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THE PROPAGATION OF POLARIZATION IN DOUBLE REFRACTION

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1. INTRODUCTION

Double refraction occurs both in uniaxial and biaxial crystals. It is caused by the non-uniformity of the characteristics of Maxwell's equations. The propagation of polarization for biaxial crystals (conical refraction) was studied in [**3**]. In this paper, we consider systems which generalize Maxwell's equations for uniaxial crystals, i.e. trigonal, tetragonal and hexagonal crystals. Then the characteristic set is a union of two hypersurfaces, tangent of exactly order one at the optical axis. We are going to assume that the characteristic set is a union of two non-radial hypersurfaces, tangent of exactly order $k_0 \geq 1$ at an involutive manifold. At the singularity, the Hamilton fields of the surfaces are parallel, and their Lie bracket vanishes of at least order k_0 there. We also assume that the principal symbol vanishes of first order on the two-dimensional kernel at the singularity, and assume a type of Levi condition.

We shall consider the $H_{(s)}$ polarization, which indicates the components of the solution, which are not in $H_{(s)}$. Outside the singularity of the characteristics, the polarization propagates along Hamilton orbits, which are liftings of the bicharacteristics. The limits of polarizations from outside the singularity, are called real polarizations. The Levi condition implies that the real polarizations are foliated by limits of Hamilton orbits. The results on the propagation depend on whether the polarization is contained in limit Hamilton orbits or not. When the polarization is contained in (limit) Hamilton orbits in a neighborhood of the singularity of the characteristic set, we can define an invariant curvature of the orbits. If this curvature satisfies a second order equation along the Hamilton field, we can prove propagation of polarization. When the polarization set is not contained in limit Hamilton orbits, we prove propagation of a more general type of polarization set.

2. SYSTEMS OF UNIAXIAL TYPE

Let $P \in \Psi^m_{phg}(X)$ be an $N \times N$ system of classical pseudodifferential operators on a C^∞ manifold X. Let $p = \sigma(P)$ be the principal symbol, $\det p$ the determinant of p and $\Sigma = (\det p)^{-1}(0)$ the characteristics of P. If

$$\Sigma_2 = \{(x,\xi) \in \Sigma : d(\det p) = 0 \text{ at } (x,\xi)\},$$

and $\Sigma_1 = \Sigma \setminus \Sigma_2$, we find that P is of real principal type at Σ_1, since the dimension of the kernel is equal to 1 there (see [**2**, Definition 3.1]). Assume,

$$\begin{aligned} &\Sigma = S_1 \cup S_2, \quad \text{where } S_j \text{ are non-radial hypersurfaces} \\ &\text{tangent at } \Sigma_2 = S_1 \cap S_2 \text{ of exactly order } k_0 \geq 1. \end{aligned} \tag{2.1}$$

This means that the Hamilton field of S_j does not have the radial direction $\langle \xi, \partial_\xi \rangle$. Also, the k_0:th jets of S_1 and S_2 coincide on Σ_2, but no k_0+1:th jet does. Observe that the surfaces need not be in involution, in the sense that their Hamilton fields satisfy the Frobenius integrability condition. Since p is homogeneous in ξ, we find that Σ_i and S_j are conical. Next we assume,

$$\Sigma_2 \text{ is an involutive manifold of codimension } d_0 \geq 2. \tag{2.2}$$

Clearly the codimension cannot be equal to 1, and by non-degeneracy Σ_2 is a manifold. In order to avoid that P essentially is a scalar operator, we put $\mathcal{N}_P = \operatorname{Ker} p$ and assume

$$\text{the dimension of } \mathcal{N}_P = 2 \quad \text{at} \quad \Sigma_2. \tag{2.3}$$

In order to make p vanish of first order on the kernel, we assume

$$d^2(\det p) \neq 0 \quad \text{at} \quad \Sigma_2. \tag{2.4}$$

It follows from the proof of [**3**, Lemma 2.2], that if (2.3) holds, then (2.4) is equivalent to the fact that $\partial_\rho p$: $\mathcal{N}_P \longmapsto \operatorname{Coker} p = \mathbf{C}^N / \operatorname{Im} p$ is a bijection, for $\rho \in N_{\Sigma_2}\Sigma$, the normal bundle. We also want to introduce a type of Levi condition on the system. In order to do that, we shall consider the limits of $\mathcal{N}_P\big|_{\Sigma_1}$ when we approach Σ_2. Let

$$\mathcal{N}_P^j = \mathcal{N}_P\Big|_{S_j \setminus \Sigma_2}, \tag{2.5}$$

and $\partial\Sigma_1 = T_{\Sigma_2}\Sigma / T\Sigma_2$.

