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The Feynman path integral as an improper integral  
over the Sobolev space

By

Daisuke Fujiwara

§ 1. Introduction.

Let  $L(\dot{x}, x) = \frac{1}{2}|\dot{x}|^2 - V(x)$ , be the Lagrangian with the potential  $V(x)$ . For the sake of simplicity of notations, we assume that the configuration space is of dimension 1. But the same results hold for n-dimensional case. The action of a path  $r: [0, t] \rightarrow \mathbb{R}$ ,  $r(0)=y$  and  $r(t)=x$ , with  $\dot{r} \in L^2(0, t)$  is the functional

$$S(r) = \int_0^t L(\dot{r}, r) ds.$$

The Feynman path integral is a formal integral over the path space  $\mathcal{P}$  :

$$\int_{\mathcal{P}} e^{ih^{-1}S(r)} \mathcal{D}[r].$$

Among various proposals to give mathematically rigorous meaning to the Feynman path integral, we adopt Itô's formulation [4] and [1].

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We denote by  $r_0$  the straight line segment joining  $y$  and  $x$ ,  
i.e.,

$$r_0(s) = \frac{s}{t} x + (1 - \frac{s}{t}) y.$$

We denote the Sobolev space  $H_0^1(0,t) = \{r : \dot{r} \in L^2(0,t), r(0) = r(t) = 0\}$  by  $\mathcal{H}$ , which is equipped with the inner product

$$(r_1, r_2)_{\mathcal{H}} = \int_0^t \dot{r}_1(s) \dot{r}_2(s) ds.$$

Let  $Q(r) = \sum_j \lambda_j e_j \otimes e_j$  be a positive quadratic form such that  $\{e_j\}_j$  is a complete orthonormal system (c.o.n.s.) and  $\{\lambda_j\}_j$  is a positive summable series. Let  $b$  be an arbitrary vector in  $\mathcal{H}$ . Let  $N(dr, b, Q)$  be the Gaussian measure on  $\mathcal{H}$  with the mean  $b$  and the variance  $Q$ . It is defined that

$$(1) \int_{\Omega} e^{ih^{-1}S(r)} \mathcal{D}[r] = \left(\frac{1}{2\pi i h t}\right)^{1/2} \lim_{n \rightarrow \infty} \prod_{j=1}^{\infty} \left(1 + \frac{n\lambda_j}{h^2}\right)^{1/2} \int_{\mathcal{H}} e^{ih^{-1}S(r_0+r)} N(dr, b, nQ),$$

if the right hand side converges.

Unfortunately, existence of the limit on the right hand side of (1) was proved only for potentials  $V(x)$  of the following two types:

$$1 \quad V(x) = ax^2 + bx + c.$$

$$2 \quad V(x) = \int e^{ix\xi} d\mu, \quad \text{where } \mu \text{ is a signed measure of finite total variation on } \mathbb{R}.$$

On the other hand Pauli [5] discussed the Feynman path integrals for potentials satisfying

$$|\text{grad } V(x)| = O(|x|) \quad \text{as } x \longrightarrow \infty.$$

We assume in the present note a little stronger condition for the potential:

$$(2) \quad |V^{(j)}(x)| < C_j, \quad \text{if } 2 \leq j.$$

We use the special quadratic form

$$Q(r) = \sum_{j=1}^{\infty} \lambda^{-j} |(r, e_j)|^2,$$

where  $\lambda > 1$  is a constant and  $\{e_j\}_j$  is the c.o.n.s. consisting of indefinite integrals of Haars' functions. Our main results is

Theorem. Let  $Q$  be as above and  $b$  be any element of  $\mathcal{H}$ . We assume that  $V$  satisfy the condition (2). If  $|t|$  is sufficiently small, then the right hand side of (1) converges to the limit independent of  $b$ . The limit  $K(t, x, y)$  is the fundamental solution of the Schrödinger equation:

$$\left\{ \left( \frac{\hbar}{i} \frac{\partial}{\partial t} \right) - \left( \frac{\hbar}{i} \frac{\partial}{\partial x} \right)^2 - V(x) \right\} K(t, x, y) = 0,$$

$$K(0, x, y) = \delta(x-y).$$

Here  $h$  is a small positive constant.

In the next section we will give more accurate statement of the result as well as outline of the proof.

§ 2. Outline of the proof.

Indefinite integrals of Haars' functions.

