, z) : \ \Psi(x, y, z) = 0, \ y \neq 0, \ x \neq \frac{z^2}{12}\}. \quad (3.5)$$

Assume that Σ_{reg} is characteristic for P, i.e if $p = \sigma^2(P)$ is its principal symbol,

$$p(d\Psi) = 0 \text{ at } \Sigma_{\text{reg}}. \quad (3.6)$$

Assume that $t(O) = 0$ and that

$$\Sigma^- = \Sigma \cap \Omega^- \quad (3.7)$$

is a smooth hypersurface of Ω^-.

Let Q be the light cone for P over O, then, see Proposition 3.3 , $Q \cap \Sigma = E \cup B$, where away from O, Σ and Q intersect transversally at E and are tangent to third order along B. Let $\mathcal{V}(\Sigma)$ and $\mathcal{V}(\Sigma, Q)$ be the Lie algebras of smooth vector fields tangent to Σ and to Σ and Q respectively.

The following is then a simple consequence of the results of [17].

Theorem 3.1 *Let $u, Du \in H^s_{loc}(\Omega), s > \frac{3}{2}$, satisfy* (1.2). *If $u, Du \in I_k L^2_{loc}(\Omega^-, \mathcal{V}(\Sigma, Q))$, then $u, Du \in I_k L^2_{loc}(\Omega, \mathcal{V}(\Sigma, Q))$.*

One deduces from Theorem 3.1

Theorem 3.2 *Let $u, Du \in H^s_{loc}(\Omega), s > \frac{3}{2}$, satisfy* (1.2). *If $u, Du \in I_k L^2_{loc}(\Omega^-, \mathcal{V}(\Sigma))$, then $u, Du \in I_k L^2_{loc}(\Omega, \mathcal{V}(\Sigma, Q))$.*

In fact the results of [17] are stronger, we show that under the hypotheses of Theorem 3.1 the solution is strongly conormal in the sense of Melrose and Ritter, [12], along B and in the sense of [16] along the cusp line L of Σ.

In this note we shall restrict ourselves to the case where u satisfies the weakly semilinear equation

$$Pu = f(z,u), \quad z \in \Omega. \tag{3.8}$$

Since it contains all new ideas involved in the proof of Theorem 3.1

I would like to acknowledge that the main new ideas in [17], originated in joint works (in progress) with R.B. Melrose, [13], and with R.B. Melrose and M. Zworski, [14]. I would like to thank them for sharing their ideas with me, for their interest and encouragement. Possible errors in this manuscript are of course my own fault.

4 Outline Of The Proof

To prove Theorem 3.1 in the case of the weakly semilinear equation (3.6) we shall introduce a family of spaces $J_k(\Omega) \subset I_k L^2_{loc}(\Omega, \mathcal{V}(\Sigma))$, $k \in \mathbf{N}_0$, satisfying the following properties:

1) $J_{k+1}(\Omega) \subset J_k(\Omega) \subset L^2_{loc}(\Omega)$, $J_0(\Omega) = L^2_{loc}(\Omega)$.
2) $J_k(\Omega)$ is a $C^\infty(\Omega)$-module.
3) $J_k(\Omega) \cap L^\infty_{loc}(\Omega)$ is a C^∞ algebra.
4) $u, Du \in J_k(\Omega) \Longrightarrow u \in J_{k+1}(\Omega)$.
5) $Pu = f \in J_k(\Omega)$, $u = f = 0$ in $\Omega_T = \Omega \cap \{t < T\}$, then $u, Du \in J_k(\Omega)$.
6) If $u, Du \in I_k L^2_{loc}(\Omega^-, \mathcal{V}(\Sigma))$ in Ω^- satisfy (3.8), then $u, Du \in J_k(\Omega^-)$.

Proof of Theorem 3.1 : Suppose that such a family of spaces $J_k(\Omega)$ has been constructed. We then proceed by an induction argument. Let $\chi \in C^\infty(\mathbf{R})$, $\chi(s) = 0$, $s < -\frac{1}{2}$, $\chi(s) = 1$, $s > 0$. We obtain from (1.8)

$$P\chi u = \chi f(z,u) + [P,\chi]u. \tag{4.1}$$

If $u, Du \in J_0(\Omega) \cap J_1(\Omega^-)$, it follows from properties 2, 3 and 4 that the right hand side of (4.1) is in $J_1(\Omega)$. Thus one deduces from property 5 that $u, Du \in J_1(\Omega)$. By the same argument it follows that if $u, Du \in J_\ell(\Omega) \cap J_{\ell+1}(\Omega^-)$, $\ell < k$, then $u, Du \in J_{\ell+1}(\Omega)$. □

