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THE POISSON SUMMATION FORMULA FOR A DIRICHLET PROBLEM WITH GLIDING AND GLANCING RAYS

by M. J. BENNETT and F. G. FRIEDLANDER

1. Introduction. Let M be a compact Riemannian manifold with smooth boundary, and let Δ be the Laplacian on M. This is an unbounded operator on $L^2(M)$ which has a self adjoint extension with domain the Sobolev space $\{u \in H^2(M): u|\partial M = 0\}$, whose spectrum subset of $\mathbb{R}^-$, say $\{0 > -\mu_1^2 \geq -\mu_2^2 \geq \cdots \to -\infty\}$. The corresponding eigenfunctions e_j are a complete orthonormal set in $L^2(M)$; in fact, they are C^∞ and satisfy

$$(1)\qquad \Delta e_j + \mu_j^2 e_j = 0, \quad e_j|\partial M = 0, \quad j = 1, 2, \ldots$$

in the classical sense. So the μ_j and e_j are, respectively, the eigenvalues and the eigenfunctions of the Dirichlet problem for Δ on M. The eigenvalues μ_j, which we take to be positive, satisfy Weyl's estimate $\#\{\mu_j \leq \tau\} = O(\tau^{\dim M})$ as $\tau \to \infty$. Hence the spectral measure

$$(2)\qquad \sigma(\tau) = \sum_{j=1}^{\infty} \delta(\tau - \mu_j)$$

is a tempered distribution.

Consider now the following initial value problem for the wave equation on $M \times \mathbb{R}$:

$$(3)\qquad \begin{cases} (\partial_t^2 - \Delta)u = 0, \quad u|_{t=0} = f \in C_c^\infty(\mathring{M}), \quad \partial_t u|_{t=0} = 0, \\ u|_{\partial M \times \mathbb{R}} = 0. \end{cases}$$

For any t, this defines a map $C_c^\infty(\mathring{M}) \ni f \to u(.,t) \in C^\infty(M)$ whose Schwartz kernel is a function $K: \mathbb{R} \to \mathcal{D}'(M \times M)$ which can be expanded as

$$(4)\qquad K(x,y,t) = \sum_{j=1}^{\infty} e_j(x)\, e_j(y) \cos \mu_j t .$$

One can also look upon this as a function $M \times M \to \mathcal{D}'(\mathbb{R})$, and as such it has a trace given by

$$(5)\qquad \operatorname{tr} K = \int K(x,x,t)\, dg_x = \sum_{j=1}^{\infty} \cos \mu_j t ,$$

where dg_x is the Riemann measure on M. By (2) one can write this identity also as

$$(6)\qquad \operatorname{tr} K = \hat{\sigma}_e(t)$$

where $\hat{\sigma}_e$ is the even part of the Fourier transform of σ ,

$$(7)\qquad \hat{\sigma}_e(t) = \tfrac{1}{2}(\hat{\sigma}(t) + \hat{\sigma}(-t)) = \tfrac{1}{2}(\sigma(\tau) + \sigma(-\tau))^{\wedge},$$

Andersson and Melrose [1] have shown that, if ∂M is everywhere geodesically concave or convex, then (6) extends the Poisson formula for compact boundaryless manifolds due to Chazarain [3] and Duistermaat and Guillemin [4] , to the Dirichlet problem for Δ . In particular, the singular support of $\hat{\sigma}_e$ is contained in the set

$\{ T \in \mathbb{R} :$ $|T|$ is the length of a closed broken geodesic on M or of a closed boundary geodesic $\}$.

Here, the broken geodesic flow includes reflection, with the usual 'equal angles' law, at the boundary, and the boundary is equipped with the induced Riemann metric. Furthermore, if $|T|$ is the length of a closed broken geodesic which meets M transversally a finite number of times, and satisfies a certain non-degeneracy condition, then Guillemin and Melrose [5] have established an extension to manifolds with boundary of the asymptotic expansions of [3] and [4] for the restriction of $\hat{\sigma}_e(t)$ to a sufficiently small neighbourhood of T.

This leaves two open questions. The first is that of the contribution of closed broken geodesics which graze the boundary; this can happen if ∂M has a geodesically concave connected component. The second one, which may be called the gliding ray problem, concerns the behaviour of $\hat{\sigma}_e$ in the neighbourhood of T when T is the length of a boundary geodesic.

We shall discuss a simple two-dimensional example which throws some light on these questions. The results are primarily due to the first author.