Definition 2.1. *We define*

$$\partial\mathcal{N}_P^j = \{(w, \varrho, z) \in \partial\Sigma_1 \times \mathbf{C}^N : \varrho \neq 0 \wedge z \in \lim_{w_k \to w} \operatorname{Ker} p(w_k)\}, \tag{2.6}$$

where the limits are taken over those $w_k \in S_j \setminus \Sigma_2$, such that $(w - w_k)/|w - w_k| \to \varrho/|\varrho|$.

It is clear that $\partial\mathcal{N}_P^j$ is closed, conical and linear in the fiber, but it may have dimension $>$ 1 at (w, ϱ). The following is the type of Levi condition we shall use. We assume

$$\partial\mathcal{N}_P^1 \bigcap \partial\mathcal{N}_P^2 = \{0\} \quad \text{at} \quad (w, \varrho) \in \partial\Sigma_1, \quad \varrho \neq 0. \tag{2.7}$$

It follows from Lemma 3.2 below, that this implies that $\partial\mathcal{N}_P^j$ is a complex line bundle over $\partial\Sigma_1 \setminus (\Sigma_2 \times 0)$. Also, (2.7) implies that $Q = {}^tP^{co}P$ satisfies the generalized Levi condition (1.3) in [4].

DEFINITION 2.2. *The system P is of uniaxial type at $w_0 \in \Sigma_2$, if (2.1)–(2.4) and (2.7) hold microlocally near w_0.*

Since these are conditions only on the principal symbol, they are invariant under conjugation by Fourier integral operators. It is easily seen that they are invariant under multiplication by elliptic systems as well. Corollary 3.3 shows that P^* is of uniaxial type at w_0, if P is.

EXAMPLE 2.3. We consider Maxwell's equations in uniaxial crystals

$$\begin{cases} \varepsilon \partial_t e - \operatorname{curl} h = 0 \\ \mu \partial_t h + \operatorname{curl} e = 0 \\ \operatorname{div}(\varepsilon e) = \operatorname{div}(\mu h) = 0. \end{cases} \tag{2.8}$$

Here e, h are distributions with values in $\mathbf{C}^3$ and ε, μ are positive definite, constant 3×3 matrices, such that $\tilde{\varepsilon} = \mu^{-1/2}\varepsilon\mu^{-1/2}$ has two different eigenvalues α, $\beta > 0$. By choosing new fiber and x variables, we may assume $\mu = Id_3$ and

$$\varepsilon = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}.$$

The system (2.8) has characteristic set included in $\{\tau \neq 0\}$. If we skip the divergence equations, which are redundant when $\tau \neq 0$, the resulting 6×6 system has determinant equal to

$$\alpha^2\beta\tau^2((\tau^2 - \psi)^2 - (\alpha^{-1} - \beta^{-1})^2(\xi_1^2 + \xi_2^2)^2/4), \tag{2.9}$$

where

$$\psi = (\alpha^{-1} + \beta^{-1})(\xi_1^2 + \xi_2^2)/2 + \alpha^{-1}\xi_3^2.$$

When $\tau \neq 0$, the 6×6 system is of uniaxial type. In fact, by choosing

$$\begin{cases} \eta_0 = \tau^2 - \psi \\ \eta_j = \xi_j, \quad j > 0, \end{cases}$$