Let  $q=2^{-n}(2k+1)$ ,  $n=1, 2, 3, \dots$  and  $k=0, 1, 2, \dots, 2^{n-1}-1$ , be any finite binary fraction. We denote  $n$  by  $n(q)$  and  $k$  by  $k(q)$ . We set  $m(q)=2^{n(q)-1} + k(q)$ . For each such  $q$ , we define  $\delta_q=2^{-n(q)}t$  and

$$e_q(s) = \begin{cases} 0, & \text{for } |qt-s| \geq \delta_q, \\ (2\delta_q)^{-1/2}(\delta_q - |s - qt|) & \text{for } |s - qt| \leq \delta_q. \end{cases}$$

The system  $\{\frac{d}{ds} e_q(s)\}_q$  is Haars' c.o.n.s. in  $L^2(0,t)$ . Hence  $\{e_q\}$  forms a c.o.n.s. in  $\mathcal{H}$ . Note that

$$\|e_q\|_{L^\infty} = 2^{-1/2} \delta_q^{1/2}, \quad \|e_q\|_{L^1} = 2^{-1/2} \delta_q^{3/2}, \quad \|e_q\|_{L^2} = \frac{1}{2} 2^{-1/2} \delta_q$$

The Quadratic form Q.

Let  $\lambda > 1$ . Then

$$Q(r) = \sum_q \lambda^{-m(q)} y_q^2, \quad \text{where } y_q = (r, e_q)_{\mathcal{H}}.$$

Splitting of  $\mathcal{H}$ .

Let  $N_0 > \lambda+2$  be so large that

$$(2.1) \quad 2^N \lambda^{-2^{N-10}} < 1 \quad \text{for any } N \geq N_0.$$

We choose  $n$  so large that

$$(2.2) \quad n^{-1} \lambda^{2^{N_0}} < 1.$$

Let  $N_n$  be the positive integer such that

$$(2.3) \quad n^{-1} \lambda^{2^{N_n}} \leq 1 < n^{-1} \lambda^{2^{N_n+1}},$$

and  $m_n$  be the integer such that

$$(2.4) \quad n^{-1} \lambda^{m_n} < 2^{-N_n/4} < n^{-1} \lambda^{m_n+1}.$$

We can easily prove from (2.1), (2.3) and (2.4) that

$$(2.5) \quad 2^{N_n-1} < m_n < 2^{N_n+1}.$$

Let  $\mathcal{H}_1 = \text{span of } \{e_q : m(q) < m_n\}$  and  $\mathcal{H}_2 = \text{span of } \{e_q : m(q) \geq m_n\}$ .

Then we have orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2,$$

which reduces the quadratic form  $Q$ . For any  $r$  in  $\mathcal{H}$ , we denote the corresponding decomposition

$$r = r_1 + r_2.$$

For any  $b$  in  $\mathcal{H}$ , the Gaussian measure  $N(dr, b, nQ)$  coincides with the product measure

$$N(dr, b, nQ) = N(dr_1, b_1, nQ_1) \times N(dr_2, b_2, nQ_2).$$

New coordinates in  $\mathcal{H}_1$ . The subspace  $\mathcal{H}_1$  is of dimension  $m_n - 1$ . We arrange points  $\{qt\}_{m(q) < m_n}$  of  $[0, t]$  in its order of magnitude. Then we obtain a division of  $[0, t]$   $0 < t_1 < t_2 < \dots < t_{m_n-1} < t$ . We set  $t_0 = 0$  and  $t_{m_n} = t$ . The subspace  $\mathcal{H}_1$  coincides with the space of all piecewise linear paths with vertices at  $s = t_j$ ,  $j = 0, 1, \dots, m_n$ . So we introduce a new

coordinates  $x_j = r(t_j)$  of  $r$  with respect to the basis  $\{w_j\}_j$ , which is given by

$$w_j(s) = \begin{cases} 0 & \text{if } s \leq t_{j-1} \text{ or } t_{j+1} \leq s, \\ 1 & \text{if } s = t_j, \\ \text{linear} & \text{for } t_{j-1} \leq s \leq t_j \text{ and } t_j \leq s \leq t_{j+1}. \end{cases}$$

Any  $r_1$  in  $\mathcal{H}_1$  can be written into two forms:

$$r_1 = \sum_{m(q) < m_n} y_q e_q = \sum_{j=1}^{m_n-1} x_j w_j.$$

### Evaluation of $It\hat{O}$ 's integral.

Let  $b = b_1 + b_2$  be decomposition of  $b$  into  $\mathcal{H}_1$  and  $\mathcal{H}_2$ .

