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PROPAGATION OF ANALYTIC SINGULARITIES UP TO NON SMOOTH BOUNDARY

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1.- Propagation for sheaves

We shall follow the notations of [K-S 1]. In particular if X is a real manifold, we denote by $D^b(X)$ the derived category of the category of complexes of sheaves with bounded cohomology, and if $F \in D^b(X)$ we denote by $SS(F)$ its micro-support. Recall that $SS(F)$ is a closed conic involutive subset of T^*X. We shall also make use of the bifunctor μhom, from $D^b(X)^0 \times D^b(X)$ to $D^b(T^*X)$, a slight generalization of the functor of Sato's microlocalization.

Let h be a real C^2-function defined on an open subset U of T^*X, H_h its hamiltonian vectir field. If $(x ; \xi)$ is a system of homogeneous symplectic coordinates, with $\omega_X = \sum_j \xi_j dx_j$, then :

$$(1.1) \qquad H_h = \sum_j \left(\frac{\partial h}{\partial \xi_j}\frac{\partial}{\partial x_j} - \frac{\partial h}{\partial x_j}\frac{\partial}{\partial \xi_j}\right) .$$

If $p \in U$ we denote by b_p^+ the positive half integral curve of H_h issued at p. We define similarly b_p^- and $b_p = b_p^- \cup b_p^+$. We also set for $* = 0,+,-$:

(1.2) $$V_* = \{p \in U \; ; \; h(p) \geq 0 \quad (* = +) \quad \text{or} \quad h(p) \leq 0 \quad (* = -) \quad \text{or} \quad h(p) = 0 \quad (* = 0)\}.$$

The following result is easily deduced from [K-S 1, Th. 5.2.1].

<u>Theorem 1.1</u>. Let F and G belong to $D^b(X)$ with $SS(G) \cap U \subset V_-$, $SS(F) \cap U \subset V_+$. Let $j \in \mathbb{Z}$ and let u be a section of $H^j(\mu hom(G,F))$ on U. Then $p \in supp(u)$ implies $b_p^+ \subset supp(u)$.

(Remark that $supp(u)$ is contained in V_0).

2.- Wave front sets at the boundary [S 1]

Let M be a real analytic manifold of dimension n, X a complexification of M, Ω an open subset of M. We introduce :

(2.1) $$C_{\Omega|X} = \mu hom(\mathbb{Z}_\Omega, \mathcal{O}_X) \otimes \underline{\omega}_{M/X}[n]$$

where $\underline{\omega}_{M/X}$ is the relative orientation sheaf.

Let π denote the projection $T^*X \longrightarrow X$, and let $B_M = R\Gamma_M(\mathcal{O}_X) \otimes \underline{\omega}_{M/X}[n]$ denote the sheaf of Sato's hyperfunctions on M. There is a natural isomorphism :

(2.2) $$\alpha : \Gamma_\Omega(B_M) \xrightarrow{\sim} \pi_* H^o(C_{\Omega|X}) .$$

Hence a hyperfunction u on Ω defines a section $\alpha(u)$ of $H^o(C_{\Omega|X})$ all over T^*X. We set :

(2.3) $$SS_\Omega(u) = supp(\alpha(u)) .$$

Since $H^{o}(C_{\Omega|X})$ is supported by the conormal boundle $T^{*}_{M}X$, $SS_{\Omega}(u)$ is a closed conic subset of $T^{*}_{M}X$. It coïncides with the classical analytical wave front set above Ω , but it may be strictly larger that its closure in $T^{*}_{M}X$ (cf. [S 1]).

Now let P be a differential operator defined on X , and assume for simplicity that the principal symbol $\sigma(P)$ never vanishes identically. Let $\mathcal{O}^{P}_{X}$ denote the sheaf of holomorphic solutions of the equation $Pf = 0$. Replacing $\mathcal{O}_{X}$ by $\mathcal{O}^{P}_{X}$ in the preceding discussion, we define :

$$(2.4) \qquad C^{P}_{\Omega|X} = \mu hom(\mathbb{Z}_{\Omega}, \mathcal{O}^{P}_{X}) \otimes \underline{\omega}_{M/X} [n] \quad .$$

Let B^{P}_{M} denote the sheaf of hyperfunction solutions of the equation $Pu = 0$. There is a natural isomorphism :

$$(2.5) \qquad \alpha : \Gamma_{\Omega}(B^{P}_{M}) \xrightarrow{\sim} \pi_{*} H^{o}(C^{P}_{\Omega|X}) \ .$$

If u is a hyperfunction on Ω solution of the equation $Pu = 0$, we set :

$$(2.6) \qquad SS^{P}_{\Omega}(u) = supp(\alpha(u)) \ .$$

Remark that

$$(2.7) \qquad SS^{P}_{\Omega}(u) \subset SS(\mathbb{Z}_{\Omega}) \cap char(P)$$

(where $char(P) = \sigma(P)^{-1}(0)$), but in general $SS^{P}_{\Omega}(u)$ is no more contained in $T^{*}_{M}X$.

