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LINEAR AND NON LINEAR FIELD EQUATIONS

par S. KLAINERMAN

The aim of this lecture is to illustrate how some recent geometric techniques which were usual to derive global existence and long time existence results for non linear wave equations ([1], [2], [3]) can be applied to tensorial field equations. We limitate ourselves here in describing the results which we have obtained in collaboration with D. Christodoulou, to the linear Maxwell and Spin - 2 equations in Minkowski space (see [4]). The latter are a linearised version of the Einstein equations in vacuum and their study important in our attempt to prove the global non linear stability of the Minkowski metric.

Consider the Minkowski space \mathbb{R}^{3+1} with canonical coordinates (x^α) $\alpha = 0,1,2,3$ and metric

$$(1) \quad ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta$$

where η is the diagonal matrix with entries $(-1,1,1,1)$. The coordinate x^0 is usually denoted by t . The following vector fields are conformal killing i.e. vector fields X so that $\mathcal{L}_X \eta$ is proportional to η .

(2i) The 4 generators of the translation group

$$T_\mu = \frac{\partial}{\partial x^\mu} \quad \mu = 0,1,2,3$$

(2ii) The 6 generators of the Lorentz group

$$\Omega_{\mu\nu} = x_\mu \frac{\partial}{\partial x^\nu} - x_\nu \frac{\partial}{\partial x^\mu}$$

where $x_\mu = \eta_{\mu\nu} x^\nu$

(2iii) The scaling vector field

$$S = x^\mu \frac{\partial}{\partial x^\mu}$$

(2iv) The 4 accelerations vector fields

$$K_\mu = -2x_\mu S + \langle x, x \rangle \frac{\partial}{\partial x^\mu}$$

with $\langle x, x \rangle = \eta_{\mu\nu} x^\mu x^\nu$.

The Lie algebra Π generated by T, Ω, S plays a very important role in what follows. Given a tensor U in \mathbb{R}^{3+1} , we define the norms

$$(3) \quad \|U(t)\|_{\Pi, s}^2 = \sum \int_{\mathbb{R}^3} |\mathcal{L}_{X_{i_1}} \dots \mathcal{L}_{X_{i_k}} U(t, x)|^2 dx$$

with the sum taken over all generator $X_{i_1} \dots X_{i_k}$, $0 \leq k \leq s$, of Π .

Here, $\mathcal{L}_{X_{i_1}} \dots \mathcal{L}_{X_{i_k}} U$ denotes the repeated Lie derivatives of U with

respect to $X_{i_1} \dots X_{i_k}$ and $|\cdot|$ denotes the euclidian norm in \mathbb{R}^{3+1} .

Also, x refers to x^1, x^2, x^3 , $dx = dx^1 dx^2 dx^3$.

The Maxwell equations in \mathbb{R}^{3+1} apply to antisymmetric 2-tensors $F_{\alpha\beta}$ which are required to satisfy the following two pairs of equations

$$(M_i) \quad F_{\alpha\beta;\gamma} + F_{\beta\gamma;\alpha} + F_{\gamma\alpha;\beta} = 0$$

$$(M_{ii}) \quad F_{\alpha\beta}^{;\beta} = 0$$

where $F_{\alpha\beta;\gamma}$ denotes the covariant differentiation of F relative to the flat Minkowski metric. The energy momentum tensor of (M_i) is given by the 2-tensor

$$(4) \quad Q_{\alpha\beta} = F_{\alpha\gamma} F_{\beta}^{\gamma} + {}^*F_{\alpha\gamma} {}^*F_{\beta}^{\gamma}$$

where *F is the Hodge dual of F i.e. ${}^*F_{\alpha\beta} = \epsilon_{\alpha\beta\gamma\delta} F^{\gamma\delta}$ and $\epsilon_{\alpha\beta\gamma\delta}$ the components of the volume 1-form of \mathbb{R}^{3+1} . We remark that Q has the following properties.

- symmetric in α, β
- traceless, i.e. $\eta^{\alpha\beta} Q_{\alpha\beta} = 0$
- satisfies the positive energy condition i.e. given any two time-like vector fields X, Y , $\langle X, X \rangle < 0$, $\langle Y, Y \rangle < 0$, both future oriented, we have :

$$Q(X, Y) = Q_{\alpha\beta} X^{\alpha} Y^{\beta} > 0$$

- $Q_{\alpha\beta}^{;\beta} = 0$

The energy momentum tensor allows one to derive energy estimates for (M). Let F be a solution of (M), and consider X a time-like vector field. Let $P^{\alpha} = Q^{\alpha\beta} X_{\beta}$ be the X -momentum of F . Then,

$$(5) \quad P_{\alpha}^{;\alpha} = \frac{1}{2} Q^{\alpha\beta} (X_{\alpha;\beta} + X_{\beta;\alpha})$$

The expression $X_{\alpha;\beta} + X_{\beta;\alpha}$ is precisely $\mathcal{L}_X \eta_{\alpha\beta}$ and thus proportional to η , if we choose X to be conformal killing. On the other hand, since Q is traceless, we conclude that any choice of a conformal killing vector field leads to a conservation law in (5) i.e.

$$(5') \quad P_{\alpha}^{;\alpha} = 0$$

Integrating (5') on slots $[0, t] \times \mathbb{R}^3$ we infer that ,

$$(6) \quad \int_{t=\text{const}} Q(T_o, X) dx = \int_{t=0} Q(T_o, X) dx$$

where $T_o = \frac{\partial}{\partial x^0} = \frac{\partial}{\partial t}$.