Using any  $r = r_1 + r_2$ , we make a new path  $r_1 + b_2$ . The action of it is

$$S(r_0 + r_1 + b_2) = \frac{1}{2} \int_0^t |\dot{r}_0|^2 ds + \|b_2\|_{\mathcal{H}}^2 + \|r_1\|_{\mathcal{H}}^2 - \int_0^t V(r_0 + r_1 + b_2) ds.$$

We can write this as

$$S(r_0 + r_1 + b_2) = \frac{1}{2} \frac{(x-y)^2}{t} + \|b_2\|_{\mathcal{H}}^2 + \sum_{j=1}^{m_n} S_j(x_j, x_{j-1}; b_2),$$

where

$$S_j(x_j, x_{j-1}; b_2) = \frac{|x_j - x_{j-1}|^2}{2\Delta t_j} - \Delta t_j \omega_j(x_j, x_{j-1}; b_2)$$

with  $\Delta t_j = t_j - t_{j-1}$  and

$$\omega_j(x_j, x_{j-1}; b_2) = (\Delta t_j)^{-1} \int_{t_{j-1}}^{t_j} V(x_j w_j + x_{j-1} w_{j-1} + r_0 + b_2) ds.$$

The Hessian Hess(S) of  $S(r_0+r_1+b_2)$  with respect to  $x_1, \dots, x_{m_n-1}$  equals  $\text{Hess}(S) = H - W = H(I-H^{-1}W)$ , where

$$H = \begin{pmatrix} \frac{1}{\Delta t_1} + \frac{1}{\Delta t_2}, & -\frac{1}{\Delta t_2}, & 0, & 0 & \dots\dots\dots \\ -\frac{1}{\Delta t_2}, & \frac{1}{\Delta t_2} + \frac{1}{\Delta t_3}, & -\frac{1}{\Delta t_3}, & 0, & \dots\dots \\ 0, & -\frac{1}{\Delta t_3}, & \frac{1}{\Delta t_3} + \frac{1}{\Delta t_4}, & -\frac{1}{\Delta t_4}, & \\ 0, & 0, & \dots\dots\dots & & \end{pmatrix}$$

and  $W$  is the Hessian of  $\sum_{j=1}^{m_n} t_j \omega_j(x_j, x_{j-1}; b_2)$ .

If  $|t|$  is small  $(I-H^{-1}W)$  is invertible. Hence the functional

$r_1 \rightarrow S(r_0+r_1+b_2)$  has only one critical point at  $r_1=r_1^*$

$= \sum_j x_j^* w_j$ . which is given by

$$\partial_{x_j} S_j(x_j^*, x_{j-1}^*) + \partial_{x_j} S_{j+1}(x_{j+1}^*, x_j^*) = 0, \quad 1 \leq j \leq m_n - 1,$$

where  $x_{m_n}^* = x_0^* = 0$ .

For any finite binary fraction  $q$  with  $m(q) > m_n$ , we define

$$\xi_q(r_0+r_1+b_2) = \int_0^t v'(r_0(s)+r_1(s)+b_2(s)) e_q(s) ds$$

and  $\xi_q^*(r_0+b_2) = \xi_q(r_0+r_1^*+b_2)$ .

The splitting  $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2$

reduces the quadratic form  $Q$  into  $Q_1$  and  $Q_2$  and it reduces the



Gaussian measure  $N(d\tau, b, nQ)$ :

$$N(d\tau, b, nQ) = N(d\tau_1, b_1, nQ_1) \times N(d\tau_2, b_2, nQ_2).$$

It is clear that

$$N(d\tau_1, b_1, nQ_1) = \prod_{j=1}^{m_n-1} \left( \frac{1}{2\pi\Delta t_j} \right)^{1/2} \exp \left( -\frac{1}{2n} \sum_{\substack{m(q) \\ m(q) < m_n}} \lambda^{m(q)} (y_q - b_q)^2 \right) \prod_{j=1}^{m_n-1} dx_j.$$

The variance of this Gaussian measure is  $nQ_1$ , which is large.

