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# Spectral Asymptotics for Hill's Equation near the Potential Maximum

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## 1. Hypotheses and General Facts on Periodic Schrödinger Operators

In this note we are interested in the spectrum near the potential maximum of a one-dimensional semiclassical Schrödinger operator

(1.1) $$P = P(h) := -\frac{h^2}{2}\frac{d^2}{dx^2} + V(x),$$

where the potential $V: \mathbb{R} \longmapsto \mathbb{R}$ satisfies the following hypotheses:

(H1) V is real analytic,

(H2) V is $2\pi$-periodic,

(H3) $V(x) \leq 0$ with equality exactly at the points $2\pi k$, $k \in \mathbf{Z}$,

(H4) $V''(0) < 0$; without loss of generality we may assume $V''(0) = -1$.

It is well known that P is selfadjoint with domain $H^2(\mathbb{R}) := \{u \in L_2(\mathbb{R}) | u', u'' \in L_2(\mathbb{R})\}$ and that P is unitarily equivalent to the direct integral

(1.2) $$\int_{[0,1[}^{\oplus} P_\vartheta \, d\vartheta;$$

where $P_\vartheta u = Pu$ on $\mathscr{H}_\vartheta^2 := \{u \in H_2^{loc} | u^{(k)}(x-2\pi) = e^{2\pi i\vartheta} u^{(k)}(x) \text{ for } k = 0,1,2\}$. So each $P_\vartheta$ can be viewed as an selfadjoint, semibounded, elliptic operator on a compact manifold, that has therefore a pure point spectrum of the form

(1.3) $$\sigma(P_\vartheta) = \{E_1(\vartheta) \leq E_2(\vartheta) \leq .......\} \qquad (\vartheta \in [0,1[).$$

The so-called bands

(1.4) $$B_k := \{E_k(\vartheta) \,|\, \vartheta \in [0,1]\}$$

are closed intervals of non-vanishing length, and build up the spectrum of P

(1.5) $$\sigma(P) = \bigcup_k B_k,$$

which in addition is absolutely continuous.

In one dimension two bands do not overlap except possibly at their endpoints, otherwise they are separated by open intervals, called gaps $G_k$. Let $\tau_{2\pi}(\mu)$ denote the operator of translation by $-2\pi$, acting on the two-dimensional space of solutions of $(P-\mu)u = 0$, defined by $(\tau_{2\pi}(\mu)u)(x) := u(x + 2\pi)$. Then we have the following simple criterion:

(1.6) $$\mu \in \sigma(P) \Longleftrightarrow \tau(\mu;h) := \frac{1}{2}\text{trace}\,(\tau_{2\pi}(\mu)) \in [-1,1].$$

## 2. Former Results in Different Regions

The applicability of several methods, for instance the WKB-method, usually employed in the study of the spectrum of a semiclassical Schrödinger operator near some level $\mu$, is strongly governed by *turning points*, i.e. zeros $x_\mu$ of $V - \mu$. We will restrict ourselves to turning points in a complex neighbourhood of the period $[0,2\pi]$. When $\mu$ is negative and sufficiently small, we typically have real *simple* turning points, that is with $V'(x_\mu) \neq 0$. Here a classical particle has to change its direction, while the region beyond such a turning point is forbidden.

In view of different geometric situations, the study is roughly divided into the following regions for the energy level $\mu$:

(I) $\quad C_0 > \mu > \varepsilon_0 > 0,$

(II) $\quad \varepsilon_0 \geq \mu \geq -\varepsilon_0,$

(III) $\quad 0 > -\varepsilon_0 > \mu > -\varepsilon_1,$

where the constants $\varepsilon_0, \varepsilon_1$ and $C_0$ are determined by the potential.

In case (I) there do not exist any real turning points, and therefore there is no obstruction to employ the standard WKB-method, such that we obtain that the gaps are of size $O(h^\infty)$, while the band lengths are of order of magnitude $O(h)$.

