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Semiclassical resonances in some simple cases.

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0. Introduction.

In this talk we report on some further developpements of the work on resonances in the semiclassical limit , started with B. Helffer in [7] . (See also [8] for a survey.) We start by recalling briefly the theory developped in [7] , which is essentially a microlocal version of the method of complex scaling initiated by Aguilar-Combes [1] and Balslev-Combes[2] . Let $P = -h^2\Delta+V(x)$, where V is analytic and real-valued , and let $p(x,\xi) = \xi^2+V(x)$ be the corresponding (principal) symbol. (All our results are actually valid for a more general class of operators.) In order to define resonances (i.e. certain complex eigenvalues) near 0 , we make the following assumptions :

(0.1) There exist smooth functions r , $R \in C^\infty(\mathbb{R}^n)$ such that $r \geq 1$, $rR \geq 1$, $\partial^\alpha r = O(r\,R^{-|\alpha|})$, $\partial^\alpha R = O(R^{1-|\alpha|})$ uniformly on $\mathbb{R}^n$ for all $\alpha \in \mathbb{N}^n$.

(0.2) There exists $C > 0$ such that V extends holomorphically to $\{ x \in \mathbb{C}^n ; |\mathrm{Im}\, x| \leq C^{-1} R(\mathrm{Re}\, x)\}$, and satisfies $|V(x)| \leq C\, r(\mathrm{Re}\, x)^2$.

(0.3) There exists a real-valued (escape-)function $G \in C^\infty(\mathbb{R}^{2n})$ with $\partial_x^\alpha \partial_\xi^\beta G = O(\tilde{r}^{1-|\beta|} R^{1-|\alpha|})$ for $|\alpha|+|\beta| \geq 1$, such that $H_p G \geq r^2/C$ in $p^{-1}(0)\setminus K$, where K is some compact set and $C > 0$ is some constant. Here $\tilde{r}(x,\xi) = (r(x)^2+\xi^2)^{\frac{1}{2}}$.

After a suitable modification of G in the region where $|\xi| \gg r(x)$, we can define certain weighted Sobolev spaces $H(\Lambda_{tG}, m)$, when $t>0$ and $h>0$ are small enough. (See [7] for details.) Here $\Lambda_{tG} \subset \mathbb{C}^{2n}$ is given by $\mathrm{Im}(x,\xi) = t\, H_G(\mathrm{Re}(x,\xi))$, and very roughly, we have $u \in H(\Lambda_{tG},1)$ iff $\tilde{u} \in L^2(e^{-2tG/h}dxd\xi)$, where $\tilde{u} = u(x,\xi)$ is a suitable FBI-transform of u. In [7], we obtained the following basic result :

<u>Theorem 0.1.</u> For $t>0$ sufficiently small, there exists $h_0>0$ and a neighborhood $\Omega \subset \mathbb{C}$ of 0 such that for $0<h<h_0$:

For all $z \in \Omega$ the operator $(P-z) : H(\Lambda_{tG},\tilde{r}^2) \to H(\Lambda_{tG},1)$ is Fredholm of index 0. Moreover, there is a discrete set $\Gamma(h) \subset \Omega$ such that $P-z$ is bijective for $z \in \Omega\setminus\Gamma(h)$, and splits in a natural way into a direct sum of one bijective operator and one nilpotent operator : $F_z \to F_z$, when $z \in \Gamma(h)$. Here $F_z \subset H(\Lambda_{tG},\tilde{r}^2) \subset H(\Lambda_{tG},1)$ is a non-trivial finite-dimensional space.

The elements of $\Gamma(h)$ are called resonances, and if $z \in \Gamma(h)$, then $\dim F_z$ is the corresponding (algebraic) multiplicity. In [7] we showed that a different choice of $t>0$ or of G gives rise to the same resonances and the same spaces F_z in some sufficiently small neighborhood of 0. We also showed that the resonances belong to the closed lower half plane.

In order to formulate the general problems and the rather special results that we have obtained so far, we first recall a simple geometric discussion from Gérard-Sjöstrand [6] (related to the geometric scattering theory, see Reed-Simon [11]).

