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Hiroshi Isozaki

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Some remarks on the multi-dimensional Borg-Levinson theorem

Hiroshi Isozaki

Department of Mathematics.
Osaka University,
Toyonaka, 560, Japan.

1.1-dim-case. The Borg-Levinson theorem is a uniqueness theorem in inverse eigenvalue problems. We first recall the 1-dim case. Consider the Dirichlet problem :

$$\begin{cases} -y'' + q(x)y = \lambda y, \ 0 \leqslant x \leqslant 1, \\ y(0) = y(1) = 0, \end{cases}$$

$q(x)$ being a real function. Let

$$\lambda_1 < \lambda_2 < \ldots$$

be the eigenvalues. Now suppose for two potentials q_1, q_2,

$$\lambda_i(q_1) = \lambda_i(q_2) \text{ for all } i \geqslant 1.$$

One can then easily see that it does not necessarily imply $q_1 = q_2$. To derive the uniqueness of the potential, we must add some auxiliary conditions.

Let $y = y(x,\lambda) = y(x,\lambda,q)$ be the solution of the Cauchy problem :

$$\begin{cases} -y'' + q(x)y = \lambda y, \ 0 \leqslant x \leqslant 1, \\ y(0) = 0, \ y'(0) = 1. \end{cases}$$

Then we have

THEOREM (Borg-Levinson). Suppose that

$$\lambda_i(q_1) = \lambda_i(q_2) \text{ for all } i \geqslant 1,$$

$$y'(1,\lambda_i,q_1) = y'(1,\lambda_i,q_2) \text{ for all } i \geqslant 1.$$

Then $q_1 = q_2$.

This is a starting point of 1-dim. inverse problems ([1],[2]). The recent article of Pöschel-Trabowitz [3] gives a deep insight. It is proved that the map

$$q \to \{\lambda_i\}_{i=1}^{\infty} \times \{\log|y'(1,\lambda_i,q)|\}_{i=1}^{\infty}$$

defines an analytic isomorphism from $L^2(0,1)$ to a Hilbert space of infinite sequences. And also, for any fixed potential p, the set defined by

$$M(p) = \{q\ ;\ \lambda_i(q) = \lambda_i(p) \text{ for } \forall i \geq 1\}$$

is a real analytic manifold (isospectral manifold) with the system of coordi- nates $\{\log|y'(1,\lambda_i,q)|\}_{i=1}^{\infty}$.

Since $y(x,\lambda_i,q)$ is an eigenfunction of $-\dfrac{d^2}{dx^2}+q(x)$ associated with the eigenvalue λ_i, one can see that there is a one to one correspondance between the potential and the eigenvalues and the normal derivatives of eigenfunctions.

2. n-dim. case. Next we turn to the n-dim. case $(n \geq 2)$. Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary S. Consider the Dirichlet problem :

$$\begin{cases} (-\Delta+q)u = \lambda u \text{ in } \Omega, \\ u|_S = 0. \end{cases}$$

Although we treat the Dirichlet problem here, all of the arguments below also hold for the Neumann or Robin boundary conditions by a suitable modification.

Let $\lambda_1 < \lambda_2 \leq \ldots$ be the eigenvalues. To derive the uniqueness theorem corresponding to the 1-dim. case, we consider the normal derivatives of eigenfunctions. However, one must be careful to choose a system of eigenfunctions, since in the multi-dimensional case eigenvalues are not simple in general.

Let m be the multiplicity of λ_i and $u_1,\ldots,u_m$ be a real-valued orthonormal eigenfunctions system associated with λ_i. We set

$$E_i = \left\{\left(\frac{\partial u_1}{\partial \nu}, \ldots, \frac{\partial u_m}{\partial \nu}\right)\right\}.$$

ν being the outer unit normal to S. One can then see that for two such systems of eigenfunctions $\{u_1,\ldots,u_m\}$, $\{v_1,\ldots,v_m\}$, there exists an orthogonal matrix $T \in O(m)$ such that

$$\left(\frac{\partial u_1}{\partial \nu}, \ldots\ldots, \frac{\partial u_m}{\partial \nu}\right) = \left(\frac{\partial v_1}{\partial \nu}, \ldots\ldots, \frac{\partial v_m}{\partial \nu}\right) T.$$

Now, this defines an equivalent relation $\sim$ in the space of functions on the boundary S. Further, it shows that for the set $\{E_i\}$, the totality of E_i, there corresponds only one equivalence class, which we denote by W_i:

$$W_i = \{E_i\}/\sim.$$

Then, we have the following theorem due to Nachman-Sylvester-Uhlmann [4].

THEOREM A. Let $q_1, q_2 \in C^\infty(\overline{\Omega})$. Suppose that

$$\lambda_i(q_1) = \lambda_i(q_2) \text{ for } \forall i \geq 1,$$
$$W_i(q_1) = W_i(q_2) \text{ for } \forall i \geq 1.$$

Then $q_1 = q_2$.