So the density of this with respect to the Lebesgue measure can be regarded as an amplitude of an oscillatory integral. On the other hand  $N(d\tau_2, b_2, nQ_2)$  has variance relatively small. The delta measure concentrated at  $\tau_2 = b_2$  is a good approximation to  $N(d\tau_2, b_2, nQ_2)$ . Thus we have

Theorem 2.1. If  $|t|$  is sufficiently small,

$$\begin{aligned} & (2\pi i h)^{-1/2} \prod_q \left( 1 + \frac{n\lambda^{-m(q)}}{i h} \right)^{1/2} \int_{\mathcal{H}} e^{i h^{-1} S(\tau_0 + \tau)} N(d\tau, nQ, b) \\ = & \exp \left( -\frac{1}{2} \sum_{\substack{m(q) \\ m(q) \geq m_n}} \frac{n\lambda^{-m(q)}}{h^2 + i h \lambda^{-m(q)}} (\xi_q^* - b_q)^2 - \sum_{\substack{m(q) \\ m(q) < m_n}} \frac{\lambda^{m(q)}}{2n} (y_q^* - b_q)^2 \right) \\ & \cdot \prod_{j=1}^{m_n} \left( \frac{1}{2\pi i h \Delta t_j} \right)^{1/2} \int_{\mathbb{R}^{m_n-1}} e^{i h^{-1} S(\tau_0 + \tau_1 + b_2)} \prod_{j=1}^{m_n-1} dx_j \\ & + \left( \frac{1}{2\pi i t h} \right)^{1/2} e^{i h^{-1} S(\tau_0 + \tau_1^* + b_2)} r_n(t, x, y), \end{aligned}$$

where  $y_q^* = (\tau_1^*, e_q)_{\mathcal{H}}$ . We have the estimate

$$\left| \partial_x^\alpha \partial_y^\beta r_n(t, x, y) \right| \leq C_{\alpha\beta}(h) 2^{-N_n/8} t.$$

The last term of the right hand side of Theorem goes to 0 as  $n$  goes to  $\infty$ . While the exponential factor of the first term tends to 1 as  $n$  goes to  $\infty$ . In fact we have

Theorem 2.2. Let  $\delta_n = 2^{-N_n t}$ . Then we have the following estimates:

$$\begin{aligned} \left| \partial_x^\alpha \partial_y^\beta \varepsilon_q^* \right| &\leq C_{\alpha\beta} \delta_n^{3/2} (1+|x|+|y|)^{(1-\alpha-\beta)^+}, \\ \left| \sum_{m(q) \geq m_n} \frac{n \lambda^{-m(q)}}{h^2 + i n h \lambda^{-m(q)}} (\varepsilon_q^* - b_q)^2 \right| &< C (1+|x|^2 + |y|^2 + \|b_2\|_{\mathcal{L}}^2) \delta_n^{3/2}, \\ \left| \partial_x^\alpha \partial_y^\beta y_q^* \right| &< C_{\alpha\beta} (1+|x|+|y| + \|b\|_{\mathcal{L}}) (1-\alpha-\beta)^+ \delta_n^{3/2}, \\ \sum_{m(q) < m_n} \frac{\lambda^{m(q)}}{n} (y_q^* - b_q)^2 &< C (1+|x|^2 + |y|^2 + \|b_2\|_{\mathcal{L}}^2) \delta_n. \end{aligned}$$

Here  $a^+$  denotes  $\max(a, 0)$ .

Theorem 2.1 and Theorem 2.2 imply that the decisive role is played by the oscillatory integral

$$J_n = \prod_{j=1}^{m_n} \left( \frac{1}{2\pi i h \Delta t_j} \right)^{1/2} \int_{\mathbb{R}^{m_n-1}} e^{i h^{-1} S(r_0 + r_1 + b_2)} \prod_{j=1}^{m_n-1} dx_j.$$

Similar integrals are treated previously in [3]. We have

Theorem 2.3. If  $|t|$  is small enough, we can rewrite

$$J_n = \left( \frac{1}{2\pi i h t} \right)^{1/2} e^{i h^{-1} S(r_0 + r_1^* + b_2)} (1 + k_n(t, x, y)),$$

For any  $K > 0$ , there exists  $C_K$  such that we have

$$\left| \partial_x^\alpha \partial_y^\beta k_n(t, x, y) \right| \leq \prod_{j=1}^{m_n} (1 + C_K \Delta t_j |t|)^{-1},$$

if  $\alpha$  and  $\beta$  are smaller than  $K$ .