Let $\varepsilon_0 > 0$ be so small, that the conclusions of (0.3) remain valid also on $p^{-1}(\varepsilon)$ for $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$. For $\rho \in p^{-1}([-\varepsilon_0, \varepsilon_0])$, put $\Phi_t(\rho) = \exp tH_p(\rho)$ for t in the maximal interval of definition $]T_-(\rho), T_+(\rho)[$, $T_\pm(\rho) \in]0, \pm\infty]$. We then define the outgoing (+) and incoming (−) tails by

$$\Gamma_\pm = \{\rho \in p^{-1}([-\varepsilon_0, \varepsilon_0]) \; ; \; \Phi_t(\rho) \not\to \infty \; , \; t \to T_\mp(\rho)\} .$$

We then have the following properties :

1^o $\Gamma_\pm$ are closed, $\Gamma_+ \cap \{G \leq T\}$ and $\Gamma_- \cap \{G \geq -T\}$ are compact for all $T \in \mathbb{R}$.

2^o For some $T_0 > 0$, we have $\Gamma_+ \subset \{G \geq -T_0\}$, $\Gamma_- \subset \{G \leq T_0\}$.

3^o $K = \Gamma_+ \cap \Gamma_-$ is compact.

4^o If $\Gamma_- \neq \emptyset$ (or if $\Gamma_+ \neq \emptyset$) then $K \neq \emptyset$.

5^o If we define the true tails $\mathcal{T}_\pm = \Gamma_\pm \setminus K$, then the symplectic volume of $\mathcal{T}_\pm$ is equal to 0.

6^o The following statements are equivalent :

(i) $\mathcal{T}_+ \neq \emptyset$, (ii) $\mathcal{T}_- \neq \emptyset$,

(iii) The set $\{\rho \in p^{-1}([-\varepsilon_0, \varepsilon_0]) \setminus K \; ; \; \mathrm{dist}(\rho, K) \leq \alpha\}$ is non empty for every $\alpha > 0$.

We also introduce $\Gamma_\pm^0 = \Gamma_\pm \cap p^{-1}(0)$, $K^0 = K \cap p^{-1}(0)$. The properties $1^o, \ldots, 4^o$ are true also with K, $\Gamma_\pm$ replaced by K^0, $\Gamma_\pm^0$. 4^o, 5^o also remain valid under the additional assumption that $dp \neq 0$ everywhere on $p^{-1}(0)$. (We then replace the symplectic volume by the corresponding Liouville measure.) We have the following unwritten theorem of [7] :

Theorem 0.2 . If $K^0 = \emptyset$, then there are no resonances in some fixed h-independent neighborhood of 0 .

The interesting problem is then to find out what happens when $K^0 \neq \emptyset$. In [7] ,(see also [4] ,) we analyzed the case of a potential well in an island . In that case the resonances are generated by tunneling through a potential barrier and they are exponentially close to the real eigenvalues of a certain self adjoint eigenvalue problem . Moreover , we have $\Gamma_+ = \Gamma_- = K$, so the true tails are empty.

We shall here describe two other simple cases , when it is possible two give a rather complete description of the resonances in certain regions. In both cases , it is rather easy to make some simple WKB-constuctions in order to guess the asymptotics of the resonances . The difficulty is rather to prove that these approximate WKB-resonances are close to actual resonances , and that there are no others . There is no place to discuss the methods of the proofs here and we refer to [6] and [12] for further details.

1. The case of a closed trajectory of hyperbolic type.

This is joint work with C. Gérard analogous to Gérard's extension [5] of Ikawa's results [9] , [10] in the case of obstacles . We assume

(1.1) $p = 0 \Rightarrow dp \neq 0$.

(1.2) K^0 is the image of a simple closed trajectory $[0,T^0] \ni t \mapsto \exp(tH_p)(\rho^0) = \gamma^0(t)$.

Let $H \subset p^{-1}(0)$ be a hypersurface which intersects γ^0 transversally at ρ^0. We then have the Poincaré map $H \to H$ obtained by following the flow of H_p once along γ^0. ρ^0 is then a fixed point and we let P^0 be the differential at ρ^0. We assume,

(1.3) γ^0 is of hyperbolic type .

This means that P^0 has no eigenvalues of modulus 1 . By the implicit function theorem , the whole situation is stable if we replace $p^{-1}(0)$ by $p^{-1}(\varepsilon)$ for $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$, if $\varepsilon_0 > 0$ is small enough . Let then $\gamma^\varepsilon : [0, T^\varepsilon] \to p^{-1}(\varepsilon)$, ρ^ε , P^ε be corresponding quantities. Let $\theta_1(\varepsilon), \ldots, \theta_{n-1}(\varepsilon)$, $1/\theta_1(\varepsilon), \ldots, 1/\theta_{n-1}(\varepsilon)$ be the eigenvalues of P^ε with $|\theta_j(\varepsilon)| > 1$. We can show the following geometrical facts:

$\Gamma_\pm$ are involutive analytic manifolds intersecting transversely along $\bar{\gamma} = \bigcup \gamma^\varepsilon$.