To define the spaces $J_k(\Omega)$ we shall introduce a blow-down map

$$\beta : X \longrightarrow \mathbf{R}^3 \tag{4.2}$$

from a manifold with corners X to $\mathbf{R}^3$ such that the lifts of Σ and Q by β intersect each other and the boundary of X transversally. We then define

$$J_k(\Omega) = \{u \in L^2_{loc}(\Omega) : \mathcal{W}^j \beta^* u \in L^2_\mu(X),\ \ j \leq k\}. \tag{4.3}$$

Where $\mathcal{W}$ is a Lie algebra and $C^\infty(X)$ module of smooth vector fields in X and μ is the lift of the Lebesgue measure of $\mathbf{R}^3$ under β. It will be a clear consequence of the definition of X and $\mathcal{W}$ that $J_k(\Omega)$, defined by (4.3), satisfies properties 1,2 and 4. It is a simple consequence of the Gagliardo-Nirenberg type of estimates of [11] that the spaces defined by (4.3) also satisfy property 3. Property 6 follows from Theorem 2.3 and from the results of [15]. The proof of property 5 is of course the most difficult one. The manifold with corners X and the algebra $\mathcal{W}$ will be constructed in Section 6.

5 Model Case

An easy computation shows that, in coordinates where (3.3) holds, Σ is invariant under the $\mathbf{R}^+$ action

$$m_s^{4-3-2}(x,y,z,t) = (s^4x, s^3y, s^2z, t),\ \ s \in \mathbf{R}^+. \tag{5.1}$$

$$\begin{aligned} Let M_r^{4-3-2}(\Omega) = \{u \in C^\infty(\Omega) : \partial_x^a \partial_y^b \partial_z^c u(0,0,0,t) = 0,\\ \forall\, a,b,c \in \mathrm{N},\ \ 4a+3b+2c \leq r\} \end{aligned} \tag{5.2}$$

be the ideal of smooth functions having Taylor series at

$$O = \{(x,y,z,t) \in \Omega;\ \ x = y = z = 0\}$$

consisting of terms of homogeneity r or greater with respect to (5.1). A differential operator P is said to have only terms of homogeneity r' or greater, with respect to (5.1), if

$$P : M_r^{4-3-2}(\Omega) \to M_{r+r'}^{4-3-2}(\Omega),\ \ r \in \mathrm{N}_0,\ \ r + r' \geq 0. \tag{5.3}$$

Simple computations show that if $P_0 = D_y^2 - D_xD_z$, then Σ_{reg} is characteristic for P_0, in general one can prove, see [17] that

Proposition 5.1 *If P and Σ are as above and (x,y,z,t) are smooth coordinates in which* (3.3) *holds, then*

$$P = a(D_y^2 - D_xD_z) + P_{-5},\ \ a \in C^\infty(\Omega),\ \ |a| > 0. \tag{5.4}$$

where P_{-5} has only terms of homogeneity -5 or greater with respect to (5.1).

Let Q_0 be the light cone for P_0 over O, then one easily finds that

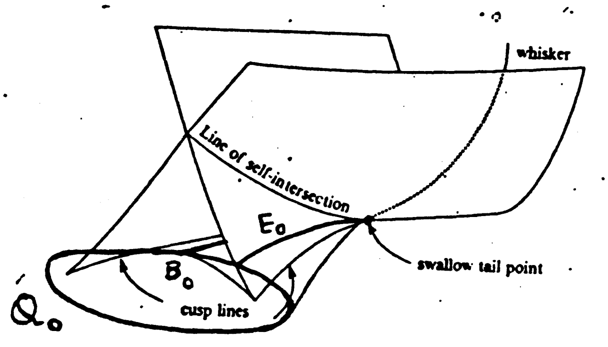
$$Q_0 = \{(x,y,z) \in \Omega : y^2 - 4xz = 0\}. \tag{5.5}$$

In this model we find that away from O, Q_0 and Σ are tangent to third order along B_0 and intersect transversally along E_0, where

$$B_0 = \{(x,y,z) \in \Omega : x = y = 0\}, \tag{5.6}$$

$$E_0 = \{(x,y,z) \in \Omega : x = \frac{3}{16}z^2,\ y^2 = -\frac{27}{32}z^3\}. \tag{5.7}$$

Fig 3:



As an immediate consequence of Propositon 5.1 one obtains

Proposition 5.2 *In the local coordinates of Proposition* 5.1 *one finds that*

$$Q = \{(x,y,z,t) \in \Omega;\ q(x,y,z,t) = 0\}, \tag{5.8}$$

where

$$q = q_0 + q',\ q_0 = y^2 - 4xz,\ q' \in M_7^{4-3-2}(\Omega). \tag{5.9}$$

See [17] for a proof. Now we deduce from it more information about the interaction of Q and Σ.