2. <u>The eigenvalue problem</u>. The manifold is a portion of a cylinder, $M = (0,d) \times (\mathbb{R}/2Y\mathbb{Z})$, where $Y > 0$ and $d > 0$, equipped with the metric $(1+x)(dx^2 + dy^2)$. So the eigenvalue problem (1) for our example can be put into the form

$$(9) \quad \begin{cases} (\partial_x^2 + \partial_y^2)\,\phi + \mu^2(1+x)\,\phi = 0 & \text{on} \quad (0,d)\times\mathbb{R}, \\ \phi\big|_{x=0} = \phi\big|_{x=d} = 0, \quad y \to \phi \ \text{has period } 2Y, \end{cases}$$

and we take $\mu > 0$.

(10) <u>Proposition</u>. With $x \in \mathbb{R}$, $\mu \in \mathbb{R}^+$, and $\eta \in \mathbb{R}$, write

$$(11) \qquad z_x = z_x(\mu,\eta) = \mu^{-4/3}(\eta^2 - (1+x)\mu^2),$$

and let $Ai(z)$, $Bi(z)$ be the standard solutions of Airy's equation $F''(z) = z\,F(z)$. (See [9], for example.) For each $m = 0, 1, \ldots$, let μ_{mj}, where $j = 1, 2, \ldots$, be the roots of

$$(12) \qquad Ai(z_d^m)\,Bi(z_o^m) - Ai(z_o^m)\,Bi(z_d^m) = 0,$$

arranged in ascending order; here $z_x^m = z_x(\mu, mY/\pi)$. Then the μ_{mj} are the eigenvalues of (9); they are simple if $m = 0$, and of multiplicity 2 if $m > 0$.

The proof is straightforward, and omitted. It is convenient to let m range over $\mathbb{Z}$ and put

$$(13) \qquad \mu_{-m,j} = \mu_{mj}, \quad m < 0,\ j = 1, 2, \ldots;$$

this takes care of the multiplicities. The spectral measure (2) is then

$$(14) \qquad \sigma(\tau) = \sum_{m=-\infty}^{\infty}\sum_{j=1}^{\infty} \delta(\tau - \mu_{mj}),$$

and the even part of its Fourier transform, (7), becomes

$$\hat{\sigma}_e(t) = \sum_{m=-\infty}^{\infty} \sum_{j=1}^{\infty} \cos \mu_{mj} t. \tag{15}$$

3. <u>The broken geodesic flow</u>. For our example, the wave equation is

$$Pu = (1+x)\,\partial_t^2 u - \partial_x^2 u - \partial_y^2 u\,.$$

The geodesic flow on T^*M is just the bicharacteristic flow of P. Leaving aside the zero section ('geodesics of zero length'), one can restrict this to $S^*M = \{(x,y,\xi,\eta) \in T^*M : \xi^2 + \eta^2 = 1 + x\}$, and t then gives the (signed) length of the geodesics, which are the bicharacteristic curves. On the covering manifold $\tilde{M} = (0,d) \times \mathbb{R}$, one can visualize these as the trajectories of a billiard ball on an infinitely long inclined billiard table whose (parallel) edges are horizontal, and perfectly reflecting.

From now on, we shall refer to the broken geodesics, both on $\tilde{M}$ and on M, as geodesics. A closed geodesic on M, of length $T \neq 0$, is the image under $\tilde{M} \to M$ of a geodesic on M such that $x(T) = x(0)$, $y(T) = y(0) + 2nY$, where $n \in \mathbb{Z}$, consisting of parabolic arcs reflected or grazing at the boundary. Here n is the winding number; one must also associate an integer $k \neq 0$ with the geodesic, where $|k|$ is the number of reflections at $x = d$, with $k > 0$ if $T > 0$, and $k < 0$ if $T < 0$. We denote such a geodesic by γ_{nk}. It will be said to be of type I if it does not meet $x = 0$, of type II if it is reflected alternately at $x = d$ and at $x = 0$, and grazing if it is tangent to $x = 0$. Geodesics of type II are of no interest for the problem in hand, and will be ignored. Elementary computations give the following:

(16) Proposition. Let λ be a real number, and put

$$(17) \qquad Y_\lambda = 2\lambda(1+d-\lambda^2)^{\frac{1}{2}}, \qquad T_\lambda = \frac{2}{3}(1+d-\lambda^2)^{\frac{1}{2}}(1+d+2\lambda^2) ;$$

let n and k be nonzero rational integers. There is a closed geodesic γ_{nk} of type I, with length $2kT_\lambda$, if there is a λ such that $1<\lambda^2<1+d$ and

$$(18) \qquad kY_\lambda = nY .$$

This has no (real) solutions if $|n/k| > (1+d)/Y$. If $|n/k| \leq (1+d)/Y$, then (18) has one solution λ_{nk} such that $\lambda_{nk}^2 \geq \frac{1}{2}(1+d)$; if also $|n/k| > 2d^{\frac{1}{2}}/Y$, then the second solution λ'_{nk} of (18), for which $\lambda'^{\,2}_{nk} < \frac{1}{2}(1+d)$, is also admissible. If $d^{\frac{1}{2}}/Y$ is a rational number, and $|n/k| = 2d^{\frac{1}{2}}/Y$, then (18) holds for $\lambda = 1$ or for $\lambda = -1$, and the corresponding γ_{nk} is grazing.

Remark. Let F^t: $S^*M \longrightarrow S^*M$ be the map obtained by letting every point of S^*M move for a time t along the lifted (broken) geodesic issuing from it, with a suitable convention for points lying above ∂M. If $\gamma \subset M$ is a closed geodesic of (signed) length T, then it is clear that the points of γ, lifted to S^*M, and their y-translates, are the fixed point set of F^T. So this set has dimension 2. One can show that it is clean, in the sense of [4] and of [5], unless γ is of type I and $|\lambda| = (\frac{1}{2}(1+d))^{\frac{1}{2}}$. Such a geodesic will be called degenerate; it occurs when the roots of (18) coincide, and one then also has

$$(19) \qquad \partial Y_\lambda / \partial \lambda = 0 .$$

4. The trace formula. In our example, the first member of (5) can be obtained without explicitly determining K by solving the initial value problem (3). One needs a technical lemma.

(20) Lemma. Let $z \in \mathbb{R}$, and put

(21) $$\chi(z) = \frac{1}{\pi}\int_z^\infty \frac{dt}{Ai^2(t) + Bi^2(t)} .$$

Then $\chi \in C^\infty(\mathbb{R})$ is positive and strictly decreasing, and one has

(22) $$\tan \pi\,\chi(z) = Ai(z)/Bi(z) \quad \text{if } Bi(z) \neq 0 .$$

Furthermore, $-\chi'(z)$ is also strictly decreasing. For z large and positive, one has $\chi(z) = O(\exp(-4z^{3/2}/3)$ and

(23) $$\pi\,\chi(-z) = \tfrac{1}{4}\pi + \tfrac{2}{3}z^{3/2} + O(z^{-3/2}) .$$

This follows from standard properties of the solutions of Airy's equation [9]. One can now reformulate Proposition (10). With z_x defined by (11), put

(24) $$\xi = \chi\circ z_d(\tau,\eta) - \chi\circ z_0(\tau,\eta), \quad (\tau,\eta) \in \mathbb{R}^+ \times \mathbb{R} .$$

Then $\xi > 0$, and $\tau \to \xi$ is strictly increasing. One can therefore invert (24) to obtain

(25) $$\tau = \mu(\xi,\eta) \in C^\infty(\mathbb{R}^+ \times \mathbb{R}),$$

and infer from (12) that the eigenvalues of the Dirichlet problem (9) are given by $\mu_{mj} =$ (j, $m\pi/Y$). So one can write (15) as

(26) $$\hat{\sigma}_e(t) = \sum_{m,j=-\infty}^{\infty} \rho(j)\cos(\mu(j,m\pi/Y)t) ,$$

where $\rho(\xi) \in C^\infty(\mathbb{R})$ is such that

(27) $$\rho = 0 \quad \text{if } \xi \leq \delta, \quad \rho = 1 \quad \text{if } \xi \geq \delta', \quad 0 < \delta < \delta' < 1.$$

The second member of (26) converges in $\mathcal{S}'(\mathbb{R})$. So, if $\phi \in \mathcal{S}(\mathbb{R})$ is real valued, one has

$$\langle \hat{\sigma}_e, \phi\rangle = \operatorname{Re} \sum_{m,j=-\infty}^{\infty} \rho(j)\,\hat{\phi}\circ\mu(j, m\pi/Y).$$

It is not hard to show that $\rho(\xi)\hat{\phi}\circ\mu(\xi,\eta)\in\mathcal{S}(\mathbb{R}^2)$. One can therefore appeal to the classical Poisson summation formula, and after some manipulations, one obtains:

(28) <u>Proposition</u>. Let $\phi\in\mathcal{S}(\mathbb{R})$ be real valued. Then

$$(29)\quad \langle\hat{\sigma}_e,\phi\rangle = \operatorname{Re}\sum_{n,k=-\infty}^{\infty}\int \sigma_{nk}(\tau)\hat{\phi}(\tau)\,d\tau = \operatorname{Re}\sum_{n,k=-\infty}^{\infty}\langle\hat{\sigma}_{nk},\phi\rangle,$$

where

$$(30)\qquad \sigma_{nk}(\tau) = \int A_{nk}(\tau,\lambda)\exp(iS_{nk}(\tau,\lambda))\,d\lambda ,$$

$$(31)\qquad S_{nk} = 2\pi k\xi(\tau,\lambda) + 2nY\lambda\tau ,$$

$$(32)\qquad \xi(\tau,\lambda) = \chi(\tau^{2/3}(\lambda^2-d-1)) - \chi(\tau^{2/3}(\lambda^2-1))$$

$$(33)\qquad (3\pi/2Y)A_{nk} =$$

$$\rho\circ\xi(\tau,\lambda)((2\lambda^2+1)\chi'(\tau^{2/3}(\lambda^2-1)) - (2\lambda^2+1+d)\chi'(\tau^{2/3}(\lambda^2-d-1))).$$

Also, $A_{nk}=0$ for $\tau\leqslant\delta''$, where $\delta''>0$ depends on the choice of ρ.

5. <u>The singularities of $\hat{\sigma}_e$</u>. These can now be examined by analysing the behaviour of $\sum\sigma_{nk}(\tau)$ as $\tau\to\infty$. Roughly speaking, the terms with $k = 0$ are related to the singularity at $t = 0$. As this is now well understood in the general case ([10], [8], [6]), it will not be discussed here.

For $k\neq 0$, it is found that the asymptotic behaviour of σ_{nk} yields information on the singularity of $\hat{\sigma}_e$ near $t = T_{nk}$, the length of the geodesic γ_{nk} of Proposition (16). We now go on to state the principal results obtained; the proofs will be published elsewhere [2]. As $\hat{\sigma}_e$ is even, we take $t>0$. We write

$$(34)\qquad \Sigma = \{T\in\mathbb{R}:\ \text{there is a closed geodesic on } M \text{ of length } |T|\}.$$

We shall use the notation, for any real number s,

$$(35) \qquad H^{s-}_{loc} = \{ f : f \in H^{t}_{loc}(\mathbb{R}) \text{ for } t < s \} .$$

We begin with the 'regular' case.

(36) <u>Theorem</u>. Let γ_{nk} be a non degenerate closed geodesic of type I, with n and k as in Proposition (16), $k > 0$. Let T_{nk} be the length of γ_{nk}, and $J \subset \mathbb{R}$ an open interval such that $J \cap \Sigma = \{T_{nk}\}$. Then there are complex numbers $a^{(m)}_{nk}$, $m = 0, 1, \ldots$ such that, for any $N \geqslant 0$,

$$(37) \qquad \hat{\sigma}_e(t)|J = \operatorname{Re} \sum_{m=0}^{N} a^{(m)}_{nk}(t-T_{nk}-i0)^{m-\frac{3}{2}} + r_N, \quad r_N \in H^{N-}_{loc} .$$

Also,

$$(38) \qquad a^{(0)}_{nk} = i^{k+\varepsilon} YT_{nk}/2\pi k^{3/2}|\partial Y_\lambda/\partial\lambda|^{\frac{1}{2}} ,$$

where λ is the appropriate solution of (18), and $\varepsilon = 1$ if $\lambda^2 < \frac{1}{2}(1+d)$, $\varepsilon = 0$ if $\lambda^2 > \frac{1}{2}(1+d)$.

The proof is in effect an application of the method of stationary phase to (30). The result is essentially that of [5] , allowing for the observation made in the remark following Proposition (16). The factor $i^{k+\varepsilon}$ incorporates the Maslov index and the changes of sign due to reflection at the boundary. The other factor in (38) is proportional to the so-called invariant volume of the relevant fixed point set of the geodesic flow on S*M.

It is clear from (19) and (38) that (37) cannot hold when the closed geodesic γ_{nk} is degenerate. In fact, the phase function which comes from (31) and (32) is then degenerate. However, this case is easy to handle. We only remark that, whereas in the non-degenerate case σ_{nk} is a classical symbol of order $\frac{1}{2}$, it is the sum of two such in the degenerate case, of orders $\frac{2}{3}$ and $\frac{1}{3}$ respectively, and omit the detailed formulae.