as new local coordinates when $\tau \neq 0$, we find

$$\Sigma \bigcap \{\tau \neq 0\} = S_1 \bigcup S_2,$$

where

$$S_j = \{\eta_0 = (-1)^j(\alpha^{-1} - \beta^{-1})(\eta_1^2 + \eta_2^2)/2\}.$$

These are non-radial, and tangent of order 2 at

$$\Sigma_2 = \{\eta_0 = \eta_1 = \eta_2 = 0\},$$

which is involutive of codimension 3. Clearly, $\partial^2_{\eta_0}(\det p) \neq 0$ at Σ_2, according to (2.9). The kernel of the principal symbol at Σ_2 is spanned by ${}^t(n_1, -\gamma n_2)$ and ${}^t(n_2, \gamma n_1)$, where n_i is the i:th unit vector in $\mathbf{R}^3$, and $\gamma = \xi_3/\tau$, $\tau \neq 0$, so the dimension is equal to 2. Thus it remains to prove (2.7). By Lemma 3.2 we only have to verify that $\partial_\varrho p : \operatorname{Ker} p \longmapsto \operatorname{Im} p$, when $\varrho \in T_{\Sigma_2}\Sigma$, since $k_0 = 1$. Clearly $T_{\Sigma_2}\Sigma$ is characterized as those $\varrho \in T_{\Sigma_2}X$, such that $\partial^2_\varrho(\det p) = 0$. Thus $T_{\Sigma_2}\Sigma$ is spanned by ∂_{ξ_1}, ∂_{ξ_2}, ∂_t, ∂_x and the radial vector field. Now if ${}^t(e,h) \in \operatorname{Ker} p$ at Σ_2, we find

$$\partial_{\xi_i} p\, {}^t(e,h) = {}^t(-n_i \times e, n_i \times h), \quad i = 1,2.$$

Since ${}^t(n_3, 0)$ and ${}^t(0, n_3)$ are in $\operatorname{Im} p$ at Σ_2, this gives (2.7).

3. The normal form

Now we shall prepare the system, when it is of uniaxial type. This makes it easier to compute the invariants of the system, and explains why (2.7) is a type of Levi condition.

PROPOSITION 3.1. *Let $P \in \Psi^1_{phg}$ be of uniaxial type at $w_0 \in \Sigma_2$. Then, by choosing suitable symplectic coordinates, we may assume $X = \mathbf{R} \times \mathbf{R}^{n-1}$, $w_0 = {}^t(0, \dots, 1)$,*

$$S_j = \{\tau = (-1)^j \beta\}, \quad j = 1,2, \tag{3.1}$$

microlocally near w_0, where β is real, homogeneous of degree 1 in ξ, and satisfies

$$c|\xi'|^{k_0+1}/|\xi|^{k_0} \leq |\beta| \leq C|\xi'|^{k_0+1}/|\xi|^{k_0}, \quad c, C > 0, \tag{3.2}$$

$(\tau, \xi', \xi'') \in \mathbf{R} \times \mathbf{R}^{d_0-1} \times \mathbf{R}^{n-d_0}$, which gives $\Sigma_2 = \{\tau = 0 \wedge \xi' = 0\}$. By multiplying P with elliptic $N \times N$ systems of order 0, we may assume

$$P \cong \begin{pmatrix} F & 0 \\ 0 & E \end{pmatrix}, \qquad \operatorname{mod} C^\infty, \tag{3.3}$$

microlocally near w_0, where $E \in \Psi^1_{phg}$ is elliptic $(N-2) \times (N-2)$ system,

$$F \cong Id_2 D_t + K(t, x, D_x), \tag{3.4}$$

is 2×2 system with $K(t, x, D_x) \in C^\infty(\mathbf{R}, \Psi^1_{phg})$, which gives $\det \sigma(F) = \tau^2 - \beta^2$.