Proposition 5.3 *With P and Σ as in Proposition* 5.1, *in a small neighborhood of O, there are smooth functions $F_i(z,t)$,* $1 \leq i \leq 3$, *such that* $Q \cap \Sigma = B \cup E$

$$B = \{x = z^3 F_1(z,t),\ y = z^2 F_2(z,t)\}, \tag{5.10}$$

$$E = \{x = \frac{3}{16}z^2 + z^3 F_3(z,t), y^2 = -\frac{27}{32}z^3 + z^4 F_4(z,t)\} \tag{5.11}$$

Away from O, Q and G meet transversally at E and are tangent of third order at B.

6 Geometric Resolution

The hypersurfaces Σ and Q will be resolved to normal crossing by iterated quasi-homogeneous blow ups. As a first step we define the 4-3-2 blow up of $\mathbf{R}^n$ along $O = (0,0,0)$.

In $\mathbf{R}^3$ consider the non-round sphere

$$S^2_{4-3-2} = \{(\omega_1, \omega_2, \omega_3);\ \ \omega_1^6 + \omega_2^8 + \omega_3^{12} = 1\}$$

and the map

$$\beta_1 : X_1 = [0,\infty) \times S^2_{4-3-2} \longrightarrow \mathbf{R}^3,\ \ \beta_1(s,\omega) = (s^4\omega_1, s^3\omega_2, s^2\omega_3).$$

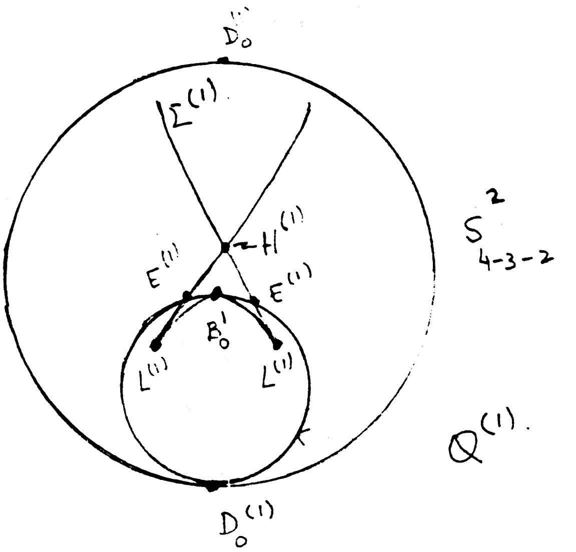
This is surjective and restricts to a diffeomorphism of $X_1 \setminus \partial X_1$ onto $\mathbf{R}^n \setminus K$. Moreover the $\mathbf{R}^+$ action (5.1) lifts to the standard multiplicative action on the factor $[0,\infty)$.

From these obeservations above it follows that the lifts of the hypersurfaces and the bicharacteristic B in the model case are:

$$\Sigma^{(1)} = \operatorname{clos}[\beta_1^{-1}(\Sigma \setminus O)] = \tag{6.1}$$
$$\{16\omega_1\omega_3^4 - 4\omega_2^2\omega_3^3 - 128\omega_1^2\omega_3^2 + 144\omega_1\omega_3\omega_2^2 + 256\omega_1^3 - 27\omega_2^2 = 0\},$$

$$Q_0^{(1)} = \operatorname{clos}[\beta_1^{-1}(Q_0 \setminus O)] = \{\omega_2^2 - 4\omega_1\omega_3 = 0\}, \tag{6.2}$$
$$B_0^{(1)} = \operatorname{clos}[\beta_1^{-1}(B \setminus O)] = \{\omega_1 = 0, \omega_2 = 0\}. \tag{6.3}$$

Fig 4:



$\Sigma^{(1)}$ has a cusp singularity at

$$L^{(1)} = \operatorname{clos}[\beta_1^{-1}(L \setminus O)] = \{\omega_1 = -\frac{1}{12}\omega_3^2, \omega_2^2 = (-\frac{2}{3}\omega_3)^3\} \tag{6.4}$$

and a self-intersection at

$$H^{(1)} = \operatorname{clos}[\beta_1^{-1}(L \setminus O)] = \{\omega_1 = -\frac{1}{4}\omega_3^2, \omega_2 = 0\}. \tag{6.5}$$

For reasons that will become clear later on, there are two "great circles" on S^2_{3-2-1} that will have to be taken into consideration. We define

$$C_1 = \{\omega_1 = 0, \ \ r = 0\}, \tag{6.6}$$
$$C_2 = \{\omega_3 = 0, \ \ r = 0\}. \tag{6.7}$$