Let L_+^ε be the sum of the eigenspaces of P^ε corresponding to the eigenvalues θ_j , and let $\mathcal{P}^{N-1}(L_+^\varepsilon)$ be the space of complex polynomials of degree $\leq N-1$ on this space . Then $P^\varepsilon|_{L_+^\varepsilon}$ induces a map $D_*^\varepsilon : \mathcal{P}^{N-1} \to \mathcal{P}^{N-1}$, which has the eigenvalues $\theta^{-\alpha} = \theta_1^{-\alpha_1} \cdots \theta_{n-1}^{-\alpha_{n-1}}$, $|\alpha| \leq N-1$.

If we introduce the action :

$$C(\varepsilon) = \int_{\gamma^\varepsilon} dx ,$$

then $C'(\varepsilon) = T(\varepsilon)$.

In [6] we also define a certain analytic function $\rho(\varepsilon)$ satisfying $\rho(\varepsilon) = |\theta_1(\varepsilon) \cdots \theta_{n-1}(\varepsilon)|^{-\frac{1}{2}}$.

<u>Theorem 1.1.</u> Let $\varepsilon_0 > 0$ be sufficiently small. Let $C_0>0$. Choose N so large that the following set does not increase if we further increase N:

$$\Gamma^0(h) = \{E \in [-\varepsilon_0,\varepsilon_0]-i[0,C_0h];\ \det(I-e^{iC(E)/h}\rho(E)D_*^E)=0\}.$$

Then if we count the elements of each set with their natural multiplicities, there is for $h>0$ sufficiently small an injective map $b(h) : \Gamma^0(h) \to \{\text{resonances of } P\}$, such that $b(h)(\mu)-\mu = o(h)$ uniformly in h and μ. The image of $b(h)$ contains all resonances in a slightly smaller rectangle, $[-\varepsilon_0+\delta h,\varepsilon_0-\delta h]-i[0,(C_0-\delta)h]$.

Notice that if E belongs to the rectangle in the definition of $\Gamma^0(h)$, then E belongs to $\Gamma^0(h)$ iff there are $k \in \mathbb{Z}$ and $\alpha \in \mathbb{N}^{n-1}$ such that

$$C(E) = 2\pi kh+ih\log\rho(E) - ih\sum\alpha_j\log\theta_j(E).$$

There is actually a more refined result:

<u>Theorem 1.2.</u> Let ε_0, C, N be as in Theorem 1.1. Then we have a classical symbol, holomorphic for (z,E) in a suitable h-independent domain:

$$F_{-+}(E,z,h) \sim \sum_0^\infty A_j(E,z)\, h^{j/2},$$

with $A_0(E,z) = I - z^{-1}\rho(E)\, D_*^E$, such that if $\Gamma^\infty(h) = \{E \in [-\varepsilon_0,\varepsilon_0]-i[0,C_0h];\ \det F_{-+}(E,e^{-iC(E)/h},h) = 0\}$, then there is an injective map $b(h) : \Gamma^\infty(h) \to \{\text{resonances of } P\}$, such that $b(h)(\mu)-\mu = O(h^\infty)$. Again the image of $b(h)$ contains all resonances in a slightly smaller rectangle.

2. The case of a non-degenerate critical point.

Here we describe the results of [12]. Not only the results, but also the proofs are close to those of [6], and the proofs are even a little simpler. In the special case of a potential maximum, intersecting results have recently and independently been obtained by Briet-Combes-Duclos [3].

We assume that K^0 is reduced to a point :

$$K^0 = \{(x_0,\xi_0)\} \quad . \tag{2.1}$$

Since the Hamilton field of p has to vanish at that point, we have $\xi_0 = 0$, and after a translation, we may also assume that $x_0 = 0$. Then we also have that $\nabla V(0) = 0$, so 0 is a critical point with critical value 0. We shall also assume that this point is non-degenerate,

$$\det V''(0) \neq 0 \ . \tag{2.2}$$

(For operators more general than the Schrödinger operators an additional assumption is necessary, but we shall not discuss this here.)