(39) Theorem. Suppose that $d^{\frac{1}{2}}/Y$ is a rational number, and that $|n/k| = 2d^{\frac{1}{2}}/Y$, $k > 0$. Then there is a closed grazing geodesic γ_{nk} of length $T_{nk} = 2n(2+d)/3Y$. Let $J \subset \mathbb{R}$ be an open interval such that $J \cap \Sigma = \{T_{nk}\}$. Then $\hat{\sigma}_e|J$ is the sum of two terms, one of which has the expansion (37), while the other one can be expanded as

$$(40) \qquad \operatorname{Re}\sum_{m=0}^{N} g_m(t-T_{nk}-i0)^{(m-4)/3} + r_N, \quad r_N \in H^{(2N-3)/6-}, \; N = 0, 1, \ldots$$

The g_m involve the (oscillatory) integrals

$$c_{km} = \frac{1}{2\pi i}\int w^m \frac{A_-^{k-1}(w)}{A_+^{k+1}(w)}\,dw$$

where $A_+(w) = \mathrm{Ai}(e^{2\pi i/3}w)$ and $A_-(w) = \mathrm{Ai}(e^{-2\pi i/3}w)$; in particular, g_0 is a multiple of $c_{k0}i^{-k}T_{nk}$.

In this case, the significant contribution to (30) comes from a neighbourhood of $\lambda = 1$ or $\lambda = -1$, and the term $\chi(\tau^{2/3}(\lambda^2-1))$ in S_{nk} cannot be handled by means of (23). However, it also follows from Lemma (20) that, if $k \in \mathbb{Z}$, then

$$\exp ik(\chi(z) - \tfrac{2\pi}{3}) = A_-^k(z)/A_+^k(z) .$$

This gives an alternative form of σ_{nk} which, with appropriate asymptotic analysis, gives (40). The 'strange constants' c_{km} resemble those which appear in the problem of forward scattering [7] and, like them, are no doubt related to the fact that Airy operators are needed for the construction of microlocal parametrices near diffractive points of the boundary.

Finally, we consider the gliding ray problem, perhaps the most interesting feature. Write $\partial^o M = \{d\}\times(\mathbb{R}/2Y\mathbb{Z})$ for the geodesically convex connected component of ∂M. Its (Riemannian) length is $L = 2Y(1+d)^{\frac{1}{2}}$. It is not a geodesic, but a limit of (broken) geodesics. Indeed, the following is easily deduced from Proposition (16):

(41) <u>Proposition</u>. The set of accumulation points of Σ is $\{\mathbb{Z}L\}$. For any $n > 0$, there is a $k_o > 0$ and a sequence γ_{nk}, $k = k_o, k_o+1, \ldots$ of non-degenerate closed type I geodesics such that $\lambda_{nk} \nearrow (1+d)^{\frac{1}{2}}$, $T_{nk} \nearrow nL$, and these γ_{nk} converge to $\partial^o M$ described n times with positive orientation. Similar statements are true for $n < 0$.

Theorem (36) holds for each γ_{nk}, but one cannot simply add the asymptotic expansions (37) in order to obtain the behaviour of $\hat{\sigma}_e(t)$ in the neighbourhood of $t = nL$. However, one easily sees from (38) and (17) $a^{(o)}_{nk} = O(k^{-2})$, so that the sum of the top order terms converges. Put

$$K_n(t) = \operatorname{Re} \sum_{k=k_o}^{\infty} a^{(o)}_{nk}(t-T_{nk}-i0)^{-3/2}$$

Then one has

(42) <u>Theorem</u>. Let n be a positive integer, and let J be an open interval such that $J \cap \Sigma = \{T_{nk} : k \geqslant k_o\}$, with k_o and T_{nk} as in Proposition (41). Then

$$\hat{\sigma}_e(t)|J = K_n(t) + O(H^{-3/4-}_{loc}) \,. \tag{43}$$

Observe that this is a genuine error estimate, as $K_n \in H^{-1-}_{loc}$; we do not know if it is the best possible.

As in the case of Theorem (39), the difficulty is that one has to work in a range of λ (a neighbourhood of $(1+d)^{\frac{1}{2}}$ or of $-(1+d)^{\frac{1}{2}}$) where the application of (23) to the phase function S_{nk} of Proposition (28) is problematical. There is a constant c such that, for any $\tau > 0$, the σ_{nk} with $k > c\tau^{1/3}$ are smooth; but one cannot control the error terms for the sum over $k \leqslant c\tau^{1/3}$. However, it turns out that one can do so for the sum of the σ_{nk} over $k \leqslant c'\tau^{1/4}$, and obtain another estimate for the range $c'\tau^{1/4} < k \leqslant c\tau^{1/3}$.

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