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Estimates for a number of negative eigenvalues of the Schrödinger operator with intensive magnetic field.

Victor Ivrii

1. In this lecture I give estimates from above and from below for a maximal dimension $N \leq \infty$ of a linear subspace $\mathcal{L} \subset C_0^\infty(X)$ on which a quadratic form

$$Q(u) = \int \left[g^{jk} (D_j - V_j)u \cdot \overline{(D_k - V_k)u} + V|u|^2 \right] dx \quad (1)$$

is negative definite; here X is a domain in \mathbb{R}^d , $d = 3$, $g^{jk} = g^{kj}$, V_j, V are real-valued and $g^{jk}, g^{jk}V_j, g^{jk}V_jV_k + V \in L_{loc}^1(X)$; we use Einstein's summation rule. Then Q is correctly defined on $C_0^\infty(X)$. Let us assume that

$$(H_1) \quad c^{-1} \leq |\xi|^{-2} g^{jk}(x) \xi_j \xi_k \leq c \quad \forall x \in X, \xi \in \mathbb{R}^d \setminus 0$$

and there are given functions $\delta, \rho = \rho_0, \rho_1, \dots$ on X such that

$$(H_2) \quad \rho_m(x) \geq 0, \delta(x) \geq 0, |\delta(x) - \delta(y)| \leq |x - y|$$

and for every point $y \in X' = \{x, \rho\delta > 1\}$ in $X \cap B(y, \delta(y))$ the following conditions are fulfilled:

$$(H_3) \quad c^{-1} \leq \rho_m(x)/\rho_m(y) \leq c, \quad |D_j \rho_m| \leq c \rho_m / \delta,$$

$$m = 0, 1, \dots, \rho_1 \geq \rho / \delta, \rho \geq \rho_2 \geq 1 / \delta,$$

$$(H_4) \quad |D^B g^{jk}| \leq c \delta^{-|B|},$$

$$|D^B F^j| \leq c \rho_1 \delta^{-|B|},$$

$$|D^B V| \leq c \rho^2 \delta^{-|B|} \quad \forall B: |B| \leq K < \infty,$$

$$(H_5) \quad \partial X \cap B(y, \delta(y)) = \{x_k = z(x_k)\} \cap B(y, \delta(y)) \quad \text{with}$$

$$|D^B z| \leq c \delta^{1-|B|} \quad \forall B: |B| \leq K$$

where $B(y, \delta)$ is an open ball with a center y and a radius δ ,

$x_k = (x_1, x_3)$ etc, $k = k(y) = 1, 2, 3$ in (H_5) , $F^j = i \varepsilon^{jk1} D_k V_1$

are components of the vector intensity of the magnetic field, ξ^{jkl} is absolutely skew-symmetric pseudo-tensor with $\xi^{123} = 1/\sqrt{g}$, $g = \det(g_{jk})$, $(g_{jk}) = (g^{jk})^{-1}$. Let $F = (g_{jk} F^j F^k)^{1/2}$ be a scalar intensity of the magnetic field. Let us further assume that

$$(H_6) \quad \forall y \in \{X', \varrho_1 > c \varrho / \gamma\} \quad \text{in } X \cap B(y, \gamma(y)) \\ F \geq c^{-1} \varrho_1,$$

$$(H_5)' \quad \forall y \in \{X', \varrho_1 > c \varrho / \gamma, v + F \leq \varepsilon \varrho^2\} \\ B(y, \gamma(y)) \subset X$$

with $\varepsilon > 0$.

Moreover, let us assume that for every $y \in X''' = \{X', \varrho_1 > c^{-1} \varrho^2, \varrho_2 < c_1^{-1} \varrho, v + F \leq \varepsilon \varrho^2\}$ in $B(y, \gamma(y))$ the following inequalities are fulfilled:

$$(H_7) \quad |D^B(v + (2j+1)F)| \leq c \varrho_2^2 \gamma^{-|B|} \quad \forall B: |B| \leq K$$

with $j = j(y) \in \mathbb{Z}^+$, $c_1 > 8c$; then X''' is a union of the disjoint domains X_j''' .