After a linear change of the x-coordinates, we may assume that

$$2\, p(x,\xi) = \sum_{1}^{n-d} \lambda_j(\xi_j^2+x_j^2) + \sum_{n-d+1}^{n} \lambda_j(\xi_j^2-x_j^2) + O(|(x,\xi)|^3) \quad , \tag{2.3}$$

near $(0,0)$, and the eigenvalues of the linearization of H_p at $(0,0)$ are then $\pm z_j$, $j=1,..,n$, where $z_j = i\,\lambda_j$, $j=1,..,n-d$, and $z_j = \lambda_j \underset{\text{def.}}{=} \mu_{j-n+d}$, $j = n-d+1,..,n$.

The H_p-flow then has a stable outgoing manifold ; L_+, of dimension d, which passes through $(0,0)$ and such that

$T_{(0,0)}(L_+)$ = the sum of eigenspaces corresponding to $\mu_1,..,\mu_d$. It is easy to show that $L_+ = \Gamma_+^0$. Similarly Γ_-^0 is the stable incoming manifold corresponding to $-\mu_1,...,-\mu_d$.

After a linear symplectic change in the last group of variables, we may write,

$$(2.4) \qquad p(x,\xi) = p'(x',\xi') + \tfrac{1}{2} Ax''\cdot\xi'' + O((x,\xi)^3) \quad ,$$

where $x' = (x_1,..,x_{n-d})$, $x'' = (x_{n-d+1},..,x_n)$, and where p' is a positive definite quadratic form, while A is a matrix with spectrum = $\{\mu_1,..,\mu_d\}$. Then $T_{(0,0)}(\Gamma_+^0)$ is spanned by the directions $\partial_{x''}$. Choose scalar products $\langle x'',y''\rangle$, $[\xi'',\eta'']$ so that $\langle Ax'',x''\rangle > 0$, $[{}^tA\xi'',\xi''] > 0$ for x'', $\xi'' \neq 0$. We can then consider a local escape function :

$$G(x,\xi) = \langle x'',x''\rangle - [\xi'',\xi''] \ .$$

It turns out that $H_pG \sim |(x,\xi)|^2$ on $p^{-1}(0)$, and that on Λ_{tG} intersected with a sufficiently small neighborhood of the origin, the function $p|_{\Lambda_{tG}}$ takes its values in a sector $\arg z \in [\theta_2-\pi,\theta_1]$, where $\theta_j > 0$, and for every fixed (sufficiently small) t, we may take θ_1 as small as we like. Furthermore, $\left|p|_{\Lambda_{tG}}\right| \sim |(x,\xi)|^2$.

We are here in a situation completely analogous to the well-known case of degenerate elliptic operators with double characteristics, if we think of Λ_{tG} as our new $\mathbb{R}^{2n}$. Let Λ_+ be the complex stable outgoing (Lagrangian) manifold of dimension n associated to the flow of $e^{-i\theta}H_p$, for $\theta>0$ small. Then $T_{(0,0)}(\Lambda_+)$ is the sum of the eigenspaces associated to the eigenvalues z_j. We then know from [13]

that Λ_+ is strictly positive with respect to Λ_{tG} .
It turns out that the resonances close to 0 correspond to WKB-functions associated to Λ_+ , and as in section 1 , we first state a simplified version of the result :

Theorem 2.1. Choose $C_0>0$ such that none of the values

$$(2.5) \qquad -ih\,\Sigma\,(\alpha_j+\tfrac{1}{2})z_j \quad , \quad \alpha \in \mathbb{N}^n$$

is on the boundary of the disc $D(0,C_0h)$. Let $\Gamma^0(h)$ be the set of values (2.4) inside the disc . We count the elements of $\Gamma^0(h)$ with their natural multiplicity . Then for sufficiently small h there is a bijection $b(h)$ from $\Gamma^0(h)$ to the set of resonances of P inside $D(0,C_0h)$, such that $b(h)(\mu)-\mu = o(h)$ uniformly with respect to μ and h .

To state the complete asymptotic result , choose complex symplectic coordinates centered at $(0,0)$; (x,ξ) , such that Λ_+ is given by $\xi = 0$ and such that the corresponding incoming manifold for $e^{-i\theta}H_p$ is given by $x=0$. Then $p = Bx\cdot\xi+O((x,\xi)^3)$, where the spectrum of B is $\{z_1,..,z_n\}$.
Then we put

$$P_0 = -i\,Bx\cdot\partial_x - \tfrac{1}{2}i\,\Sigma\,z_j \ .$$

The eigenvalues of P_0 in the space $\mathcal{P}^N$ of polynomials of degree $\leq N$ are then the values $-i\,\Sigma\,(\alpha_j+\frac{1}{2})z_j$ with $|\alpha| \leq N$. With C_0 as before , we fix N so large that no such values with $|\alpha| > N$ are in the disc $D(0,C_0)$.