Convergence as  $n$  goes to  $\infty$ . In order to discuss convergence of the oscillatory integral  $J_n$ . We compare  $S(r_0 + r_1 + b_2)$  with the action of piecewise classical orbit. Let

$$\Delta; 0 = t_0 < t_1 < \dots < t_{m_n-1} < t_{m_n} = t$$

be the division of the  $[0, t]$  as above. Let  $r_\Delta$  be the piecewise classical orbit

$$\begin{aligned} r_\Delta(s) + \partial_x V(r_\Delta) &= 0 & \text{for } t_{j-1} < s < t_j, \\ r_\Delta(t_j) &= x_j + r_0(t_j), & \text{for } j=0, 1, \dots, m_n, \end{aligned}$$

with  $x_0 = x_{m_n} = 0$ . The action along  $r_\Delta$  is

$$S_{cl}(t, x, x_{m_n-1}, \dots, x_1, y) = \sum_{j=1}^{m_n-1} S_j(t_j, x_j, x_{j-1}),$$

where

$$S_j(t_j, x_j, x_{j-1}) = \int_{t_{j-1}}^{t_j} L(r_\Delta(s)) ds.$$

The critical point of  $S_{cl}(t, x, x_{m_n-1}, x_{m_n-2}, \dots, x_1, y)$  with respect to vertices  $x_{m_n-1}, x_{m_n-2}, \dots, x_1$  is unique if  $|t|$  is small. The critical level of it equals the classical action

$$S_{cl}(t, x, y) = \int_0^t L(r_{cl}(s)) ds,$$

where  $r_{cl}$  is the classical orbit joining  $r(0)=y$  and  $r(t)=x$ .

Theorem 2.4. If  $|t|$  is small, we have

$$\prod_{j=1}^{m_n} \left( \frac{1}{2\pi i \hbar \Delta t_j} \right)^{1/2} \int_{\mathbb{R}^{m_n-1}} e^{i\hbar^{-1} S_{cl}(t, x, x_{m_n-1}, \dots, x_1, y)} \prod_{j=1}^{m_n-1} dx_j \\ = \left( \frac{1}{2\pi i \hbar t} \right)^{1/2} e^{i\hbar^{-1} S_{cl}(t, x, y)} (1 + k^\#(\Delta, t, x, y)).$$

For any  $K > 0$ , There exists a constant  $C_K > 0$  such that

$$\left| \partial_x^\alpha \partial_y^\beta k^\#(\Delta, t, x, y) \right| < \prod_{j=1}^{m_n} (1 + C_K \Delta t_j |t|) - 1,$$

if  $\alpha$  and  $\beta < K$ .

The next theorem compares  $k^\#(\Delta, t, x, y)$  with  $k_n(t, x, y)$ .

Theorem 2.5. For any  $K > 0$ , there exists a constant  $C_K$  such that

$$\left| \partial_x^\alpha \partial_y^\beta (k^\#(\Delta, t, x, y) - k_n(t, x, y)) \right| < C \delta_n^{1/8} (1 + |x| + |y| + \|b_2\|_{\mathcal{H}}).$$

If we make the division of interval finer,  $k^\#(\Delta, t, x, y)$  forms a Cauchy sequence. More precisely, we have

Theorem 2.6. Let  $\Delta_1$  be a subdivision of  $\Delta$ , then

$$\left| \partial_x^\alpha \partial_y^\beta (k^\#(\Delta, t, x, y) - k^\#(\Delta_1, t, x, y)) \right| \leq C_K |t| |\Delta|,$$

where  $|\Delta| = \max_{1 < j < m_n} |\Delta t_j|$ . We set

$$k(t, x, y) = \lim_{|\Delta| \rightarrow 0} k^\#(\Delta, t, x, y).$$

Then  $k(t, x, y)$  satisfies

$$\left| \partial_x^\alpha \partial_y^\beta k(t, x, y) \right| < C t.$$

Combining all these, we can prove

Theorem 2.7.

$$\begin{aligned} & \lim_{n \rightarrow \infty} (2\pi i \hbar)^{-1/2} \prod_q \left(1 + \frac{n\lambda}{i\hbar}\right)^{-m(q)} 1/2 \int_{\mathcal{H}} e^{i\hbar^{-1}S(r_0+r)} N(d_r, nQ, b) \\ & = \left(\frac{1}{2\pi i \hbar}\right)^{1/2} e^{i\hbar^{-1}S_{cl}(t, x, y)} (1+k(t, x, y)), \end{aligned}$$

which is the fundamental solution of the Schrödinger equation.