More generally we find, see [17]

Proposition 6.1 *In local coordinates in which* (3.1) *and* (5.8) *hold the lifts* $\Sigma^{(1)}, Q^{(1)}$ *and* $B^{(1)}$ *of the hypersurfaces and the bicharacteristic to* X_1 *are diffeomorphic, on* X_1*, to the model* $\Sigma^{(1)}, Q_0^{(1)}$ *and* $B^{(1)}$ *under a diffeomorphism fixing* ∂X_1 *pointwise. Conversely any diffeomorphism preserving* (3.1), (5.8) *and* O*, lifts to a diffeomorphism of* X_1 *near* ∂X_1 *preserving* $\Sigma^{(1)}$ *and* $Q^{(1)}$

The full resolution of the geometry is obtained by blow ups of the three (really six) submanifolds $L^{(1)}$, $D_0^{(1)} = Q^{(1)} \cap C_2$ and $B^{(1)}$. There are local coordinates (s, X, Y, T) near $L^{(1)}$ with

$$\Sigma^{(1)} = \{Y^3 = X^2\}, \tag{6.8}$$

near $D_0^{(1)}$ with

$$Q^{(1)} = \{X = Y^2\}, C_2 = \{X = 0, r = 0\}. \tag{6.9}$$

near $B^{(1)}$ with

$$Q^{(2)} = \{X = 0\}, \ \ \Sigma^{(1)} = \{X = Y^4\}, \ \ C_1 = \{X = Y^2, r = 0\}. \tag{6.10}$$

Thus $\Sigma^{(1)}$ can be resolved to normal crossing by a $3-2$ blow-up of $L^{(1)}$, thus set

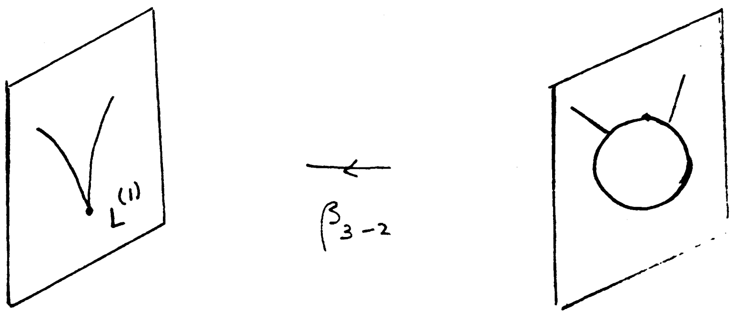
$$S^1_{3-2} = \{(\theta_1, \theta_2) \in \mathbf{R}^2; \theta_1^4 + \theta_2^6 = 1\} \tag{6.11}$$

and in local coordinates (6.8) we construct the map

$$\beta_{3-2} : [0,\infty)_s \times [0,\infty)_r \times S^1_{3-2} \times \mathbf{R}^{n-3} \to X_1 \tag{6.12}$$

$$\beta_{3-2}(s,r,\theta) = (r, s^3\theta_1, s^2\theta_2). \tag{6.13}$$

Fig 5:



It will also be necessary to blow-up $D_0^{(1)}$ with homogeneity 2-1-1, thus let

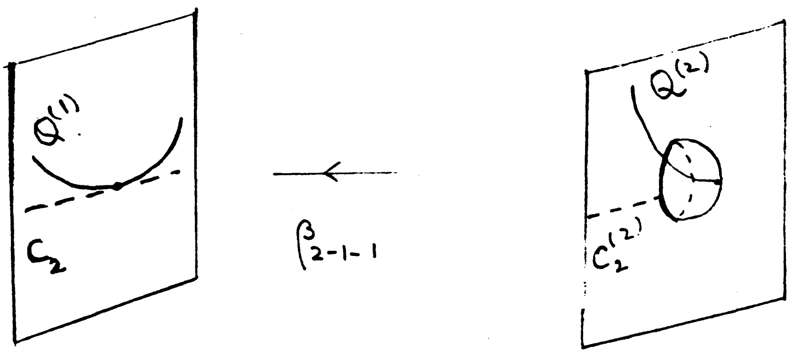
$$S^2_{2-1-1} = \{(\theta_1,\theta_2,\theta_3) \in \mathbf{R}^2; \theta_1^2 + \theta_2^4 + \theta_3^4 = 1\} \tag{6.14}$$

and in local coordinates (6.9) construct the map

$$\beta_{2-1-1} : [0,\infty)_s \times [0,\infty)_R \times S^1_{2-1} \times \mathbf{R}^{n-3} \to X_1 \tag{6.15}$$