Theorem 2.2. There exists a matrix $F_{-+}(z,h) : \mathcal{P}^N \to \mathcal{P}^N$, depending holomorphically on $z \in D(0,(C_0+\delta)h)$ (for some $\delta > 0$), which is a classical symbol in h with an asymptotic expansion $F_{-+}(z,h) \sim \sum_0^\infty A_j(z)\, h^{j/2}$, where $A_0 = P_0 - z$, such that the following holds :

Let $\tilde{\Gamma}(h)$ be the set of roots in $D(0,C_0 h)$ of $\det F_{-+}(E/h,h)$, counted with their natural multiplicity. Then for sufficiently small h, $\tilde{\Gamma}(h)$ is equal to the set of resonances of P inside $D(0,C_0 h)$.

3. Examples of resonances , which are second order poles for the resolvent.

Here we only give a rough sketch and refer to [12] for detailed statements and proofs. We shall produce our examples by a perturbation argument . In $\mathbb{R}^2$, we consider the unperturbed Scrödinger operator

$$(3.1) \qquad P_0 = -h^2\Delta + V_0(x) \quad ,$$

where $V_0(x) = -x^2$. (This potential is very large near infinity , but enters into the general framework of [7] , besides the arguments of this section work equally well if V_0 is a rotation invariant analytic function with $V_0(x) = -1+o(1)$ as $x \to \infty$ in a domain $|\mathrm{Im}\, x| \leq C^{-1}|\mathrm{Re}\, x|$, such that 0 is an absolute and non-degenerate maximum on $\mathbb{R}^n$ with $V_0(0)=0$.) The resonances of P_0 are then

$$(3.2) \qquad -ih(2+2(\alpha_1+\alpha_2)) \quad , \quad \alpha=(\alpha_1,\alpha_2) \in \mathbb{N}^2 \quad .$$

Here the "first" resonance $-2ih$ is simple , but the next one ; $\lambda_0(h) = -4ih$ is double . Now perturb the potential:

$$V = V_0 + (hq_2(x)+q_4(x))e^{-x^2/2} ,$$

where q_j are real j-homogeneous polynomials and q_4 is sufficiently small so that the theory of [7] applies with the same standard escape function for $P=-h^2\Delta+V$ as for P_0. The double resonance $\lambda_0(h)$ then splits into two possibly equal resonances , of distance at most $o(h)$ from $\lambda_0(h)$, and if we let F be the corresonding 2-dimensional sum of eigenspaces , then the matrix of $P|_F$ for a suitable basis in F is given by

$$\lambda_0 I+h^2\tilde{M}(q_2,q_4,h) = \lambda_0 I+h^2 M(q_2,q_4)+O(h^3) . \tag{3.3}$$

Here M is a real-linear function of (q_2,q_4) , which can take arbitrary values in the space of complex symmetric 2x2 - matrices , while $\tilde{M}$ is a smooth function of (q_2,q_4) , with $\tilde{M}-M = O(h)$ in the C^∞ sense . We may assume that q_4 is allowed to be so large that we may have $M(q_2,q_4)$ take any value in some neighborhood of

$$M_0 = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$$

in the space of complex symmetric 2x2-matrices . Otherwise we could just replace M_0 by a small positive multiple .

Now the complex 2x2-matrices near M_0 with double eigenvalues form a hypersurface H , and the elements of H are of the form $\lambda+N$, with $N^2=0$, $N\neq 0$. It is easy to see that if we restrict (q_2,q_4) to a suitable 2-dimensional real plane , then the corresponding matrices $\tilde{M}$ form a smooth real 2-dimensional surface , intersecting H tranversally at a point near M_0 . The conclusion is then that for all sufficiently small values of h , we can find q_2 , q_4 such that P has a resonance $\lambda(h)$ of multiplicity 2 with $\lambda(h)-\lambda_0(h)=o(h)$, such that if F is the corresponding 2-dimensional space , then $P|_F = \lambda(h)+N(h)$, where $N^2=0$, $N\neq 0$. In particular , $(P-z)^{-1}$ has a second order pole at $\lambda(h)$. It seems to have been an open question wether such resonances exist.

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