According to the positive energy condition, $Q(T_0, X)$ is everywhere positive if X is time-like. The only two choices of conformal killing time-like vector fields are $X = T_0$ and $X = K_0$. In fact, let $\bar{K}_0 = K_0 + T_0$. Then, according to (6), we have

$$(6') \quad \int_{t=\text{const}} Q(T_0, \bar{K}_0) dx = \int_{t=0} Q(T_0, \bar{K}_0) dx$$

According to (6'), we introduce the norm

$$(7) \quad \|F(t)\|^{\#} = \left(\int_{\mathbb{R}^3} Q(T_0, \bar{K}_0) dx \right)^{1/2}$$

where Q is the energy momentum tensor of F . Also, we define

$$(7') \quad \|F(t)\|_{\Pi, s}^{\#2} = \sum \| \mathcal{L}_{X_{i_1}} \dots \mathcal{L}_{X_{i_k}} F(t) \|^{\#2}$$

with the sum extended over all choices of vector fields X_1, \dots, X_k , $0 \leq k \leq s$, among the generators of Π . Now, due to the conformal equivalence of the equations (M), we can easily check that if F is a solution, the so is $\mathcal{L}_X F$ for any conformal vector field X . As a consequence, we conclude that

$$(8) \quad \|F(t)\|_{\Pi, s}^{\#} = \|F(0)\|_{\Pi, s}^{\#}$$

and finite if the right hand side is finite. Since the right hand side depends only on initial conditions for F at time $t=0$, we conclude that $\|F(t)\|_{\Pi, s}^{\#}$ can be made globally finite, by requiring appropriate conditions at infinity, for the initial data. Finally, we concise this feet to derive uniform decay properties for F . To state our theorem, we need to introduce null frames in \mathbb{R}^{3+1} . Thus, let $e_+ = \partial_t + \partial_r$, $e_- = \partial_t - \partial_r$ and e_l, e_r vector fields orthogonal to e_+, e_- , and to each other, and of length one. The vector field $\frac{\partial}{\partial r}$ is the radial vector field $\frac{x^i}{|x|} \partial_i$ with $|x|^2 = \sum_{i=1}^3 (x^i)^2$.

We decompose F relative to the null frame according to :

$$(9) \quad \begin{aligned} \alpha_A &= F_{A+} \quad , \quad \underline{\alpha}_A = F_{A-} \quad , \quad A = 1, 2 \\ \rho &= F_{+-} \quad \quad \sigma = {}^* F_{+-} \end{aligned}$$

Here, α and $\underline{\alpha}$ are vectors tangent to the spheres $|x| = \text{const}$ in \mathbb{R}^3 while ρ and σ are scalars. Clearly they determine the full tensor F .

The finiteness of the norm $\|F(t)\|_{\Pi, s}^{\#}$ can be used to prove the following

Theorem 1 : Let F be a solution of the Maxwell equations (M) with initial conditions at $t=0$ for which the norm $I_s = \|F(0)\|_{\Pi, s}^{\#}$, $s \geq 2$, is finite, then

$$(i) \quad |F(t, x)| \leq C(1+t)^{-5/2} I_2$$

for any $t \geq 0$, $x \in \mathbb{R}^3$, $|x| \leq \frac{t}{2} + 1$

$$(ii) \quad |\underline{\alpha}(t, x)| \leq C(1 + |t - |x||)^{-3/2} (1 + t + |x|)^{-1} I_2$$

$$|(\rho, \sigma)(t, x)| \leq C(1 + |t - |x||)^{-1/2} (1 + t + |x|)^{-2} I_2$$

$$|\alpha(t, x)| \leq C(1 + t + |x|)^{-5/2} I_2$$

for any $t \geq 0$, $x \in \mathbb{R}^3$, $|x| \geq \frac{t}{2} + 1$.

Similar estimates can be derived for the derivatives of F in the interior, $|x| \leq \frac{t}{2} + 1$, or for the derivatives $\alpha, \underline{\alpha}, \rho, \sigma$ relative to the null frame e_+, e_-, e_1, e_2 in the exterior $|x| \geq \frac{t}{2} + 1$ (see [4]).