We need some further preparation of P, since the system (3.4) need not satisfy the Levi condition (2.7). First we have to introduce symbol classes adapted to β in (3.2). Let

$$m(\xi) = 1 + |\xi'|^{k_0+1}\langle\xi\rangle^{-k_0}, \tag{3.5}$$

where $\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$, thus $m \approx 1 + |\beta|$. Put

$$g(dx, d\xi) = |dx|^2 + |d\xi'|^2/(\langle\xi\rangle^\mu + |\xi'|)^2 + |d\xi''|^2/\langle\xi\rangle^2 \quad \text{at } (x,\xi), \tag{3.6}$$

where $\mu = k_0/(k_0+1)$, which gives $h^2 = \sup g/g^\sigma = (\langle\xi\rangle^\mu + |\xi'|)^{-2} \leq 1$. We find that g is σ temperate, m is a weight for g, and $\beta \in S(m,g)$ (see [4]).

LEMMA 3.2. *Let*

$$P = Id_2 D_t + K(t, x, D_x) \tag{3.7}$$

be 2×2 *system with* $K \in C^\infty(\mathbf{R}, \Psi^1_{phg})$, *and assume* $k = \sigma(K)$ *has determinant* $\det k \equiv -\beta^2$ *and trace* $\operatorname{tr} k \equiv 0$. *Then* P *is of uniaxial type if and only if* $k \in C^\infty(\mathbf{R}, S(m, g))$.

Thus P is of uniaxial type if and only if $|k| \le C|\beta|$. Also we find that $\partial\mathcal{N}_P^j$ has one-dimensional fiber over any $(w, \varrho) \in \partial\Sigma_1$, $\varrho \ne 0$, when P is of uniaxial type.

COROLLARY 3.3. *If* $P \in \Psi^m_{phg}$ *is an* $N \times N$ *system of uniaxial type at* $w_0 \in \Sigma_2$, *then the adjoint* P^* *is.*

The projection of $\partial\mathcal{N}_P^j$ on $\mathcal{N}_P\big|_{\Sigma_2}$, $j = 1, 2$, may have intersection of dimension greater than zero, according to the following

EXAMPLE 3.4. Let $k_0 = 1$, $p = \tau Id + k$ with $k \in C^\infty(\mathbf{R}, S(m, g))$ equal to

$$k = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_2 & -\alpha_1 \end{pmatrix},$$

where $\alpha_1 = (\xi_1^2 - \xi_2^2)/|\xi|$ and $\alpha_2 = 2\xi_1\xi_2/|\xi|$. When $\xi_1^2 = \xi_2^2 \ne 0$ and $\tau = (-1)^j(\xi_1^2 + \xi_2^2)/|\xi|$, we find that the fiber of $\mathcal{N}_P$ is spanned by ${}^t(-1, (-1)^j\alpha_2/|\alpha_2|)$.

4. INVARIANTS OF THE SYSTEM.

Now we assume P to be of uniaxial type, and on the form in Proposition 3.1 and Lemma 3.2. Thus it is a 2×2 system with principal symbol $p = \tau I + k(t, x, \xi)$, where $k \in C^\infty(\mathbf{R}, S(m, g))$ is homogeneous, has determinant $-\beta^2$ and trace 0. Let $\partial\Sigma_1 = T_{\Sigma_2}\Sigma/T\Sigma_2$ and let $\partial\mathcal{N}_P^j$ be defined by (2.6). If we put

$$\mathcal{N}_{\mathbf{R}} = \pi^*(\partial\mathcal{N}_P^1 \cup \partial\mathcal{N}_P^2) \subset \Sigma_2 \times \mathbf{C}^2, \tag{4.1}$$

where $\pi : \partial\Sigma_1 \to \Sigma_2$ is defined by $\pi(w, \varrho) = w$, we get the real polarizations in the case of Maxwell's equations for uniaxial crystals. On Σ_1, P is of real principal type and $\mathcal{N}_P^j$ is foliated by Hamilton orbits, which are liftings of bicharacteristics of Σ_1 (see [**2**]). Now we shall analyze what happens when we approach Σ_2. We say that a sequence of C^∞ curves converges, if there exist parametrizations that converge in C^∞. A sequence of Hamilton orbits converges, if it does as a sequence of curves in $T^*X \times \mathbf{P}^1_{\mathbf{C}}$.