In case (III) we have at least two real turning points $b_\mu^-$, $b_\mu^+$ near 0. Since further turning points between $b_\mu^+$ and $b_\mu^- + 2\pi$, would hinder a systematic study, we will exclude the corresponding $\mu$-regions. In other words, we assume that there is only *one well* $I_\mu$ over the period interval:

$$(2.1) \qquad I_\mu := [b_\mu, b_\mu + 2\pi] = \{x | V(x) \leq \mu; x \in [0,2\pi]\}.$$

This situation has been investigated by Harrell [Ha], Simon [Si] and Outassourt [Ou]. We only mention here that [Ou] applies the method of the interaction matrix due to Helffer/Sjöstrand [He,Sj 1] in order to compute precise asymptotic formulas for the width $B_p(h)$ of the p-th band, concretely

$$(2.2) \qquad B_p(h) = h^{\frac{1}{2p}} \frac{\pi^{-\frac{1}{2}}}{p!} 2^{p+3} e^{(2p+1)A} e^{-S(\mu)/h} \left(1 + O_p(h)\right),$$

where A is determined by the potential and

$$(2.3) \qquad S(\mu) := \int_{b_\mu^-}^{b_\mu^+} \left(2\left[V(x) - \mu\right]\right)^{\frac{1}{2}} dx$$

is the *Agmon distance* between the wells $I_\mu$ and $I_\mu + 2\pi$ with $\mu$ contained in the band.

Now the zone, given by case (II), is the region under consideration in this note. Then the situation concerning the turning points is as follows: When one is passing from $\mu < 0$ to $\mu > 0$, one has a change from two real turning points to two purely imaginary turning points near the origin, where for $\mu = 0$ there is exactly one double turning point at the origin.

(Clearly the same situation near the origin is given in the case of a double well potential V, recently studied by Gérard/Grigis [Gé,Gr] and Horn [Ho].)

One approach for the treatment of equations with turning points is given by *R.E. Langer's method of the comparison equation.* In our case (II) this comparison equation is given by *Weber's equation*

$$(2.4) \qquad -\frac{d^2u}{dx^2} - \left(\frac{x^2}{2} - \frac{S(\mu)}{\pi h}\right)u = 0.$$

Here (for $\mu > 0$) $S(\mu)$ is defined by

$$(2.5) \qquad S(\mu) := i\int_{a_\mu^-}^{a_\mu^+} \left(2|V(y) - \mu|\right)^{\frac{1}{2}} dy.$$

Weinstein/Keller [We,Ke] use this method in order to compute asymptotically a fundamental system of solutions of the Schrödinger equation and, with respect to which they determine the translation matrix, such that they obtain the beautiful formula

$$(2.6) \qquad \tau(\mu;h) \sim \left(1 + e^{2S(\mu)/h}\right)^{\frac{1}{2}} \cos\left\{\tfrac{1}{h}C(\mu)\right\},$$

where

$$(2.7) \qquad C(\mu) := \int_0^{2\pi} \left(2[\mu - V(x)]_+\right)^{\frac{1}{2}} dx.$$

The role of the "$\sim$" is not quite clear, but it seems that their study is only valid up to the second order. Nevertheless following Lynn/Keller [Ly,Ke] it should be possible to carry out the study up to the order $O(h^\infty)$. Finally they estimate very briefly the size of the bands $B_k(h)$ and the gaps $G_k(h)$ and get $|B_k(h)| \sim \frac{1}{2}|G_k(h)|$ in the region $\mu \le 0$, which does not coincide with our results.

### 3. Formula for the Trace and Theorems

Our analysis will yield

$$(3.1) \quad \tau(\mu;h) = \left(1 + e^{-2\pi\frac{\mu'}{h}}\right)^{\frac{1}{2}} \cos\left\{\tfrac{1}{h}\left[C(\mu) + \mu_0'(\log|\mu_0'| - 1) - \mu'\log h\right] + \arg\left[\Gamma\left(\tfrac{1}{2} - i\tfrac{\mu'}{h}\right)\right] + hr(\mu;h)\right\} + O\left(e^{-\frac{\varepsilon_0}{h}}\right).$$

Explanation of this formula:

* The error term $O\left(e^{-\frac{\varepsilon_0}{h}}\right)$ is due to the method and uniform with respect to $\mu$.
* $r(\mu;h)$ is an analytic symbol of order 0 (in the sense of Sjöstrand [Sj 1]).