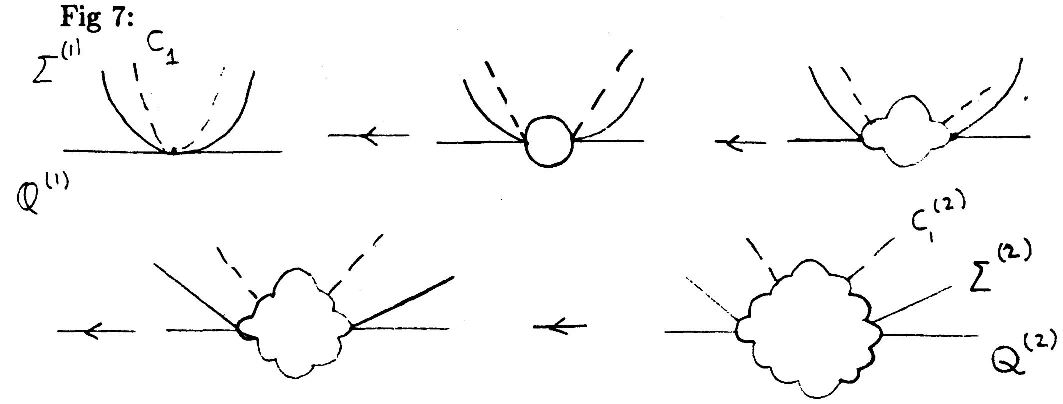
$$\beta_{2-1}(s,R,\omega,t) = (R, s^2\theta_1, s\theta_2, s\theta_3, t). \tag{6.16}$$

Fig 6:



To resolve $Q^{(1)}$, $\Sigma^{(1)}$ and C_1 to normal crossing it will be more coneve-nient to use four normal blow-ups as in [12]. Since $Q^{(1)}$ and $\Sigma^{(1)}$ are tangent to third order at $B^{(1)}$, if C_1 did not have to be taken into consideration, one could use a 4-1 nonhomogeneous blow-up to resolve $Q^{(1)}$ and $\Sigma^{(1)}$ to normal crossing, but C_1 destroys the 4-1 homogeneity.

Fig 7:



Since $D_0^1, L^{(1)}$ and $B^{(1)}$ are disjoint we can use these maps to replace small neighborhoods of $D_0^{(1)}$, $L^{(1)}, B^{(1)}$ by their respective blow ups and so define the manifold with corners X and a blow down map $\beta_2 : X \to X_1$. Let

$$\beta = \beta_2 o \beta_1 : X \to \mathbf{R}^n \tag{6.17}$$

Denote

$$\begin{aligned} Q^{(2)} &= \text{clos}[\beta_2^{-1}(Q^{(1)} \setminus (B^{(1)} \sqcup D_0^{(1)}))], \\ \Sigma^{(2)} &= \text{clos}[\beta_2^{-1}(\Sigma^{(1)} \setminus (L^{(1)} \sqcup B^{(1)}))] \\ L^{(2)} &= \text{clos}[\beta_2^{-1}(L^{(1)})], \\ B^{(2)} &= \text{clos}[\beta_2^{-1}(B^{(1)})], \\ C_1^{(2)} &= \text{clos}[\beta_2^{-1}(C_1 \setminus B^{(1)})], \\ C_2^{(2)} &= \text{clos}[C_2 \setminus D_0^{(1)}]. \end{aligned}$$

The circle $C_2^{(2)}$ does not continue into the boundary face introduced by the 2-1-1 blow-up.

The manifold with corners X has twelve boundary hypersurfaces which meet transversally pairs or triples. Let ρ_L, $\rho_B^j, 1 \leq j \leq 8$, ρ_D and ρ_K be respectively the defining functions of $\beta^{-1}(L)$, each of the eight hypersurfaces of $\beta^{-1}(B), \beta^{-1}(D)$ and $\beta^{-1}(K)$ (These functions are assumed to be extended smoothly past the surfaces they define).

Proposition 6.2 *Under the C^∞ map $\beta : X \to \mathbf{R}^n$ the lifts*

$$\beta^*(M) = \mathrm{clos}[\beta^{-1}(M \setminus [K \cup L \cup B])), \tag{6.18}$$

for $M = Q, \Sigma$ are smooth hypersurfaces that intersect the boundaries of X transversally. Any C^∞ diffeomorphism of X_1 preserving $\Sigma^{(1)}, Q^{(1)}$ $D_0^{(1)}$ and ∂X_1 lifts to a C^∞ diffeomorphism of X preserving all boundaries and all the hypersurfaces.