PROPOSITION 4.1. *We find that* $\partial\Sigma_1 \setminus (\Sigma_2 \times 0)$ *is foliated by limits of bicharacteristics in* Σ_1, *called limit bicharacteristics, and* $\partial\mathcal{N}_P^1 \cup \partial\mathcal{N}_P^2$ *is foliated by limits of Hamilton orbits, which are line bundles over limit bicharacteristics.*

The limit Hamilton orbit through $(w, \varrho, z) \in \partial\mathcal{N}_P^j$, $\varrho \ne 0$, is obtained by taking the limit of the Hamilton orbits through $(w_k, \operatorname{Ker} p(w_k))$, where $S_j \setminus \Sigma_2 \ni w_k \to w$ and $(w - w_k)/|w - w_k| \to \varrho/|\varrho|$. The projection of the limit bicharacteristics on Σ_2 have tangent proportional to the Hamilton field of S_j at Σ_2. In the case when the polarization set is a union of (limit) Hamilton orbits, we need conditions in a neighborhood of Σ_2. The following lemma will help us compute the invariants of the orbits.

LEMMA 4.2. *We find that* $\mathcal{N}_P^j$ *extends to a* C^∞ *line bundle over* S_j, $j = 1$ *or* 2, *if and only if* $\mathcal{N}_{P^*}^k = \mathcal{N}_{P^*}\Big|_{S_k \setminus \Sigma_2}$, $k \neq j$, *extends to a* C^∞ *line bundle over* S_k.

Now we want to characterize the elements in $\mathcal{N}_\mathbf{C} = \mathcal{N}_P\Big|_{\Sigma_2} \setminus \mathcal{N}_\mathbf{R}$, in terms of the degree of vanishing of the principal symbol. In order to do that, we must extend the polarization vector $z_0 \in \operatorname{Ker} p(w_0)$, $w_0 \in \Sigma_2$, to a neighborhood, but the result will be independent of the extension.

PROPOSITION 4.3. *We find* $(w_0, z_0) \in \mathcal{N}_\mathbf{C}$, $w_0 \in \Sigma_2$, *if and only if*

$$|pz(w)| \geq |\beta(w)|, \quad c > 0, \quad w \in \Sigma, \tag{4.2}$$

near w_0, *for any homogeneous* C^∞ *extension* $z(w)$ *of* z_0 *to a neighborhood of* w_0.

We also want to characterize those sections of $\mathbf{C}^2$ over Σ_2, which are tangent to limit Hamilton orbits. We shall consider C^∞ sections over bicharacteristics, but the result will only depend on the first jet of the section.

PROPOSITION 4.4. *Let* $\Gamma_0 \subset \Sigma_2$ *be a bicharacteristic of* S_j, *and* $\Gamma_0 \ni w \longmapsto z_0(w) \in \mathbf{C}^2$ *a* C^∞ *section. Then no lifting of* $z_0(w)$ *is tangent to a limit Hamilton orbit at* w_0, *if and only if*

$$|p_1(w)| + |p_2(w)| + |\{p_1, p_2\}(w)| \geq c|\beta(w)|, \quad c > 0, \quad w \in \Sigma, \tag{4.3}$$

near w_0, ${}^t(p_1, p_2) = pz$, *for any* C^∞ *homogeneous extension* $z(w)$ *of* z_0 *to a neighborhood of* w_0.

Thus, either $z_0 \notin \mathcal{N}_\mathbf{R}$, or no lifting of z_0 to $\partial\mathcal{N}_P^j$ is tangent to a limit Hamilton orbit. We shall now define the $H(s)$ polarization set (see [**5**]).

DEFINITION 4.5. *For* $u \in \mathcal{D}'(X, \mathbf{C}^N)$, *we define*

$$WF^s_{pol}(u) = \bigcap \mathcal{N}_B, \tag{4.4}$$

where $\mathcal{N}_B = \operatorname{Ker} \sigma(B)$, *and the intersection is taken over those* $B \in \Psi^0_{phg}$, *such that* $Bu \in H_{(s)}$.