Let

$$\gamma_1 = \gamma^2 |\nabla(v/F)| \varrho_1 / \varrho^2 + \\ \gamma \min_{j \in \mathbb{Z}^+} |v + (2j+1)\mu h F|^{1/2} / \varrho \quad (\mu = h = 1 \text{ here})$$

on X' and $\gamma_2 = \gamma_1 \varrho / \varrho_2$ on X''' .

Finally, let us assume that

$$(H_8) \quad Q(u) \geq c^{-1} \int (|\nabla u|^2 - w|u|^2) dx \quad \forall u \in C_0^\infty(X'')$$

where $X'' = \{X, \varrho \gamma < 2\}$, $w \in L_{loc}^1(X)$, $w \geq 0$.

Our the first principal result is

Theorem 1. Let conditions (H_1) - (H_8) be fulfilled. Then

$$\mathcal{N} - CR_1 - C'R_2 \leq N \leq \mathcal{N} + C(R_1 + R_3) + C'R_2$$

where

$$\mathcal{N} = (1/2 \pi^2) \sum_{j=0}^{\infty} \int_{X^j} (V + (2j+1)F)^{-1/2} F \sqrt{g} dx ,$$

$$z_{\pm} = \max(\pm z, 0), \quad R_1 = \sum_{t=1}^5 R_{1t}, \quad R_2 = \sum_{t=1}^2 R_{2t},$$

$$R_{11} = \int_{\{X^1 \setminus X_0''', V+F \leq \varepsilon \rho^2 \gamma\}} \rho^2 \gamma^{-1} dx ,$$

$$R_{12} = \int_{\{X^1 \setminus X''', V+F \leq \varepsilon \rho^2, \gamma_1 \geq \rho^{-1/2+\sigma} \gamma^{1/2+\sigma}\}} \rho_1 \gamma_1^{-1} dx ,$$

$$R_{13} = \int_{\{X^1 \setminus X'''; V+F \leq \varepsilon \rho^2, \gamma_1 \leq \rho^{-1/2+\sigma} \gamma^{1/2+\sigma}\}} \rho_1 \rho^{1/2+\sigma} \gamma^{-1/2+\sigma} dx ,$$

$$R_{14} = \sum_{j=0}^{\infty} \int_{\{X_j''', V+(2j+1)F \leq \varepsilon \rho_2^2, \gamma_2 \geq \rho_2^{-1/2+\sigma} \gamma^{1/2+\sigma}\}} \rho^2 \gamma_2^{-1} dx ,$$

$$R_{15} = \sum_{j=0}^{\infty} \int_{\{X_j''', V+(2j+1)F \leq \varepsilon \rho_2^2, \gamma_2 \leq \rho_2^{-1/2+\sigma} \gamma^{1/2+\sigma}\}} \rho^2 \rho_2^{1/2+\sigma} \gamma^{-1/2+\sigma} dx ,$$

$$R_{21} = \int_{X^1} \rho^3 \rho_1^{-s} \gamma^{-2s} dx ,$$

$$R_{22} = \int_{X'''} \rho^2 \rho_2^{-s} \gamma^{-1-s} dx ,$$

$$R_3 = \int_{X''} W^{3/2} dx ,$$

here and in what follows $\xi > 0$, $\sigma > 0$, s are arbitrary and $C = C(c)$, $C' = C'(c, c_2, \xi, \sigma, s)$, $K = K(\sigma, s)$ in $(H_4), (H_7)$.

Remark 2. If conditions $(H_1)-(H_8)$ are fulfilled and if $\mathcal{N} + R_1 + R_2 + R_3 < \infty$ then Q is semi-bounded from below on $L^2(X)$ and hence it generates a self-adjoint Schrödinger operator $A = (D_j - V_j)g^{jk}(D_k - V_k) + V$ on X with the Dirichlet boundary condition; then N is a dimension of its invariant negative subspace.