Let $L_c^2(X)$ be the space of compactly supported square integrable functions in X with respect to the measure $\mu = \beta^*(dxdydz)$. Then the blow down map β gives an isomorphism

$$\beta^* : L_c(\mathbf{R}^n) \leftrightarrow L_c^2(X). \tag{6.19}$$

Let $\mathcal{W}$ be the Lie algebra and smooth vector fields W on X satyisfying the following properties:
1) W is tangent to all boundary hypersurfaces.
2) W is tangent to $\beta^*(\Sigma)$ and to $\beta^*(Q)$.
3) W is tangent to $C_2^{(2)}$.
4) In local coordinates (r, s, X) in which $\rho_K = r$ and $C_1^{(2)} = \{r = X = 0\}$, $\mathcal{W}$ is spanned by $r\partial_r$, $s\partial_s$, $X\partial_X$, $r^2\partial_X$.
We then define for any integer k

$$J_k(\Omega) = \{u \in L_c^2(\Omega) : \beta^* u \in I_k L_c^2(X, \mathcal{W})\} \tag{6.20}$$

As a consequence of Propositions 6.1 and 6.2 it follows that the spaces $J_k(\Omega)$ are independent on the choices of coordinates. Moreover from the Gagliardo-Nirenberg type inequalities of [15] one obtains

Proposition 6.3 *For any $k \in \mathbf{N}$, $J_k(\Omega) \cap L^\infty_{loc}(\Omega)$ is a C^∞ algebra, i.e for any $f \in C^\infty(\mathbf{R}^m)$ and $u_i \in J_k(\Omega) \cap L^\infty(\Omega), 1 \le i \le m$,*

$$f(u_1, ..., u_m) \in J_k(\Omega) \cap L^\infty_{loc}(\Omega). \tag{6.21}$$

By writing the generators of $\mathcal{V}(\Sigma, Q)$ and their lift under the map β it is not hard to see that

$$J_k(\Omega) \subset I_k L^2_{loc}(\Omega, \mathcal{V}(\Sigma, Q)) \tag{6.22}$$

7 The Linear Propagation Theorem

In this section we sketch the proof that the spaces $J_k(\Omega)$ satisfy

Theorem 7.1 *Let $f \in J_k(\Omega)$, $f = 0$ in Ω^-. Let $u \in H^1_{loc}(\Omega)$, $u = 0$ in Ω^-, satisfy*

$$Pu = f. \tag{7.1}$$

Then $u, Du \in J_k(\Omega)$.

Lemma 7.1 *Let $\phi \in C_0^\infty(X_1)$, $\phi = 1$ in sufficiently small neighborhoods of $L^{(1)}$, $E^{(1)}$ and $H^{(1)}$, $\phi = 0$ outside slightly bigger neighborhoods. There exist $v_1, Dv_1 \in J_k(\Omega)$ such that*

$$\beta_1^*(Pv_1) - \phi\beta_1^* f \in I_k L^2_{loc}(X_1, \partial X_1) \tag{7.2}$$

The proof o Lemma 7.1 is based on the fact that the lift of the operator P by the map β_1 is of real principal type in the totally characteristic sense, see [10], in some directions near $L^{(1)}, E^{(1)}$ and $H^{(1)}$. One can then use the calculus of totally characteristic Fourier Integral Operators of [10] to transform the operator, the characteristic surfaces and their intersections into model cases. Lemma 7.1 is then a consequence of the mapping properties of these operators.

Lemma 7.2 *Let $g \in L^2_{loc}(\Omega)$ be such that*

$$\beta^* g \in I_k L^2_{loc}(X, \partial X_1). \tag{7.3}$$

Then there exists $v_2, Dv_2 \in J_k(\Omega)$ such that $Pv_2 = g$.

The proof of Lemma 7.2 is considerably simpler than the one of Lemma 7.1, it is based on a commutator argument.

7.1 Marked Lagrangian Distributions

Let $\Lambda \subset T^*\Omega$ be a smooth conic closed Lagrangian and let $S_2 \subset S_1 \subset \Lambda$ be conic smooth hypersurfaces. Denote

$$\mathcal{M}(\Lambda)_1 = \{A \in \Psi^1(\Omega) : a = \sigma^1(A) = 0 \text{ at } \Lambda_\Lambda, \tag{7.4}$$

$$H_a \text{ tangent to } S_1 \text{ and to } S_2\} \tag{7.5}$$

and define

$$I_k L^2_c(\Omega, \mathcal{M}(\Lambda)_1) = \{u \in L^2_c(\Omega) : \mathcal{M}(\Lambda)_1^j u \subset L^2_{loc}(\Omega),\ j \le k\}. \tag{7.6}$$

A detailed study of these distributions can be found in [8]. As mentioned in Section 2, the marked Lagrangian Distributions were first introduced by Melrose in [9] to study the cusp case.