5. THE PROPAGATION OF POLARIZATION

We shall now state the results on the propagation of polarization. First we consider the case where there is no polarization. Since $\Sigma = S_1 \cup S_2$, where the hypersurfaces are tangent at Σ_2, which is involutive, Σ has a well-defined Hamilton flow, which is tangent to Σ_2. The orbits of the flow is called the bicharacteristics of Σ. Let $s^*(w) = \{\sup s : u \in H(s) \text{ at } w\}$ be the regularity function, $w \in T^*X \setminus 0$.

THEOREM 5.1. *Let $P \in \Psi^m_{phg}(X)$ be an $N \times N$ system of uniaxial type at $w_0 \in \Sigma_2$. Assume that $u \in \mathcal{D}'(X, \mathbf{C}^N)$ satisfies $s^*_{Pu} > s - m + 1$ at w_0. Then $\min(s^*_u, s)$ is constant on the bicharacteristics of Σ near w_0.*

Next, we are going to consider the limit Hamilton orbit case. As before, we assume $P \in \Psi^m_{phg}$ is of uniaxial type at $w_0 \in \Sigma_2$. Let $\mathcal{V} \subset \mathcal{N}_P$ be a C^∞ line bundle over S_j, for some j. Then $\mathcal{V}$ is a union of (limit) Hamilton orbits over S_j, and $(\pi^*)^{-1}\mathcal{V}\Big|_{\Sigma_2} \subset \partial\mathcal{N}^j_P$. We shall define an invariant curvature of $\mathcal{V}$. Choose $v \in S^0$ homogeneous of degree 0 in ξ, and w_1, $w_2 \in S^{1-m}$ homogeneous of degree $1-m$, such that v spans $\mathcal{V}$ over S_j, w_1 spans $\mathcal{N}_{P^*}$ over S_k, $k \neq j$, and w_1, w_2 span $\mathcal{N}_{P^*}$ over Σ_2. This is possible, according to Lemma 4.2. Let $V \in \Psi^0_{phg}$, $W_i \in \Psi^{1-m}_{phg}$ have principal symbols v, w_i, $i = 1, 2$. Put

$$P_i = W_i^* P V \in \Psi^1_{phg}, \tag{5.1}$$

and $p_i = \sigma(P_i)$. Clearly, $p = (p_1, p_2) = 0$ on S_j, and $p_1 = 0$ on S_k also. By condition (2.4), we find $dp \neq 0$ at Σ_2, and since $dp_1 = 0$ at Σ_2, we obtain $dp_2 \neq 0$. Thus we can find $C \in \Psi^0_{phg}$, $\sigma(C) = 0$ at Σ_2, so that

$$P_1 + CP_2 = K \in \Psi^0_{phg}. \tag{5.2}$$

Let $R_i = \{a \in \Psi^0_{phg} : \sigma(a) = 0 \text{ on } S_i\}$, for $i = 1, 2$.

PROPOSITION 5.2. *We find that $\sigma(K)$ is independent of the choice of V and W_i, modulo R_i, $i = 1, 2$, and elliptic factors.*

This makes it possible to make the following

DEFINITION 5.3. *We call $\kappa = \sigma(K)\big|_{S_j}$ the curvature of the C^∞ line bundle $\mathcal{V} \subset \mathcal{N}_P$ over S_j.*

Clearly, the k_0:th jet $j^{k_0}\kappa$ of the curvature at Σ_2 is well-defined, modulo invertible transformations corresponding to elliptic factors. Now we can state the result on the propagation of polarization sets in the limit Hamilton orbit case. Let π_0: $T^*X \times \mathbf{C}^N \longmapsto T^*X$ be the projection along the fiber.