### Appendix

In proving above results, we use the following stationary phase method in a space of large dimension. We consider the following oscillatory integrals

$$I(S, a, \nu) = \prod_{k=1}^L \left(\frac{\nu}{2\pi i \tau_k}\right)^{1/2} \int_{\mathbb{R}^{L-1}} e^{i\nu S(x_L, \dots, x_0)} a(x_L, \dots, x_0) \prod_{j=1}^{L-1} dx_j.$$

The phase function  $S(x_L, \dots, x_0)$  is a real valued function of the form

$$S(x_L, x_{L-1}, \dots, x_0) = \sum_{j=1}^{L-1} S_j(x_j, x_{j-1}),$$

with

$$S_j(x_j, x_{j-1}) = \frac{1}{2t_j} (x_j - x_{j-1})^2 + t_j \omega_j(x_j, x_{j-1}), \quad j=1, 2, \dots, L.$$

We assume further that

$$\sup_{x, y} \left| \partial_x^\alpha \partial_y^\beta \omega_j(x, y) \right| \leq C_{\alpha\beta},$$

if  $0 \leq \alpha, \beta \leq K$  and  $\alpha + \beta \geq 2$ .

If  $T = t_1 + t_2 + \dots + t_L$  is small enough, the critical point  $(x_{L-1}^*, x_{L-2}^*, \dots, x_1^*)$  of the phase is the unique solution of

$$\partial_{x_j} S_{j+1}(x_{j+1}^*, x_j^*) + \partial_{x_j} S_j(x_j^*, x_{j-1}^*) = 0, \quad j=1, 2, \dots, L-1,$$

where  $x_L^* = x_L$  and  $x_0^* = x_0$ . We use the following notation

$$a(\overline{x_L}, x_0) = a(x_L, x_{L-1}^*, \dots, x_1^*, x_0).$$

Let  $k < l$  and  $x_{k+1}^*, \dots, x_{l-1}^*$  be the partial critical point, i.e.,

$$\partial_{x_j} S_{j+1}(x_{j+1}^*, x_j^*) + \partial_{x_j} S_j(x_j^*, x_{j-1}^*) = 0, \quad \text{for } j=k+1, \dots, l-1,$$

where  $x_k^* = x_k$  and  $x_l^* = x_l$ . Then we set

$$a(x_L, \dots, \overline{x_l}, x_k, \dots, x_0) = a(x_L, \dots, x_l, x_{l-1}^*, \dots, x_{k+1}^*, x_k, \dots, x_0).$$

We assume that for any sequence of positive integers

$$j_0=0 < j_1 < j_2 < \dots < j_r < L$$

and positive  $K$  we have the estimate

$$\left| \partial_{x_0}^{\alpha_0} \partial_{x_{j_1}}^{\alpha_1} \dots \partial_{x_{j_r}}^{\alpha_r} \partial_{x_L}^{\alpha_L} a(\overline{x_L, x_{j_s}, \overline{x_{j_s-1}, x_{j_s-1}}, \dots, \overline{x_{j_1}, x_0}) \right| < A_K,$$

if each  $\alpha_j \leq K$ ,  $j=0, 1, \dots, r$  and  $L$ .

Theorem. Under the assumptions above there is a positive constant  $\delta$  such that if  $T=t_1+t_2+\dots+t_L < \delta$

$$I(S, a, \nu) = \left( \frac{\nu}{2\pi i T} \right)^{1/2} e^{i\nu S(\overline{x_L, x_0})} \det(I+H^{-1}W)^{-1/2} (a(\overline{x_L, x_0}) + r(x_L, x_0)),$$

where  $r(x_L, x_0)$  satisfies the estimate:

for any  $\alpha_0, \alpha_L \leq K$ ,

$$\left| \partial_{x_0}^{\alpha_0} \partial_{x_L}^{\alpha_L} r(x_L, x_0) \right| \leq A_{3K+10} \left( \prod_{j=1}^L (1 + C_K \nu^{-1} |t_j T|) - 1 \right),$$

Constants  $\delta$  and  $C_K$  are independent of the dimension  $L$  of the space.  $H$  is the matrix

$$H = \begin{pmatrix} \frac{1}{t_1} + \frac{1}{t_2}, & -\frac{1}{t_2}, & 0, & 0, & \dots, & 0 \\ -\frac{1}{t_2}, & \frac{1}{t_2} + \frac{1}{t_3}, & -\frac{1}{t_3}, & 0, & \dots, & 0 \\ 0, & -\frac{1}{t_3}, & \frac{1}{t_3} + \frac{1}{t_4}, & -\frac{1}{t_4} & & \\ 0, & 0, & \dots & & & \end{pmatrix}$$

and  $W$  is the Hessian matrix of  $\sum_j t_j \omega_j(x_j, x_{j-1})$  at the

critical point of  $S(x_L, x_{L-1}, \dots, x_1, x_0)$  with respect to  $x_{L-1}, x_{L-2}, \dots, x_1$ .

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