(3.2) $\mu' := F(\mu;h) = f_0(\mu) + hf_1(\mu) + h^2 f_2(\mu) + \ldots\ldots$ is a classical analytic symbol (c.a.s); here $\mu_0' := f_0(\mu)$, and it can be shown that $f_1 \equiv 0$. It can be shown that

(3.3) $S(\mu) = -\pi\mu' + O(h^2)$ and

* $\mu \longmapsto C(\mu) + \mu_0'(\log|\mu_0'| - 1)$ is analytic for $\mu$ sufficiently small.

We still have to take into account the different asymptotic behaviour of the $\Gamma$-function in the following regions:

(i) Let be $|\mu| \le Ch$ for C arbitrarily large, but fixed. Viewing here the $\Gamma$-function as a holomorphic function, that is real on the real axis, we have $\arg\left[\Gamma\left(\frac{1}{2} - i\frac{\mu'}{h}\right)\right] = O\left(\frac{\mu'}{h}\right)$.

(ii) Let C be large enough. Then by the complex version of Stirling's formula we have in the region $Ch \le |\mu'| \le \frac{1}{C}$: $\arg\left[\Gamma\left(\frac{1}{2} - i\frac{\mu'}{h}\right)\right] = \frac{\mu'}{h}\left(1 - \log\left(\frac{|\mu'|}{h}\right)\right) + \frac{h}{\mu'}F\left(\frac{\mu'}{h}\right)$, where F is the real part of a function, that is holomorphic and bounded in $\left|\frac{\operatorname{Im} z}{\operatorname{Re} z}\right| \le \frac{1}{C}$.

These observations allow it to simplify the phase of the cosine such that we get the following theorems:

**Theorem 1:** *Let $C > 0$ be arbitrarily large, but fixed. Then the spectrum of P in $[-Ch,Ch]$ for h sufficiently small is the union of disjoint closed bands. Let $\mu'$ be defined as above for $\mu \in [-Ch,Ch]$. If $\mu'$ lies in a gap, then the length of this gap is given by*

$$\frac{2h}{\left\{\log\left(\frac{1}{h}\right)\right\}}\left(\arccos\left[\left(1 + e^{-2\pi\frac{\mu'}{h}}\right)^{-\frac{1}{2}}\right]\right) + O\left(\frac{h}{\left(\log\left(\frac{1}{h}\right)\right)^2}\right),$$

*If $\mu'$ lies in a band, the length of this band is*

$$\frac{2h}{\left\{\log\left(\frac{1}{h}\right)\right\}}\left(\arcsin\left[\left(1 + e^{-2\pi\frac{\mu'}{h}}\right)^{-\frac{1}{2}}\right]\right) + O\left(\frac{h}{\left(\log\left(\frac{1}{h}\right)\right)^2}\right).$$

In particular: *If $\mu' = o(h)$, then we see that the length of the gaps is tending to the length of the bands.*

**Theorem 2:** *If $C > 0$ is large enough and $h > 0$ is sufficiently small, then the spectrum of P in $\left[-\frac{1}{C},-Ch\right]$ is the union of bands $B_k$ separated by open gaps $G_k$ with*

$$|B_k| = \frac{2h}{C'(\mu)}\left(1 + O\left(\frac{h^2}{\mu^2}\frac{1}{\log\left(\frac{1}{|\mu|}\right)}\right)\right)\arcsin\left[\left(1 + e^{-2\pi\frac{\mu'}{h}}\right)^{-\frac{1}{2}}\right]$$