THEOREM 5.4. *Let $P \in \Psi^m_{phg}$ be an $N \times N$ system of uniaxial type at $w_0 \in \Sigma_2$, and let $A \in \Psi^0_{phg}$ be a $1 \times N$ system such that the dimension of the fiber of $\mathcal{N}_A \cap \mathcal{N}_P$ is equal to 1 at w_0, and $M_A = \pi_0(\mathcal{N}_A \cap \mathcal{N}_P \setminus 0)$ is a hypersurface near w_0. Let κ be the curvature of $\mathcal{N}_A \cap \mathcal{N}_P$ over M_A, and assume that the k_0:th jet*

$$j^{k_0}(D^2\kappa + c_1 D\kappa + c_0\kappa) \equiv 0 \quad \text{at} \quad \Sigma_2, \tag{5.3}$$

near w_0, for some $c_j \in C^\infty(M_A)$, where $0 \neq D$ is the Hamilton field of M_A. Then, if $u \in \mathcal{D}'(X, \mathbf{C}^N)$ satisfies $\min(s^*_{Pu} + m - 1, s^*_u + 1) > s$ at w_0, we find that $\min(s^*_{Au}, s)$ is constant on the bicharacteristics of M_A near w_0.

In this case $M_A = S_j$, for some j, the dimension of $\mathcal{N}_A \cap \mathcal{N}_P$ is equal to one over S_j, $\mathcal{N}_A \cap \mathcal{N}_P\Big|_{\Sigma_2} = \pi^*\partial\mathcal{N}^j_P$, and $\mathcal{N}_A \cap \mathcal{N}_P$ is a union of (limit) Hamilton orbits. Condition (5.3) means that $D^2\kappa + c_1 D\kappa + c_0\kappa$ vanishes of order $k_0 + 1$ at Σ_2 near w_0.

6. The non-tangential case

Now we shall study the case when no lifting of $\mathcal{N}_A \cap \mathcal{N}_P\big|_{\Sigma_2}$ is tangent to a limit Hamilton orbit. Then the transport equations will become non-homogeneous, the terms will be in $S(1, g)$. Therefore we have to define this symbol class invariantly.

DEFINITION 6.1. *Let $\Omega \subset T^*X \setminus 0$ be an involutive, conical manifold, and let $1/2 \le \nu \le 1$. Then $S^m_{\Omega,\nu}$ is the set of $a(x,\xi) \in C^\infty(T^*X \setminus 0)$ satisfying*

$$|L_1 \dots L_j V_1 \dots V_k a(x,\xi)| \le C_{jk} \langle \xi \rangle^{m+k(1-\nu)}, \quad \forall j,k, \tag{6.1}$$

for all normalized, homogeneous vector fields L_i and V_i, such that $L_i\big|_\Omega$, $i = 1, \dots, j$ are tangent to Ω.

Clearly, since Ω is involutive, the order of differentiation does not matter in (6.1), because commutators will never raise k. Outside a conical neighborhood of Ω, we get the usual symbol classes $S^m_{1,0}$. The definition of $S^m_{\Omega,\nu}$ is independent of the choice of homogeneous, symplectic coordinates. With the choice of coordinates as in Proposition 3.1, we get the earlier symbol classes.

LEMMA 6.2. *If the coordinates in $T^*X \setminus 0$ are chosen so that*

$$\Omega = \{(x,\xi) \in T^*X \setminus 0 : \xi' = 0\}, \tag{6.2}$$

where $\xi = (\xi', \xi'')$, then we find $S^m_{\Omega,\nu} = S(\langle\xi\rangle^m, g_\nu)$, where

$$g_\nu(dx, d\xi) = |dx|^2 + |d\xi'|^2/\left(\langle\xi\rangle^\nu + |\xi'|\right)^2 + |d\xi''|^2/\langle\xi\rangle^2 \quad at \quad (x,\xi). \tag{6.3}$$

Observe that, when $\Omega = \Sigma_2 = \{\tau = 0 \wedge \xi' = 0\}$ and the symbols are independent of τ, we find $S^0_{\Sigma_2,\mu} = C^\infty(\mathbf{R}, S(1,g))$ when $|\tau| \le c|\xi'|$. We shall define new polarization sets with respect to these symbol classes.

