Journées Équations aux dérivées partielles

Linda P. Rothschild

Existence of solutions for transversally elliptic left invariant differential operators on nilpotent Lie groups

Journées Équations aux dérivées partielles (1983), p. 1-6
<http://www.numdam.org/item?id=JEDP_1983____A12_0>

Existence of solutions for transversally elliptic left invariant differential operators on nilpotent Lie groups.

Linda Preiss Rothschild
University of Wisconsin - Madison

1. Introduction and notation. We describe here some recent results, obtained jointly with Lawrence Corwin [3] on solvability of left invariant differential operators on nilpotent Lie groups. For related results see [2], [8], [9], [10], [11], [14], [15], [16], [17].

We consider first operators on a 2-step nilpotent Lie group G, i.e. we assume that the Lie algebra $\mathfrak{G}$ is a vector space direct sum $\mathfrak{G} = \mathfrak{G}_1 + \mathfrak{G}_2$ with $[\mathfrak{G}_1, \mathfrak{G}_1] = \mathfrak{G}_2$ and $[\mathfrak{G}_2, \mathfrak{G}] = (0)$. For $\eta \in \mathfrak{G}_2^*$ let B_η be the bilinear form $B_\eta(X_1, X_2) = \eta([X_1, X_2])$ for $X_1, X_2 \in \mathfrak{G}_1$. B_η assumes its maximal rank on a Zariski open subset in $\mathfrak{G}_2^*$. Recall that to every $\ell \in \mathfrak{G}^*$ we may associate, by the Kirillov theory, an irreducible unitary representation π_ℓ of G, realized on a Hilbert space of the form $L^2(\mathbb{R}^k)$ for some k.

By a transversally elliptic operator on G we shall mean a left invariant differential operator L on G which is an elliptic polynomial on $\mathfrak{G}_1$, i.e.

$$L = L_m + L_{m-1} + \cdots + L_0, \tag{1.1}$$

with L_j homogeneous of degree j, and L_m an elliptic polynomial on $\mathfrak{G}_1$.

2. Necessary conditions for local solvability. We give the following criterion, which generalizes known results [2] for homogeneous operators i.e. those for which $L_j \equiv 0$, $j < m$.

Theorem 1. Let L be a left invariant operator on G which is transversally elliptic. Assume that there is a non-empty open set $V \subset \mathfrak{G}^*$ such that

(2.1) $$\ker \pi_\ell(L^\tau) \neq 0 \quad \text{for all} \quad \ell \in V,$$

or, equivalently,

(2.2) $$\ker L^\tau \cap L^2(G) \neq 0 .$$

Then L is not locally solvable.

The idea of the proof is as follows. First, if B_η is degenerate for all η, then [1] may be applied to show that the hypothesis is vacuous. So assume B_η nondegenerate for η in a Zariski open set. We show, using microlocal constructions as in [6] that there is a pseudo-differential operator Π not of order $-\infty$ such that $L^\tau\Pi$ is of order $-\infty$. Now for any distribution σ for which $Lv - \sigma = 0$ in an open set U for some distribution v, $\Pi^\tau(Lv - \sigma)$ is smooth, and hence $\Pi^\tau\sigma$ is smooth. Hence σ cannot be arbitrary.

3. Sufficient conditions for solvability on H-groups. G is called an H-group if B_η is nondegenerate for $\eta \neq 0$. We prove the following converse to Theorem 2 for H-groups. A globally defined differential operator P is uniformly semi-globally solvable if there is an integer r such that for every bounded open neighborhood U of 0 there exists a distribution σ_U of order at most r such that $L\sigma_U = \delta$ in U.

Theorem 2. If G is an H-group and L a left invariant transversally elliptic operator on G then L is uniformly semi-globally solvable if

(2.1) *and* (2.2) *do not hold.*

The proof of Theorem 2 is somewhat similar to that of the corresponding result [17] in the case where L is homogeneous. Both rely on the theorem of Lojasiewicz which says that one can divide a distribution by a non-zero analytic function.

Corollary. If L_m *is locally solvable, then* L *is locally solvable.*

4. *Existence of global fundamental solutions.* Here we allow G to be any connected Lie group, not necessarily nilpotent.

Theorem 3. Let L *be a left invariant differential operator on* G *which is uniformly semi-globally solvable. Suppose that* G *is* L-*convex. Then* L *has a global fundamental solution; i.e. there is a distribution* σ *on* G *for which* $L\sigma = \delta$.