*for arbitrary $\mu \in B_k$, where $\mu'$ is given as above and $C(\mu)$ is defined by (2.7).*
*The distance between the centers of two consecutive bands is:*

$$\left(1 + O\left(\frac{h}{|\mu|}\frac{1}{\log\left(\frac{1}{|\mu|}\right)}\left(\frac{h}{|\mu|} + \frac{1}{\log\left(\frac{1}{|\mu|}\right)}\right)\right)\right)\frac{\pi h}{C'(\mu)}.$$

Remark: If C is very large, we conclude from our remarks above

$$\arcsin\left[\left(1 + e^{-2\pi\frac{\mu'}{h}}\right)^{-\frac{1}{2}}\right] \sim e^{-\frac{S(\mu)}{h}},$$

consequently

$$|B_k| \sim \frac{2h}{C'(\mu)}e^{-\frac{S(\mu)}{h}}\left(1 + O\left(\frac{h^2}{\mu^2}\frac{1}{\log\left(\frac{1}{|\mu|}\right)}\right)\right).$$

This corresponds to the size of splitting in Theorem 3.1 of Gérard/Grigis, in view of the fact that $C'(\mu)$ is the half of the period of the Hamilton flow on the surface $\{p = \mu\}$.

**Theorem 3:** *If $C > 0$ is large enough and $h > 0$ is sufficiently small, then the spectrum of $P$ in $[Ch, \frac{1}{C}]$ is the union of bands $B_k$ separated by open gaps $G_k$ with*

$$|G_k| = \frac{2h}{C'(\mu)}\left(1 + O\left(\frac{h^2}{\mu^2}\frac{1}{\log\left(\frac{1}{\mu}\right)}\right)\right)\arcsin\left[\left(1 + e^{2\pi\frac{\mu'}{h}}\right)^{-\frac{1}{2}}\right].$$

*for arbitrary $\mu \in G_k$ and with $\mu'$ and $C(\mu)$ as above.*

*The distance between the centers of two consecutive gaps is:*

$$\left(1 + O\left(\frac{h}{\mu}\frac{1}{\log\left(\frac{1}{\mu}\right)}\left(\frac{h}{\mu} + \frac{1}{\log\left(\frac{1}{\mu}\right)}\right)\right)\right)\frac{\pi h}{C'(\mu)}.$$

Here we notice that $C'(\mu)$ is the time that needs a classical particle of energy $\mu$ for passing over the period. So in view of the behaviour of the amplitude of $\tau(\mu;h)$ we conclude that up to this modification the bands and the gaps exchange their roles.

## Description of the method

### 4. Reduction to a Normal Form - The Branching Model

From now on we will make an extensive use of the microlocal theory due to Sjöstrand (see [Sj 1]). The essential ideas and the terminology can be found in the appendices of [He,Sj 2] and [Mz].

The operator P given by (1.1) is now viewed as an h - pseudodifferential operator, whose (principal) symbol is

(4.1) $$p(x,\xi) = \frac{1}{2}\xi^2 + V(x).$$

p has a non-degenerate saddle point at (0,0). So we can apply the results of appendix b of [He,Sj 2]: There exists a real analytic canonical transformation $\varkappa$ from a neighbourhood of (0,0) onto a neighbourhood of (0,0) and a realvalued function $f_0(t)$, defined in a neighbourhood of 0 such that

(4.2) $$f_0(0) = 0, \qquad f_0'(0) = 1$$

and

(4.3) $$f_0 \circ p \circ \varkappa = p_0,$$

where

(4.4) $$p_0(x,\xi) := x\xi$$

is the (principal) symbol of the dilation generator $P_0 := \frac{1}{2}(xhD + hDx)$. We also have

(4.5) $$d\varkappa|_{(0,0)} = \varkappa_{\frac{\pi}{4}} = \text{the rotation by the angle } \frac{\pi}{4} \text{ around } (0,0).$$

Furthermore, there exist a realvalued (formal) classical analytic symbol

(4.6) $$F(t;h) = \sum f_j(t)\, h\ ,$$

defined for t in a neighbourhood of 0, and a formal unitary Fourier integral operator U associated to the canonical transformation $\varkappa$, mapping functions defined microlocally in some fixed neighbourhood of (0,0) (in a sense that can be made precise by means of FBI-tranformations) to functions defined microlocally in some other fixed neighbourhood of (0,0) such that

(4.7) $\quad U^{-1} F(P;h)U = P_0.$

In this equation F(P;h) is defined by a functional calculus based on Cauchy's integral formula such that both members may be considered as analytic pseudodifferential operators with symbols defined in a neighbourhood of (0,0) and hence are acting on functions defined microlocally in some fixed neighbourhood of (0,0).