DEFINITION 6.3. *If $u \in \mathcal{D}'(X, \mathbf{C}^N)$ we say that $(w_0, z_0) \notin WF^{s,\mu}_{pol}(u)$ if there exists a conical neighborhood U of w_0 and $a \in S^0_{\Sigma_2,\mu}$, $\mu = k_0/(k_0+1)$, such that $a^w(x,D)u \in H_{(s)}$ and*

$$|a(x,\xi)z_0| \ge c > 0, \quad when\ (x,\xi) \in U\ and\ |\xi| \ge 1. \tag{6.4}$$

*Here a^w is the Weyl operator, see [**7**, Section 18.5].*

Clearly, $S^0_{\Sigma_2,\mu} \subset S^0_{\mu,0}$, $1/2 \le \mu < 1$, so the usual calculus applies to $S^0_{\Sigma_2,\mu}$. Conjugation with elliptic, homogeneous Fourier integral operators only changes Weyl operators having symbols in $S^0_{\mu,1-\mu}$, with symbols in $S^{(1-3\mu)/2}_{\mu,1-\mu} \subset S^{-1/4}_{\mu,1-\mu}$, according to [**6**, Theorem 9.1]. Thus the definition is independent of the choice of symplectic coordinates. By choosing coordinates so that $\Sigma_2 = \{\tau = 0 \wedge \xi' = 0\}$, we get an asymptotic expansion for the calculus, according to Lemma 6.2. Thus we obtain

$$\pi_0(WF^{s,\mu}_{pol}(u) \setminus 0) = WF_{(s)}u, \tag{6.5}$$

where $\pi_0 : T^*X \times \mathbf{C}^N \longmapsto T^*X$ is the projection along the fiber. Now we let $0 \ne D$ be the Hamilton field of Σ, and $\exp(tD)$ the Hamilton flow, $t \in \mathbf{R}$.

THEOREM 6.4. *Let $P \in \Psi^m_{phg}$ be an $N \times N$ system of uniaxial type at $w_0 \in \Sigma_2$, and let $A \in \Psi^0_{phg}$ be a $1 \times N$ system, such that the dimension of the fiber of $\mathcal{N}_A \cap \mathcal{N}_P$ is equal to 1 at w_0, and no lifting of $\mathcal{N}_A \cap \mathcal{N}_P\big|_{\Sigma_2}$ is tangent to a limit Hamilton orbit over w_0. If $u \in \mathcal{D}'(X, \mathbf{C}^N)$ satisfies $\min(s^*_{Pu} + m - 1, s^*_u + \mu) > s$ at w_0, $\mu = k_0/(k_0+1)$, and $s^*_{Au} > s$ in $\exp(tD)w_0$, $0 < t < \varepsilon$, for some $\varepsilon > 0$, then $WF^{s,\mu}_{pol}(u) \subset \mathcal{N}_A \cap \mathcal{N}_P$ at w_0.*

The conditions mean that, for any lifting of $\mathcal{N}_A \cap \mathcal{N}_P\big|_{\Sigma_2}$ to $\partial\Sigma_1 \times \mathbf{C}^N$, either it is not in $\partial\mathcal{N}^j_P$, or it is not tangent to the limit Hamilton orbit through the lifting.

7. THE DISTRIBUTION OF POLARIZATION

We are also interested in the distribution of the singularities of the solution, when we have a polarization condition.

THEOREM 7.1. *Let $P \in \Psi^m_{phg}$ be an $N \times N$ system of uniaxial type at $w_0 \in \Sigma_2$, and let $A \in \Psi^0_{phg}$ be a $1 \times N$ system such that the dimension of $\mathcal{N}_A \cap \mathcal{N}_P$ is equal to 1 at w_0. If $u \in \mathcal{D}'(X, \mathbf{C}^N)$ satisfies $\min(s^*_{Pu} + m - 1, s^*_{Au}) > s$ at w_0, then $WF^s_{pol}(u)$ is a union of C^∞ line bundles in $\mathcal{N}_A \cap \mathcal{N}_P$ over the bicharacteristics of Σ in $M_A = \pi_0(\mathcal{N}_A \cap \mathcal{N}_P \setminus 0)$ near w_0.*

This means precisely that $WF_{(s)}u$ is a union of bicharacteristics of Σ in M_A. When the conditions in Theorem 5.4 are satisfied, we obtain that $WF^s_{pol}(u)$ is a union of (limit) Hamilton orbits near $w_0 \in \Sigma_2$.

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