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# Some problems in inverse scattering theory

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# Some problems in inverse scattering theory.

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We shall consider the Schrödinger operator $H_v = -\Delta + v(x)$ in $\mathbf{R}^n$ ,where $n = 3, 5, \ldots$ . We assume that $v \in \mathcal{V}$, i.e.

(1) $$\int (1+|x|)^{|\alpha|-(n-2)} |v^{(\alpha)}(x)|\, dx < \infty$$

for any $\alpha$.

Some of the main problems we consider are the following:

(a) Analysis of bound states and poles of the scattering matrix.
(b) Backward scattering.
(c) The characterization problem for scattering matrices.

This talk will be a continuation of the authors lecture at École Polytechnique [6] , and we shall mainly give some comments to (a).

We shall study families of intertwining operators $A$ such that

(2) $$H_v A = A H_0$$

or equivalently

(3) $$(\Delta_x - \Delta_y - v(x))A(x,y) = 0.$$

(We shall always identify operators with their distribution kernels .) Let $\mathcal{M}$ be the set of all $U(x,y) \in L^1_{loc}$ such that

$$\|U\|_{\mathcal{M}} = \max\left\{ \sup_x \int |U(x,y)|\, dy,\ \sup_y \int |U(x,y)|\, dx \right\} < \infty.$$

Then $\|U\|_{L^p \to L^p} \leq \|U\|_{\mathcal{M}}$ for $1 \leq p \leq \infty$ if $U \in \mathcal{M}$. We let $\mathcal{M}_\theta$ be the subspace of $\mathcal{M}$ consisting of $U$ such that $\langle y-x, \theta\rangle \geq 0$ in its support. Here $\theta \in S^{n-1}$ and $\mathcal{M}_{\theta,\lambda}$ is the set of $U$ in $\mathcal{M}_\theta$ such that

$$e^{-\lambda\langle y-x,\theta\rangle} U(x,y) \in \mathcal{M}_\theta.$$

The spaces $\mathcal{M}$, $\mathcal{M}_\theta$ and $\mathcal{M}_{\theta,\lambda}$ are Banach algebras. Finally $\tilde{\mathcal{M}}_{\theta,\lambda}$ is defined by the following conditions:

$$\int |U(x,y)|\, dy \to 0 \text{ as } |x| \to \infty,\ x/|x| \to \theta$$

and

$$\int |U(x,y)|\, dx \to 0 \text{ as } |y| \to \infty,\ y/|y| \to -\theta.$$

**Example.** If $q \in L^1(\mathbf{R}^n)$ we let $[q]$ be the convolution operator with kernel $q(x-y)$. If $\langle x, \theta\rangle \leq 0$ in the support of $q$, then $(I - [q])^{-1}$ exists in $I + \mathcal{M}_{\theta,\lambda}$ when $\lambda$ is large.

THEOREM 1. *Let $v \in \mathcal{V}$ be real valued and $\theta \in S^{n-1}$. Then there is a unique $A_\theta \in \cup_{\lambda \geq 0} I + \tilde{\mathcal{M}}_{\theta,\lambda}$ such that $H_v A = A H_0$. Moreover, $A^*_{-\theta} \circ A_\theta = I$.*

The distribution $A_\theta$ is constructed as the infinite sum $\sum_0^\infty U_N$, where $U_0(x,y) = \delta(x-y)$, and

$$U_{N+1} = E_\theta * (vU_N),$$

Here $(vU_N)(x,y) = v(x)U_N(x,y)$, and $E_\theta$ is the fundamental solution for $\Delta_x - \Delta_y$, which is uniquely determined from the following conditions:

(i) $\langle y-x, \theta\rangle \geq 0$ in the support of $E_\theta$,
(ii) $E_\theta(x+t\theta, y+t\theta) \to 0$ in $\mathcal{D}'(\mathbf{R}^n \times \mathbf{R}^n)$ as $|t| \to \infty$.
(iii) $E_\theta = \sum c_{\alpha,\beta} \partial_x^\alpha \partial_y^\beta h_{\alpha,\beta}$, where $\phi(x-y)h_{\alpha,\beta}(x,y) \in \mathcal{M}$ for any $\phi \in C_0$.

THEOREM 2. *There exists a family of $L^1$ functions $q_\theta$ in $\mathbf{R}^n$ which depend continuously on $\theta$ and are supported in the set where $\langle x, \theta\rangle \leq 0$, such that*

$$A_\theta(I - [q_\theta]) \in I + \mathcal{M}_\theta.$$

COROLLARY 3. *Assume that $v \in C_0^\infty$. Then the scattering matrix $S_k(\theta,\theta')$is analytic in the upper half-plane $\Im k \geq 0$ after multiplication by $1 - \widehat{q_{\theta'}}(-k)$.*

**Sketch of proof .** One first constructs $B_\theta \in I + \tilde{\mathcal{M}}_{\theta,0}$ so that

$$B_\theta^{-1} H_v B_\theta = H_0 + \sum_1^N f_j \otimes g_j,$$

where $f_j$ and $g_j$ are in $L^1$ together with all their derivatives.

Next one defines the $L^1$ functions $q_{jk}$ by the formula

$$q_{jk}(y) = \int (\check{f}_j * g_k)(x) E_\theta(x,y)\, dx. \tag{4}$$

Set $[Q] = [q_{jk}]$, where the right-hand side is considered as a $N \times N$ matrix of convolution operators, and define the vector valued function $\vec{h} = (h_1, \ldots, h_N)$ by the equation

$$\vec{h}^{co}(I - [Q])\vec{g},$$

where ${}^{co}(I - [Q])$ denotes the co-factor matrix of $I - [Q]$. We can now define the $L^1$ function $q = q_\theta$ by the equation

$$\det(I - [Q]) = I - [\check{q}].$$

It is easy to see that $\langle x, \theta\rangle \leq 0$ in the support of $q_\theta$. Set

$$C_\theta = I - [q_\theta] + F_\theta,$$

where $F_\theta = \sum_1^N E_\theta * (f_j \otimes h_j)$. Then $H_v(B_\theta C_\theta) = (B_\theta C_\theta)H_0$. Therefore , if we set

$$R(x,y) = A_\theta^{-1} B_\theta C_\theta - \delta(x-y),$$

then $(\Delta_x - \Delta_y)R = 0$ and $\langle y - x, \theta\rangle \geq 0$ in its support.From a uniqueness result for $\Delta_x - \Delta_y$ one then finds that $R$ is constant in the direction of $(\theta,\theta)$, i.e. $R(x + t\theta, y + t\theta) = R(x,y)$ when $t$ is any real number. Since $R + [q] \in \tilde{\mathcal{M}}_{\theta,\lambda}$ we conclude that $R + [q] = 0$. Hence

$$A_\theta(I - [q_\theta]) = B_\theta C_\theta \in I + \tilde{\mathcal{M}}_{\theta,0}$$

and the proof is complete.

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