In the second part of this lecture, I will indicate how similar results, based on the same ideas, can be used to derive decay estimates for the Spin-2 equations. These are equations satisfied by 4-tensors $W_{\alpha\beta\gamma\delta}$ which have all the symmetry properties of the Riemann curvature tensor of metric satisfying the Einstein vacuum equations. Namely,

$$(i) \quad W_{\alpha\beta\gamma\delta} = -W_{\beta\alpha\gamma\delta} = -W_{\alpha\beta\delta\gamma}$$

$$W_{\alpha\beta\gamma\delta} = W_{\gamma\delta\alpha\beta}$$

$$(ii) \quad W_{\alpha\beta\gamma\delta} + W_{\alpha\gamma\delta\beta} + W_{\alpha\delta\beta\gamma} = 0$$

$$(iii) \quad W_{\alpha\beta\gamma}^{\beta} = 0$$

The Spin-2 equations are

$$(Sp) \quad W_{\alpha\beta\gamma\delta;\epsilon} + W_{\alpha\beta\delta\epsilon;\gamma} + W_{\alpha\beta\epsilon\gamma;\delta} = 0$$

As the Maxwell equations, the Spin-2 equations are conformal invariant. In particular, for any solution W and any conformal vector field X , $\mathcal{L}_X W$ is also a solution. What corresponds to the energy momentum tensor for the Maxwell equations is now a 4-tensor Q defined by

$$(10) \quad Q_{\alpha\beta\gamma\delta} = W_{\alpha\mu\gamma\nu} W_{\beta\delta}^{\mu\nu} + {}^*W_{\alpha\mu\gamma\nu} {}^*W_{\beta\delta}^{\mu\nu}$$

with ${}^*W_{\alpha\beta\gamma\delta} = \epsilon_{\alpha\beta}^{\mu\nu} W_{\mu\nu\gamma\delta}$ the Hodge dual of W .

One can prove that Q satisfied the following properties

- Q is symmetric and traceless relative to all pair of indices
- Q satisfied the positive energy condition i.e. given any X, Y time like and future oriented :

$$Q(X, X, Y, Y) = Q_{\alpha\beta\gamma\delta} X^{\alpha} X^{\beta} Y^{\gamma} Y^{\delta} > 0$$

$$\bullet Q_{\alpha\beta\gamma\delta}^{;\delta} = 0$$

whenever W is a solution of (Sp).

One can now proceed as in the derivation of the energy identities for the Maxwell equations to show that

$$(11) \quad \int_{t=cste} Q(T_o, T_o, \bar{K}_o, \bar{K}_o) dx = \int_{t=0} Q(T_o, T_o, \bar{K}_o, \bar{K}_o) dx \quad .$$

Or, introducing the norm

$$(12) \quad \|W(t)\|^{\#} = \left(\int_{\mathbb{R}^3} Q(T_o, T_o, \bar{K}_o, \bar{K}_o) dx \right)^{1/2}$$

With Q the energy momentum tensor of W , and also,

$$(12') \quad \|W(t)\|_{\Pi, s}^{\#} + \left(\sum \mathcal{L}_{X_{i_1}} \dots \mathcal{L}_{X_{i_k}} W(t) \right)^{\#2} \Big)^{1/2}$$

for any generators X_{i_1}, \dots, X_{i_k} , $0 \leq k \leq s$ of Π ,

$$(13) \quad \|W(t)\|_{\Pi, s}^{\#} = \|W(0)\|_{\Pi, s}^{\#}$$

We now decompose W relative to the same null frame introduced above, and introduce

$$(14) \quad \begin{aligned} \alpha_{AB} &= W_{A+B+} & , & & \underline{\alpha}_{AB} &= W_{A-B-} \\ \beta_A &= \frac{1}{2} W_{A+ \dot{-} +} & & & \underline{\beta}_A &= \frac{1}{2} W_{A- \dot{-} -} \\ \rho &= \frac{1}{4} W_{+ - \dot{-} + -} & & & \sigma &= \frac{1}{4} * W_{+ - \dot{-} + -} \quad . \end{aligned}$$

Clearly, $\alpha, \underline{\alpha}, \beta, \underline{\beta}, \rho, \sigma$ completely determine W , and we can prove the following

Theorem 2 : Let W be a solution of (Sp) with initial conditions at $t = 0$ for which $I_s = \|W(0)\|_{\Pi, s}^{\#} < +\infty$ for some $s \geq 2$. Then

$$(i) \quad |W(t, x)| \leq C (1+t)^{-7/2} I_2$$

for any $t \geq 0$, $x \in \mathbb{R}^3$, $|x| \leq \frac{t}{2} + 1$

$$(ii) \quad |\underline{\alpha}(t, x)| \leq C (1 + |t - |x||)^{-5/2} (1 + t + |x|)^{-1} I_2$$

$$|\underline{\beta}(t, x)| \leq C (1 + |t - |x||)^{-3/2} (1 + t + |x|)^{-2} I_2$$

$$|(\rho, \sigma)(t, x)| \leq C (1 + |t - |x||)^{-1/2} (1 + t + |x|)^{-3} I_2$$

$$|\alpha(t, x)| \leq C (1 + t + |x|)^{-7/2} I_2$$

for any $t \geq 0$, $x \in \mathbb{R}^3$, $|x| \geq \frac{t}{2} + 1$.

Similar estimates can be derived for the derivatives of W (see [4]).

The spirit of these linear estimates can be adjusted to treat the non linear Einstein equations. This, I hope, will be done in a series of papers together with D. Christodoulou.

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