Let $\Gamma: u \longmapsto \bar{u}$ be the complex conjugation, $\mathcal{F}_h$ the h-Fourier-transformation and put $A_0 := \mathcal{F}_h\Gamma$. Then, since $[P_0,A_0] = [P,\Gamma] = 0$, we are able to modify the proof of (4.7) such that we find U (as above) satisfying

(4.8) $\quad \Gamma U = UA_0.$

U may be represented more explicitly by the (formal) expression

(4.9) $\quad Uu(x) = 2^{\frac{1}{4}} e^{i\frac{\pi}{8}} (2\pi h)^{-\frac{1}{2}} \int e^{\frac{i}{h}\psi(x,y)} \sigma(x,y;h)u(y)\, dy,$

where the phase function $\psi$ is analytic near (0,0) and is generating $\varkappa$:

(4.10) $\quad \varkappa: (y,-\psi'_y(x,y)) \longmapsto (x,\psi'_x(x,y)),$

where by (4.5)

(4.11) $\quad \psi(x,y) = -\frac{x^2}{2} + \sqrt{2}\,xy - \frac{y^2}{2} + O((x,y)^3).$

(4.8) implies

(4.12) $\quad$ (i) $y = \psi'_y(x,\psi'_y(x,y)),$ $\qquad$ (ii) $\psi'_x(x,y) = -\psi'_x(x,\psi'_y(x,y)).$

$\sigma(x,y;h) \sim \sigma_0(x,y) + h\sigma_1(x,y) + h\,\sigma_2(x,y) + \ldots.$ is a c.a.s. with $\sigma_0(0,0) = 1$, and from (4.8) we get

(4.13) $$\frac{\overline{\sigma_0(x,y)}}{|\psi''_{yy}(x,y)|^{\frac{1}{4}}} = \frac{\sigma_0(x,\psi'_y(x,y))}{|\psi''_{yy}(x,\psi'_y(x,y))|^{\frac{1}{4}}}.$$

We will sketch now, why $f_1 \equiv 0$. Let be $P = p(x,\xi) + p_1(x,\xi)h + \ldots.$ a real-valued classical analytic symbol, defined near (0,0). Assume that (0,0) is a saddle point for p with critical value 0. In view of the definition of f(P), when f is a holomorphic function near 0, we get for the Weyl-symbol of f(P)

(4.14) $\quad \sigma(f(P)) = f(P(x,\xi;h)) + O(h^2).$

Furthermore we may replace $\varkappa$ by an *h-dependent* canonical transformation $\tilde{\varkappa}_U$, such that

(4.14) $\quad F(P(x,\xi;h);h)\circ\tilde{\varkappa}_U = p_0 + O(h^2).$

We apply this to action integrals, i.e. integrals of the form $I_q(\mu) := \int \xi dx$, where the integration is taken over some closed not necessarily real curve in $q^{-1}(\mu)$. Let $\mu$ and $\mu'$ be related by (3.2), and let $\tilde{p}_0$ be the left hand side of (4.14). We then have

$$(4.15) \qquad 2\pi i\mu_0' = I_{p_0}(\mu_0') = I_p(\mu) = I_{\tilde{p}_0}(\mu') = I_{p_0}(\mu') + \mathcal{O}(h^2) = 2\pi i\mu' + \mathcal{O}(h^2),$$

hence $f_1 \equiv 0$. Transforming $I_p(\mu)$ into an integral between turning points we verify (3.3).