Theorem 1 is a more refined and general version of the principal theorems announced in [1,2]. Moreover, under a certain condition of a global nature concerning integral curves of the vector field (F^1, F^2, F^3) one can derive a more precise estimates. If A depends on parameters then theorem 1 implies asymptotics of N with respect to these parameters (see e.g. [1,2]).

2. The following assertion is the crucial step in the proof of theorem 1:

Theorem 3. Let

$$A_{\mu, h} = (hD_j - \mu V_j)g^{jk}(hD_k - \mu V_k) + V \quad (2)$$

with the Dirichlet boundary condition be a self-adjoint semi-bounded Schrödinger operator with the discrete spectrum and with the polynomial growth of the eigenvalue counting function $N(\lambda)$ as $\lambda \rightarrow \infty$; here $h \in (0, 1]$, $\mu \geq 1$; let $e(x, y, \lambda, \mu, h)$ be a Schwartz kernel of its spectral projector. Let $y \in X$ and in $B(y, 1) \subset X$ conditions $(H_1)-(H_6)$ be fulfilled with $\gamma = \rho = \rho_1 = 1$; moreover, let $\psi \in C_0^K(B(y, 1/2))$, $0 \leq \psi \leq 1$, $|D^B \psi| \leq c_2 \gamma^{-|B|} \quad \forall B: |B| \leq K$. Then

(i) The following estimate holds:

$$\mathcal{R} = \left| \int (e(x, x, 0, \mu, h) - S(x, \mu h)h^{-d}) \psi^2(x) dx \right| \leq$$

$$\begin{aligned}
& ch^{-2} \left(1 + \mu h \int \gamma_1^{-1} dx \right) + \\
& \quad \left\{ B(y,1), \gamma_1 \geq h^{1/2-\sigma} \right\} \\
& \mu h^{1/2-\sigma} \text{ mes } \left\{ B(y,1), \gamma_1 \leq h^{1/2-\sigma} \right\} + c'h^{-1}
\end{aligned}$$

where

$$S(x, \mu h) = (1/2 \pi^{d-1}) \sum_{j=0}^{\infty} (V + (2j+1) \mu h F)^{(d-2)/2} \mu h F \sqrt{g}.$$

(ii) If $V + \mu h F \geq \varepsilon$ in $X \cap B(y,1)$ (and not necessarily $B(y,1) \subset X$ here) then

$$e(x, x, 0, \mu, h) \leq c'h^s \mu^{-s} \quad \forall x \in X \cap B(y, 1/2).$$

(iii) If $V = -(2j+1)F + \zeta^2 V'$ with $\zeta \in (h, 1]$ and $j \in \mathbb{Z}^+$ and if V' satisfies $(H_4)_3$ with $\gamma = \varrho = 1$ then for $\mu = h^{-1}$

$$\begin{aligned}
\mathcal{R} \leq & ch^{-2} \left(1 + \int \gamma_2^{-1} dx \right) + \\
& \left\{ B(y,1), \gamma_2 \geq (h/\zeta)^{1/2-\sigma} \right\}
\end{aligned}$$

$$(h/\zeta)^{-1/2-\sigma} \text{ mes } \left\{ B(y,1), \gamma_2 \leq (h/\zeta)^{1/2-\sigma} \right\} + c'h^{-1} \zeta^{-1};$$

here $\varrho_2 = \zeta$ in the definition of γ_2 .