Let $\Lambda_\Sigma = \text{clos}N^*(\Sigma_{reg})$, $\Lambda_Q = \text{clos}N^*(Q \setminus O)$. It is weel known that Λ_Σ and Λ_Q are smooth conic Lagrangian submanifolds of $T^*\mathbf{R}^3$. Let $\Lambda_B = N^*B$ and let $\Lambda_O = T^*_O\mathbf{R}^3$, denote $S_1 = \Lambda_\Sigma \cap \Lambda_B = \Lambda_Q \cap \Lambda_B = \Lambda_\Sigma \cap \Lambda_Q$ and $S_2 = \Lambda_\Sigma \cap \Lambda_O$. Let $S_3 = \Lambda_0 \cap \Lambda_Q$ and let $I_k L^2_{loc}(\Omega, \mathcal{M}(\Lambda_0)_3)$ be the space of marked Lagrangian distributions to Λ_0 marked by S_3 and S_2.

In coordinates where (3.1) holds one obtains that $\mathcal{M}(\Sigma)_1$ is the $\Psi^0(\Omega)$ span of

$$V_1 = 4x\partial_x + 3y\partial_y + 2z\partial_z, \; V_2 = (2xz - \frac{3}{4}y^2)\partial_x - \frac{1}{2}yz\partial_y + 4x\partial_z, \quad (7.7)$$

$$P_1 = z(\partial_y^2 - \partial_x\partial_z), \; P_2 = y(\partial_y^2 - \partial_x\partial_z), \quad (7.8)$$

$$P_3 = 4\partial_z^2 + 2z\partial_y^2 + y\partial_y\partial_x, \; P_4 = (\partial_y^2 - \partial_x\partial_z)\partial_z, \quad (7.9)$$

$$P_5 = (\partial_y^2 - \partial_x\partial_z)\partial_y. \quad (7.10)$$

Times elliptic factors of the appropriate orders. The space of marked Lagrangian distributions to the swallowtail marked by S and S_1 is however too small for our purposes, we shall need a slighty bigger one. Let $P_5' = (3\partial_y^2 - 8\partial_x\partial_z - 12z\partial_x^2)^3\partial_y^2$ and define the space of "supermarked" Lagrangian distributions to Λ_Σ S andS_1 as

$$I_{3k}L^2_c(\Omega, \mathcal{M}(\Lambda_\Sigma)_1)^s = \{u \in L^2_c(\Omega) : V_1^{\alpha_1}V_2^{\alpha_2}P_1^{\ell_1}P_2^{\ell_2}P_3^{\ell_3}P_4^{\ell_4}P_5'^{\ell_5}u \in H_c^{-\ell}(\Omega), \; \ell = \ell_1 + \ell_2 + \ell_3 + \ell_4 + 6\ell_5 \leq 3k\}. \quad (7.11)$$

Where the superscript s is for "supermarked". The spaces of supermarked Lagrangians was introduced by M. Zworski in [18] where a more detailed description of those spaces is given. One defines the space $I_kL^2_c(\Omega, \mathcal{M}(\Sigma)_1)^s$ for all integers k by complex interpolation. One can easily show that

$$I_kL^2_c(\Omega, \mathcal{M}(\Lambda_\Sigma)_1) \subset I_kL^2_c(\Omega, \mathcal{M}(\Lambda_\Sigma)_1)^s. \quad (7.12)$$

Let

$$M_k(\Omega) = I_kL^2_c(\Omega, \mathcal{M}(\Lambda_\Sigma)_1)^s + I_kL^2_c(\Omega, \mathcal{M}(\Lambda_Q)_1) + I_kL^2_c(\Omega, \mathcal{M}(\Lambda_B)_1) + I_kL^2_c(\Omega, \mathcal{M}(\Lambda_O)_3) \quad (7.13)$$

be the space of marked Lagrangian distributions to Σ, Q and B.

Lemma 7.3 *Let $g \in J_k(\Omega)$ be such that $\beta^* g$ is supported away from $E^{(1)}$, $H^{(1)}$ and $L^{(2)}$, then $g \in M_k(\Omega)$.*

The proof of Lemma 7.3 is quite long and consists basically of lifting the generators of each of the components of M_k under the map β. Now we are going to use the same idea as in the case of the cusp, first we prove a propagation theorem for $M_k(\Omega)$ and then use again the equation to show that the solution is in fact in the smaller space $J_k(\Omega)$. By commutator methods one can prove

Lemma 7.4 *Let $f \in M_k(\Omega)$, there exist $v_3, Dv_3 \in M_k(\Omega)$ such that $Pv_3 = f$.*