## 5. Treatment of the Equation $(P - \mu)u = 0$

We will now use the following special solutions of the equation $(P_0 - \mu')v = 0$:

$$(5.1) \qquad u_\pm^0(x) = H(\pm x)\,|x|^{-\frac{1}{2} + i\frac{\mu'}{h}}, \qquad\qquad w_\pm^0 = B_0 u_\pm^0,$$

where $B_0 := \Gamma\mathcal{F}_h$. Any solution $v \in \mathcal{S}'$ of $(P_0 - \mu')v = 0$ is of the form $v = \alpha_+ u_+^0 + \alpha_- u_-^0 = \beta_+ w_+^0 + \beta_- w_-^0$, where the coefficients are related by

$$(5.2) \qquad \begin{pmatrix}\beta_+\\ \beta_-\end{pmatrix} = B_{\mu'/h}\begin{pmatrix}\alpha_+\\ \alpha_-\end{pmatrix}.$$

B is a unitary, symmetric matrix; so it is only necessary to know the matrix element $b_{11}$:

$$(5.3) \qquad b_{11} = e^{i(\frac{\mu'}{h}\log h - \frac{\pi}{4})}\, e^{\frac{\pi\mu'}{2h}}\,(2\pi)^{-\frac{1}{2}}\,\Gamma\!\left(\tfrac{1}{2} + i\tfrac{\mu'}{h}\right).$$

With respect to our microlocal framework we see that $u_+^0$, $w_+^0$, $u_-^0$ and $w_-^0$ are defined microlocally in some $\mu'$-independent neighbourhood $\Omega'$ of $(0,0)$ and that they are uniformly (with respect to $\mu'$) microlocally concentrated to small neighbourhoods of $\{(y,0) \mid y \ge 0\} \cup \{(0,\eta) \mid \eta \in \mathbb{R}\}$, $\{(0,\eta) \mid \eta \ge 0\} \cup \{(y,0) \mid y \in \mathbb{R}\}$, $\{(y,0) \mid y \le 0\} \cup \{(0,\eta) \mid \eta \in \mathbb{R}\}$, $\{(0,\eta) \mid \eta \le 0\} \cup \{(y,0) \mid y \in \mathbb{R}\}$ respectively, where these neighbourhoods may be taken arbitrarily small, if we choose $|\mu'|$ sufficiently small.

Now we put

$$(5.4) \qquad \begin{aligned} u_{++} &:= Uu_+^0 & \qquad u_{--} &:= Uu_-^0 \\ u_{+-} &:= Uw_-^0 & \qquad u_{-+} &:= Uw_+^0. \end{aligned}$$

We know that these $u_{\pm\pm}$ are solutions of

$$(5.5) \qquad (P - \mu)u = 0,$$

microlocally defined in a neighbourhood $\Omega$ of $(0,0)$. The equation (5.5) is valid uniformly with respect to $\mu$ (small enough) in the sense that after applying an FBI-transform we get an analogue of (5.5), valid locally and with a uniformly exponentially decreasing error. Furthermore the $u_{++}$, $u_{-+}$, $u_{--}$ and $u_{+-}$ are microlocally concentrated to small neighbourhoods of $\gamma_{++}^\Omega \cup \gamma_{-+}^\Omega \cup \gamma_{+-}^\Omega$, $\gamma_{-+}^\Omega \cup \gamma_{--}^\Omega \cup \gamma_{++}^\Omega$, $\gamma_{--}^\Omega \cup \gamma_{+-}^\Omega \cup \gamma_{-+}^\Omega$ and $\gamma_{+-}^\Omega \cup \gamma_{++}^\Omega \cup \gamma_{--}^\Omega$ respectively, where $\gamma_{\pm\pm}^\Omega := \{(x,\xi) \in p^{-1}(0) \cap \Omega \mid \pm x \ge 0,\ \pm\xi \ge 0\}$ is one of the four bicharacteristic segments going out from $(0,0)$. So the microlocal theory of [Sj 1] tells us that the $u_{\pm\pm}$ are even well-defined as functions on an interval containing 0 in its interior, up to errors $r_{\pm\pm}(x,h)$, which are uniformly (w.r.t. $\mu$) of exponential decrease in some complex neighbourhood of 0, and satisfying (5.5) up to errors of the same type.