(iv) Moreover, if $V' \geq \varepsilon$ in $B(y,1)$ then

$$e(x, x, 0, \mu = h^{-1}, h) \leq c'h^{-2} (h/\zeta)^s \quad \forall x \in B(y, 1/2).$$

The proof of theorem 3 is complicated and it is based on the quasiclassical microlocal analysis of the non-stationary Schrödinger equation with parameters μ, h . Without conditions $(H_5)', (H_6)$ the similar assertion holds for $\mu \leq c, d \geq 2$ and it is the basis of the proof of the principal theorems of [3]. When all these asymptotics are established we generalize them first to arbitrary balls and then complete the proof of theorem 1 by means of an appropriate partition of unity and Rosenblyum variational estimate for an eigen-

value counting function for operator generated by Q in $L^2(X'', \mathcal{J}dx)$ with an admissible weight function \mathcal{J} . One can find the similar procedure in [4].

3. Let us consider now the case $d = 2$. Certainly, now $F = F^3$, $F^3 = i(D_1V_2 - D_2V_1)/\sqrt{g}$. This case is not completely investigated yet. However I have proved the following

Theorem 4. Let $d = 2$ and all the conditions of theorem 3 be fulfilled. Then

(i) If $\gamma_1 \geq c^{-1}$ then

$$\mathcal{R} \leq c \mu^{-1} h^{-1} + c \cdot \mu^{-2}.$$

(ii) Assertion (ii) of theorem 3 holds.

(iii) If $V = -(2j+1)F + \zeta^2 V'$ with $\zeta \in (h, 1]$ and $j \in \mathbb{Z}^+$ and if V' satisfies $(H_4)_3$ with $\gamma = \rho = 1$ and if $\gamma_2 \geq c^{-1}$ (here $\rho_2 = \zeta$ in the definition of γ_2) then

$$\mathcal{R} \leq c \zeta^{-2} + c \cdot h \zeta^{-1}.$$

(iv) On the other hand, if

$$|V + (2j+1) \mu h F| \geq \zeta^2 \geq c \mu^{-2} \quad \forall j \in \mathbb{Z}^+ \quad \forall x \in B(y, 1)$$

then

$$\begin{aligned} |e(x, x, \lambda_2, \mu, h) - e(x, x, \lambda_1, \mu, h)| &\leq c \cdot (h/\zeta)^s \\ \forall x \in B(y, 1/2) \quad \forall \lambda_1, \lambda_2 \in (-\zeta^2/c, \zeta^2/c) &\quad (3). \end{aligned}$$

(v) Moreover, if

$$\begin{aligned} |V + (2j+1) \mu h F + \mu^{-2} R V^2 / 8F| &\geq \zeta^2 \geq \varepsilon \mu^{-2} \\ \forall j \in \mathbb{Z}^+ \quad \forall x \in B(y, 1) &\end{aligned}$$

where R is a scalar curvature associated with the metrics (g_{jk}/F) and if $\mu^{-1} + \mu h \leq \delta = \delta(c, \varepsilon)$ then inequality (3) holds.

Let us note that in the two-dimensional case the presence of the intensive magnetic field can improve the remainder estimate; on the

other hand, in this case the gaps in the quasiclassical limit of the spectrum can appear. The role of Landau levels $E_j = V + (2j+1)\mu hF$ is more important in the two-dimensional case than in three-dimensional one; there is a correction $\Delta E_j = \mu^{-2} R(E_j - V)^2 / 8F$ to these levels.

4. Finally, it is well-known that if $X = \mathbb{R}^d$, $d = 2, 3$, g^{jk} , F^j are constant and $V = 0$ then

$$e(x, x, \lambda, \mu, h) =$$

$$(1/2 \pi^{d-1}) \sum_{j=0}^{\infty} (\lambda - (2j+1)\mu hF)_+^{(d-1)/2} \mu h^{1-d} F \sqrt{g};$$

in particular,

$$\sigma(A) = \sigma_{ac}(A) = [\mu hF, \infty) \quad \text{for } d = 3,$$

$$\sigma(A) = \sigma_{ess}(A) = \sigma_{pp}(A) = \{(2j+1)\mu hF, j \in \mathbb{Z}^+\}$$

for $d = 2$.

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