This theorem seems to be a direct generalization of the 1-dimensional Borg-Levinson theorem. So, it is natural to ask : does the map $q \rightarrow \{\lambda_i\} \times \{W_i\}$ define an isomorphism ? Can $\{W_i\}$ be the coordinates of the isospectral set of potentials ? The answer is always negative. In fact, we have

THEOREM B. Let $q_1, q_2 \in C^\infty(\overline{\Omega})$. Suppose that there exists an $N > 0$ such that

$$\lambda_i(q_1) = \lambda_i(q_2) \text{ for } \forall i > N,$$
$$W_i(q_1) = W_i(q_2) \text{ for } \forall i > N.$$

Then $q_1 = q_2$.

In other words, if λ_i and W_i are equal except for a finite number of indices i, the potentials are equal. It also shows that the totality of λ_i and W_i is too much to determine the potential. It is a common belief that, in contrast to the 1-dim. case, the multi-dimensional inverse eigenvalue problem has a sort of rigidity. Here one can find an exemple.

3. Proof of Theorem B. We sketch the proof of theorem B. Let $N(\lambda)$ be the Neumann operator, mamely,

$$N(\lambda)f = \frac{\partial v}{\partial \nu},$$

where v satisfies

$$\begin{cases} (-\Delta+q)v = \lambda v, \\ v|_S = f. \end{cases}$$

We introduce the following notation :

$$(f,g) = \int_\Omega f(x)\,\overline{g(x)}\,dx,$$

$$<f,g> = \int_S f(x)\overline{g(x)}\,dS,$$

$$\varphi_{\lambda,\omega}(x) = e^{i\sqrt{\lambda}\,\omega.x}, \ \omega \in S^{n-1}, \ \lambda \in \mathbb{C}.$$

Let $S(\lambda,\theta,\omega)$ be defined by

$$S(\lambda,\theta,\omega) = <N(\lambda)\varphi_{\lambda,\omega}, \overline{\varphi_{\lambda,-\theta}}>$$

The crucial fact is the following lemma.

LEMMA C. If $\lambda \neq$ eigenvalue,

$$S(\lambda,\theta,\omega) = -\frac{\lambda}{2}(\theta-\omega)^2 \int_\Omega e^{-i\sqrt{\lambda}(\theta-\omega)x}dx$$

$$+ \int_\Omega e^{-i\sqrt{\lambda}(\theta-\omega)x} q(x)dx$$

$$- (R(\lambda)\, q\, \varphi_{\lambda,\omega}, q\, \overline{\varphi_{\lambda,-\theta}}),$$

where $R(\lambda) = (-\Delta_D+q-\lambda)^{-1}$.

Note that the above expression is similar to the S-matrix in scattering theory. Now, we recall the Born approximation.

Let $\mathbb{R}^n \ni \xi \neq 0$ be arbitrarily fixed. Take $\eta \in S^{n-1}$ such that $n^\perp\xi$. For a a large parameter N, we define

$$\begin{cases} \theta_N = C_N\eta + \xi/2N, \ C_N = (1-|\xi|^2/4N^2)^{1/2}, \\ \omega_N = C_N\eta - \xi/2N, \\ \sqrt{t_N} = N + i. \end{cases}$$

They have the following properties :

$$\theta_N, \ \omega_N \in S^{n-1},$$

$$\sqrt{t_N}(\theta_N - \omega_N) \to \xi, \text{ as } N \to \infty,$$

$$\text{Im } t_N \to \infty \text{ as } N \to \infty,$$

$$\text{Im } \sqrt{t_N}\ \theta_N, \ \text{Im} \sqrt{t_N}\ \omega_N \text{ are bounded.}$$

Invoking these properties, one can easily show

THEOREM D.

$$\lim_{N\to\infty} S(t_N, \theta_N, \omega_N) = -\frac{|\xi|^2}{2} \int_\Omega e^{-ix\xi} dx + \int_\Omega e^{-ix\xi} q(x)\, dx.$$

So, one can reconstruct the potential from $S(\lambda, \theta, \omega)$.

Now, we prove Theorem B. $N(\lambda)$ has, formally, the integral kermel :

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_i - \lambda} \left(\frac{\partial \varphi_i}{\partial \nu}\right)(x) \left(\frac{\partial \varphi_i}{\partial \nu}\right)(y),$$

where φ_i is the eigenfunction. In view of this expression one can show that the assumption of Theorem B implies

$$\|N(\lambda, q_1) - N(\lambda, q_2)\|_{B(L^2(S^{n-1}))} \leqslant C/|\lambda|$$

for large $|\lambda|$. Theorem B then follows from this inequality and theorem D.

From the very proof, one can see that the potential is uniquely determined by the asymptotic properties of the eigenvalues and eigenfunctions.

4. Variable coefficient case. We briefly mention the variable coefficients t case.

Consider the operator

$$H = -\sum_{i,j} \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial}{\partial x_j}) + q(x).$$

Assume that for $|\alpha| \leqslant N$, N is choosen large enough,

$$\sup_{x \in \Omega} |\partial_x^\alpha (a_{ij}(x) - \delta_{ij})| \equiv \delta << 1.$$

Then the above Theorem B, Lemma C, Theorem D also hold in this case. The proof relies on the method of asymptotic solutions and Fourier integral operators. Note that we are fixing a_{ij} and seeking q(x).

The above results may be extended to higher order elliptic operators and elliptic systems.

References

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