Now we will study $u_{\pm\pm}$ more closely outside 0. Using the definition of $A_0, B_0$, (4.8), (5.1) and that $\mathcal{F}_h^2 u_+^0(x) = u_+^0(-x) = u_-^0(x)$, we obtain

$$(5.6) \qquad u_{-+} = \Gamma u_{--}, \qquad\qquad u_{+-} = \Gamma u_{++}.$$

Hence it is enough to study $u_{++}$ and $u_{--}$.

According to the definition of $u_+^0$ and the expression (4.9) defining U, we write formally

$$(5.7) \qquad u_{++}(x) = 2^{\frac{1}{4}} e^{i(\frac{\pi}{8}+\lambda)} (2\pi h)^{-\frac{1}{2}} \int_0^\infty e^{\frac{i}{h}(\psi(x,y)+\mu_0' \log y)} \sigma(x,y;h) e^{\frac{i}{h}(\mu'-\mu_0')\log y} |y|^{-\frac{1}{2}} \, dy.$$

The critical point $y_c(x,\mu)$ of the phase $y \longmapsto \psi(x,y) + \mu_0' \log y$ is uniquely determined and holomorphic near $x_0 > 0$, $\mu = 0$, and the critical value $\varphi(x,\mu)$ satisfies the eiconal equation

$$(5.8) \qquad p(x,\varphi_x') - \mu = 0.$$

Near $x_0 > 0$ we can decompose $u_{++}$ into the sum of two functions microlocally concentrated near $\gamma_{++}^\Omega$ and $\gamma_{+-}^\Omega$ respectively. Since $(x,\varphi_x'(x,\mu))$ lies on $\gamma_{++}^\Omega$, the first one is precisely that one, we obtain by writing down the stationary phase expansion of (5.7) associated to the critical point $y_c(x,\mu)$. Thus near $x_0 > 0$ the contribution to $u_{++}$ from a neighbourhood of $\gamma_{++}^\Omega \cap \Pi_x^{-1}(x_0)$ (where $\Pi_x$ is the projection $(x,\xi) \longmapsto x$) is of WKB-form:

$$(5.9) \qquad u_{++}(x) = e^{\frac{i}{h}\varphi(x,\mu)} b(x,\mu;h)$$

with an c.a.s. b (in the $(x,\mu)$-space) of order 0, satisfying $\arg b_0 = -\frac{\pi}{8}$. In the same manner we get

$$(5.10) \qquad u_{--}(x) = e^{\frac{i}{h}\tilde{\varphi}(x,\mu)} d(x,\mu;h)$$

near $\gamma_{--}^\Omega \cap \Pi_x^{-1}(-x_0)$ with a c.a.s. d of order 0, satisfying $\arg d_0 = -\frac{\pi}{8}$. Here $\tilde{\varphi}$ is another solution of (5.8), that (like $\varphi$) can be written down explicitly.

(5.9) and (5.10) extend to $\gamma_{++}^\Omega \cap \Pi_x^{-1}(\dot{I}_\mu)$, $\gamma_{--}^\Omega \cap \Pi_x^{-1}(\dot{I}_\mu - 2\pi)$ respectively, for each of the transport equations , determining the $b_j$ (resp. $d_j$), can be solved over the whole interior of the corresponding well.

## 6. Computation of the Translation matrix

First we remark that $u_{++}$ and $u_{--}$ are independent in the sense that the Wronskian satisfies:

$$(6.1) \qquad |W(u_{++},u_{--})| \geq \frac{1}{C_\varepsilon} e^{-\frac{1}{h}(\eta(\mu)+\varepsilon)},$$

uniformly on a neighbourhood of $[0,2\pi]$ for every $\varepsilon > 0$, where $\eta$ is a continuous function with $\eta(0) = 0$. So it makes sense to compute the translation matrix with respect to $u_{++}$, $u_{--}$, which describes the exact operator of translation acting on the solution space of $(P-\mu)u = 0$ up to an exponentially small error.