I don't know if $SS^{P}_{\Omega}(u) \cap T^{*}_{M}X = SS_{\Omega}(u)$, but this is true when $M \setminus \Omega$ is convex (locally, up to analytic diffeomorphisms).

Of course the preceding discussion extends to solutions of general systems of differential equations (cf. [S 1]) .

Now assume $\partial\Omega = N$ is a real analytic hypersurface and let Y be a complexification of N in X. Assume P of order m, Y is non characteristic for P, and a normal vector field to N in M is given, so that the induced system $(\mathcal{D}_X/\mathcal{D}_X P)_Y$ is isomorphic to $\mathcal{D}_Y^m$; (as usual, $\mathcal{D}_X$ denotes the ring of differential operators).

Let ρ and ϖ denote the natural maps associated to $Y \longrightarrow X$:

$$T^*Y \xleftarrow{\rho} Y \times_X T^*X \xrightarrow{\varpi} T^*X \quad . \tag{2.8}$$

Let $u \in \Gamma(\Omega ; \mathcal{B}_M^P)$ be a hyperfunction on Ω solution of $Pu = 0$, and let $b(u) \in \Gamma(N ; \mathcal{B}_N^m)$ be its traces. Recall (cf. [S 1], [S 2]) :

Theorem 2.1. In the preceding situation, one has :

$$SS_N(b(u)) = \rho\, \varpi^{-1}\, SS_\Omega^P(u) \quad .$$

In other words, the analytic wave front set of $b(u)$ is exactly the projection of $SS_\Omega^P(u)$. Remark that if $\mathrm{char}(P) \cap SS(\mathbb{Z}_\Omega)$ is contained in T_M^*X, $SS_\Omega^P(u)$ may be replaced by $SS_\Omega(u)$ in Theorem 2.1.

Remark moreover that $b(u)$ does not make sense when $\partial\Omega$ is not smooth, but $SS_\Omega(u)$ always does.

3.- Transversal propagation for non smooth boundaries

Let M be a real analytic manifold, X a complexification of M, Ω an open subset of M.

If $x \in M$, the cone $N_x(\Omega)$ is defined in [K-S 1]. Recall that $N_x(\Omega)$ is an open convex cone of T_xM, and $\theta \in N_x(\Omega)$, $\theta \neq 0$ implies that there exists a convex open cone γ (in a system of local coordinates around x) such that $\theta \in \gamma$ and $\Omega + \gamma \subset \Omega$.

We shall have to consider the real underlying structure of T^*X. Recall that if ω_X is the complex canonical 1-form on T^*X, this real symplectic structure in defined by $2\mathrm{Re}\,\omega_X$.

If h is a real C^2-function on T^*X, we denote by $H_h^{\mathbb{R}}$ its real Hamiltonian vector field.

If $(z\,;\,\zeta)$ is a system of homogeneous holomorphic symplectic coordinates on T^*X, such that $\omega_X = \sum_j \zeta_j\, dz_j$, and $z = x + iy$, $\zeta = \xi + i\eta$, then

$$(3.1) \qquad H_h^{\mathbb{R}} = \sum_j \left(\frac{\partial h}{\partial \xi_j}\frac{\partial}{\partial x_j} - \frac{\partial h}{\partial x_j}\frac{\partial}{\partial \xi_j} + \frac{\partial h}{\partial y_j}\frac{\partial}{\partial \eta_j} - \frac{\partial h}{\partial \eta_j}\frac{\partial}{\partial y_j}\right).$$

Now let P be a differential operator on X, u a hyperfunction on Ω, solution of the equation $Pu = 0$. Let $p \in T_M^*X$, $x_o = \pi(p)$.

<u>Theorem 3.1</u>. Assume :

a) $\mathrm{Im}\,\sigma(P)\big|_{T_M^*X} = 0$

b) $\pi(H^{\mathbb{R}}_{\mathrm{Im}\,\sigma(P)}(p)) \in N_{x_o}(\Omega)$.

Let b_p^+ be the positive half integral curve of $H^{\mathbb{R}}_{\mathrm{Im}\,\sigma(P)}$ issued at p. Then $p \in SS_\Omega(u)$ implies $b_p^+ \subset SS_\Omega(u)$.

Proof

We may assume X is open in $\mathbb{C}^n$ and $M = X \cap \mathbb{R}^n$. Then there exists a convex open cone γ such that $\Omega + \gamma \subset \Omega$ (in a neighborhood of x_o) and $\pi(H^{\mathbb{R}}_{\mathrm{Im}\,\sigma(P)}(p)) \in \gamma$. This last condition implies :

$$<d_\xi \; \mathrm{Im}\,\sigma(P)(x,i\eta),\xi> \; \geq c|\xi|$$

for some $c > 0$, and all $\xi \in \gamma^o$ (γ^o is the polar set to γ). Hence :

$$\text{(3.2)} \qquad \mathrm{Im}\,\sigma(P)(x\,,\,\xi + i\eta) \leq 0$$

for $(x, \xi + i\eta)$ in a neighborhood of p, $\xi \in \gamma^{oa}$, where $\gamma^{oa} = -\gamma^o$.