Then one proves an elliptic regularity type of Theorem which states that

Lemma 7.5 *Let $v, Dv \in M_k(\Omega)$ be such that $Pv \in J_k(\Omega)$. Then $v, Dv \in J_k(\Omega)$.*

When one lifts $v \in M_k(\Omega)$ under the map β one finds that it may be singular at some circles at the boundary of X, but it turns out that the lift of operator P under the map β is elliptic in some directions of ${}^bT^*X$ over those circles and therefore one concludes that if v satisfies the inclusion $Pv \in J_k(\Omega)$, then $v \in J_k(\Omega)$. This is the reason why one has to include the great circles in the definition of the spaces, since the hypersurfaces $\{x = 0\}$ and $\{z = 0\}$ are characteristic for P_0 the lift of the operator P could not the be elliptic on circles $C_1^{(2)}$ and $C_2^{(2)}$.

Conclusion of the proof of Theorem 7.1:

Let v_1, v_2 and v_3 be as in Lemmas 7.1, 7.2 and 7.3 and $w = u - v_1 - v_2 - v_3$. Then

$$Pw = 0, \quad w \in J_k(\Omega) \ \text{ in } \ t < 0. \tag{7.14}$$

Let

$$\mathcal{M}(\Lambda_Q \cup \Lambda_\Sigma) = \{A \in \Psi^1(\Omega) : a = \sigma^1(A) = 0 \ \text{ on } \ \Lambda_Q \cup \Lambda_\Sigma\} \tag{7.15}$$

Equation (7.14) implies that

$$w, Dw \in I_k L^2_{loc}(\Omega^-, \mathcal{M}(\Lambda_Q \cup \Lambda_\Sigma)). \tag{7.16}$$

By commutator methods one can easily show that

$$w, Dw \in I_k L^2_{loc}(\Omega, \mathcal{M}(\Lambda_Q \cup \Lambda_\Sigma)). \tag{7.17}$$

By the arguments used in the proof of Lemma 7.3 one can show that

$$I_k L^2_{loc}(\Omega, \mathcal{M}(\Lambda_Q \cup \Lambda_\Sigma)) \subset J_k(\Omega). \tag{7.18}$$

This concludes the proof of Theorem 7.1.

References

[1] V.I.Arnol'd. *Wave fronts evolution and equivariant Morse lemma. Comm. Pure Appl. Math. 28 (1976) 557-582.*

[2] V.I.Arnol'd. *Classical Mechanics. Springer Verlag GTM vol. 60, 1984*

[3] M.Beals. *Regularity of nonlinear waves associated with a cusp. Preprint.*

[4] J-M.Bony. *Interaction des singulariteés pour les équations aux derivées partielles non-lineaires. Sem. Goulaouic- Meyer-Schwartz, exp no. 22 (1979-1980).*

[5] J-M.Delort *Conormalite des ondes semilineaires le long des caustiques, Preprint.*

[6] G.Lebeau *Probleme de Cauchy semilineaire en 3 dimensions d'espace. Un resultat de finitude. Jour. Func Anal 78(1988) 185-196.*

[7] G.Lebeau *Equations des ondes seminlineaires II. Controle des singularites et caustiques semilineaires. Invent. Math. 95 (1989), 277-323.*

[8] R. Melrose. *Marked lagrangian distributions. In Preparation.*

[9] R. Melrose. *Semilinear equations with cusp singularities. Preprint.*

[10] R. Melrose. *Transformation of boundary value problems. Acta Matematica 147 (1981) 149-236.*

[11] R.Melrose and N.Ritter *Interactions of progressing waves for semilinear wave equations Ann. of Math. 121 (1985) 187-213.*

[12] R. Melrose and N. Ritter. *Interaction of nonlinear progressing waves for semilinear wave equations II. Arkiv For Matematik vol 25 (1987) 91-114.*

[13] R. Melrose and A. Sá Barreto. *Non-linear interaction of a cusp and a plane. In Preparation.*

[14] R. Melrose, A. Sá Barreto and M. Zworski *Semilinear diffraction of conormal waves. In Preparation.*

[15] A. Sá Barreto *On the Interactions Of Conormal Waves. To appear in the Proceedings of IMA.*

[16] A. Sá Barreto *Second microlocal ellipticity and propagation of conormality for semilinear wave equations. To appear in Journ. Of Funct. Anal.*

[17] A. Sá Barreto *Evolution of conormal waves with swallowtail singularities. In preparation.*

[18] M. Zworski *Propagation of sub-marked Lagrangian singularities. Inpreparation.*

DEPARTMENT OF MATHEMATICS
PURDUE UNIVERSITY
WEST LAFAYETTE, IN 47907.