Since $u_{++}$ and $\tau_{2\pi}u_{-+}$ are WKB-solutions along $\gamma_{++}$, we have there

(6.2) $\qquad u_{++} = t\tau_{2\pi}u_{-+} \qquad$ with $\qquad t = e^{id(\mu)/h}s(\mu;h),$

where s is an analytic symbol of order 0 and $d(\mu)$ a real valued function. By a normalization argument of [He,Sj 2] (see also [Sj 2]) we can prove that

(6.3) $\qquad |t| = 1.$

Fixing some $x_0 \in ]0,2\pi[$, we get (for $|\mu|$ small enough):

(6.4)
$$\begin{aligned}\arg t &= \frac{1}{h}\left(\varphi(x_0,\mu) + \tilde{\varphi}(x_0 - 2\pi,\mu)\right) + \arg b(x_0,\mu) + \arg d(x_0,\mu)\\ &= \frac{1}{h}\left(\varphi(x_0,\mu) + \tilde{\varphi}(x_0 - 2\pi,\mu)\right) - \frac{\pi}{4} + hr(\mu;h),\\ &= \frac{1}{h}\left(C(\mu) + \mu_0'(\log|\mu_0'| - 1)\right) - \frac{\pi}{4} + hr(\mu;h),\end{aligned}$$

where $C(\mu)$ is given by (2.7) and $r(\mu;h)$ is a c.a.s. of order 0.

Recalling (5.2) and the fact that the matrix B is symmetric, we get

(6.5) $\qquad u_{++} = b_{11}u_{-+} + b_{12}u_{+-}.$

So microlocally near $\gamma_{+-} \cap \Pi_x^{-1}(\dot{I}_\mu)$ we have

(6.6) $\qquad u_{++} = b_{12}u_{+-} = b_{12}\overline{u_{++}},$

where in the last member we think of $u_{++}$ as defined microlocally near $\gamma_{++} \cap \Pi_x^{-1}(\dot{I}_\mu)$.
Combining this with (6.2), we see that microlocally near $\gamma_{+-} \cap \Pi_x^{-1}(\dot{I}_\mu)$

(6.7) $\qquad u_{++} = b_{12}\bar{t}\,\tau_{2\pi}u_{--}.$

The next work to do is to extend $u_{++}$ further to the right, to a neighbourhood of $2\pi$. Such an extension should be of the form

(6.8) $\qquad u_{++} = t\tau_{2\pi}u_{-+} + \tilde{t}\tau_{2\pi}u_{+-} = s\tau_{2\pi}u_{++} + \tilde{s}\tau_{2\pi}u_{--} \qquad$ (near $2\pi$).

Here the coefficient t is imposed by (6.2), and since from (6.6) $\tilde{s} = b_{12}\bar{t}$, we get by (5.2)

(6.9) $\qquad s = \dfrac{t - b_{12}^2\bar{t}}{b_{11}}.$

The same considerations give

(6.10) $\qquad u_{--} = -b_{12}\bar{t}\tau_{2\pi}u_{++} + b_{11}\bar{t}\tau_{2\pi}u_{--},$

such that the corresponding translation matrix is determined as:

(6.11)
$$\tilde{T}(\mu;h) = \begin{pmatrix} \dfrac{t - b_{12}^2\bar{t}}{b_{11}} & -b_{12}\bar{t} \\ b_{12}\bar{t} & b_{11}\bar{t} \end{pmatrix}.$$

Taking into account the properties of B we easily find

(6.12) $\qquad \tilde{\tau}(\mu;h) = \frac{1}{2}\operatorname{trace}\tilde{T}(\mu;h) = \operatorname{Re}\left(\frac{t}{b_{11}}\right).$

Inserting finally (6.4) and (5.3) we verify (3.1).

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