Since $\Omega + \gamma \subset \Omega$, we have (cf. [K-S 1]) :

$$SS(\mathbb{Z}_\Omega) \subset T^*_M X + \gamma^{oa} \quad .$$

Thus :

$$\text{(3.3)} \qquad \mathrm{Im}\,\sigma(P) \leq 0 \quad \text{on} \quad SS(\mathbb{Z}_\Omega)$$

in a neighborhood of p.

Now let $u \in \Gamma(\Omega\,;\,B_M)$ be a solution of the equation $Pu = 0$. Then u defines a section $\alpha(u) \in \Gamma(T^*X\,;\,H^n(\mu hom(\mathbb{Z}_\Omega, \mathcal{O}^P_X))$ and $p \in SS_\Omega(u)$ implies $p \in SS^P_\Omega(u)$, that is, $p \in \mathrm{supp}(\alpha(u))$. Since $SS(\mathcal{O}^P_X) = \mathrm{char}(P) \subset \{\mathrm{Im}\,\sigma(P) = 0\}$, we may apply Theorem 1.1 and we obtain :

$$b^+_p \subset SS^P_\Omega(u) \quad .$$

But $b^+_p \setminus \{p\}$ is contained in $\pi^{-1}(\Omega)$ and $SS^P_\Omega(u) = SS_\Omega(u) = SS_M(u)$ above Ω. Thus $b^+_p \subset SS_\Omega(u)$, which is the desired result.

4.- Diffraction

We keep the notations of §3, but we assume :

(4.1) $$\Omega = \{x \in M \; ; \; x_1 > 0\}$$

(4.2) $$\sigma(P) = \zeta_1^2 - g(z \; , \; \zeta')$$

where $z = (z_1, z')$, $\zeta = (\zeta_1, \zeta')$.

Moreover we assume :

(4.3) a) $\frac{\partial}{\partial x_1} g < 0$ at p or b) $\frac{\partial}{\partial x_1} g \equiv 0$.

Theorem 4.1. Under these hypotheses, if $p \in SS_\Omega(u)$ then b_p^+ or b_p^- is contained in $SS_\Omega(u)$, in a neighborhood of p.

The idea of the proof is the following.

If $\zeta_1 \neq 0$ at p, the result is a particular case of Theorem 3.1 . Otherwise define for $* = 0,1,-$:

$$\Omega_* = \{z \in X \; ; \; x_1 > 0, \; y' = 0, \; y_1 \in \mathbb{R} \; (* = 0)$$
$$\text{or } \; y_1 \geq 0 \; (* = +) \; \text{ or } \; y_1 \leq 0 \; (+ = -)\}$$

Thus $\mathrm{Im}\,\sigma(P)$ is negative (resp. positive) on $SS(\mathbb{Z}_{\Omega+})$ (resp. $SS(\mathbb{Z}_{\Omega-})$) in a neighborhood of p. Then one can apply Theorem 1.1 to $\mu hom(\mathbb{Z}_{\Omega*} \, , \mathcal{O}_X^P)$, $* = +$ or $-$, and one obtain that if $u|_{b_p}$ has compact support, then $u \in H^{n-1}(\mu hom(\mathbb{Z}_{\Omega_o}, \mathcal{O}_X^P))$, and it is not difficult to conclude using the holomorphic parameter z_1 (cf. [S 2]) .

Remark that Theorem 4.1 has been first obtained by Kataoka [Ka] (under hypothesis (4.3) a)) then refined by G. Lebeau [Le].

<u>An application</u> : Let $(x_1,\ldots,x_n)$ be the coordinates on $\mathbb{R}^n$, and let Ω_1 and Ω_2 be two open half spaces. Set $\Omega = \Omega_1 \cup \Omega_2$ and let u be a hyperfunction on Ω. One can easily prove :

$$SS_{\Omega}(u) = SS_{\Omega_1}(u) \cup SS_{\Omega_2}(u) \quad . \tag{4.4}$$

Now assume $\Omega_i = \mathbb{R} \times \Omega_i'$, $(i = 1,2)$ and u satisfies the wave equation $Pu = 0$, where $P = D_1^2 - \sum_{j=2}^{n} D_j^2$.

Applying Theorem 4.1 we get that $p \in SS_{\Omega}(u) \Longrightarrow b_p^+$ or b_p^- is contained in $SS_{\Omega}(u)$, where b_p^+ and b_p^- are the half bicharacteristic curves of $\operatorname{Im} \sigma(P)$.

<u>Problem</u> : to extend this result to the case where $\mathbb{R}^n \setminus \Omega = \mathbb{R} \times A$, and A is any convex closed subset of $\mathbb{R}^{n-1}$. Remark that if $\mathbb{R}^n \setminus \Omega = \mathbb{R} \times A$, where A is polyedral, and if $p \in SS_{\Omega}(u)$, $b_p^+ \setminus \{p\} \subset \pi^{-1}(\Omega)$ then $b_p^+ \subset SS_{\Omega}(u)$, in view of Theorem 3.1.

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