The proof of Theorem 3 involves a construction similar to that used in proving that L-convexity implies global solvability. The theorem gives a new result even for homogeneous operators.

Corollary 1. Let L *be a homogeneous left invariant differential operator on a nilpotent Lie group* G *with dilations. If* L *is locally solvable at* 0 *then* L *has a global fundamental solution.*

Corollary 2. If L *is a transversally elliptic operator on an* H-*group which satisfies the hypothesis of Theorem* 2 *then* L *has a global fundamental solution.*

5. <u>Global criteria for hypoellipticity.</u> The various global criteria for local solvability for homogeneous differential operators on nilpotent groups, e.g. $\ker L^{\tau} \cap L^2(G) = (0)$, suggest that the representation-theoretic criterion of Helffer-Nourrigat [5] may be reformulated. Indeed, using a recent Liouville-type theorem of Geller [4] one may obtain the following.

<u>Theorem 4.</u> (Geller, Helffer-Nourrigat). *Let G be a stratified nilpotent Lie group and L a homogeneous left invariant differential operator on G. Then L is hypoelliptic if and only if there is no non-constant bounded function f on G such that $Lf = 0$.*

References

[1] Boutet de Monvel, L., Grigis, A., and Helffer, B., Paramétrixes d'Opérateurs Pseudo-Differentiels à Caracteristiques Multiples, Astérisque 34-35 (1976), pp. 93-121.

[2] Corwin, L., and Rothschild, L. P., Necessary Conditions for Local Solvability of Homogeneous Left Invariant Differential Operators on Nilpotent Lie Groups, Acta Math. 147 (1981), pp. 265-288.

[3] Corwin, L., and Rothschild, L. P., Solvability of Transversally Elliptic Differential Operators on Nilpotent Lie Groups, preprint.

[4] Geller, D., preprint.

[5] Helffer, B. and Nourrigat, J., Caracterisation des Operateurs Hypoélliptiques Homogènes Invariants à Gauche sur une Groupe de Lie Nilpotent Gradué. Comm. P. D. E. 4 (8) (1979) 899-958.

[6] Grigis, A., and Rothschild, L. P., A Criterion for Analytic Hypoellipticity for a Class of Differential Operators with Polynomial Coefficients, Annals of Math., to appear.

[7] Kato, T., Perturbation Theory for Linear Operators, 2nd ed., Springer-Verlag, Berlin 1976.

[8] Lévy-Bruhl, P., Résolubilité de Certain Operateurs Invariants du Second Ordre sur les Groupes de Lie Nilpotents de Rang Deux, Bull. Sci. Math. 104 (1980), pp. 369-391.

[9] Lévy-Bruhl, P., Résolubilité Locale d'Opérateurs Homogénes Invariants Gauche sur Certains Groupes de Lie Nilpotents de Rang Trois, preprint.

[10] Lévy-Bruhl, P., Application de la Formule de Plancherel à la Resolubilité d'Opérateurs Invariants à Gauche sur des Groupes de Lie Nilpotent d'Ordre Deux, Bull. Sci. Math (2) 106 (1982) pp. 171-191.

[11] Lévy-Bruhl, P., Resolubilité Locale d'Operateurs Non Homogènes sur des Groupes de Lie Nilpotents Gradués, preprint.

[12] Lojasiewicz, S., Sur le Problème de Division, Studia Math. 18 (1959), pp. 87-136.

[13] Melin, A., Parametrix Constructions for some Classes of Right-Invariant Differential Operators on the Heisenberg Group. Comm. P. D. E. (to appear).

[14] Métivier, G., Equations aux Derivées Partielles sur les Groupes de Lie Nilpotents, Sem. Bourbaki, Exposé #583, 1981.

[15] Rothschild, L. P., Local Solvability of Second Order Differential Operators on Nilpotent Lie Groups, Ark. Math., 19 (1981), pp. 145-175.

[16] Rothschild, L. P., and Stein, E. M., Hypoelliptic Differential Operators and Nilpotent Lie Groups, Acta Math. 137 (1976), pp. 247-320.

[17] Rothschild, L. P. and Tartakoff, D., Inversion of Analytic Matrices and Local Solvability of Some Invariant Differential Operators on Nilpotent Lie Groups, Comm. P. D. E. 6 (1981), pp. 625-650.