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S. REMPEL
BERT-WOLFGANG SCHULTZE

Resolvents of boundary problems for pseudo-differential operators without the transmission property

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Resolvents of Boundary Problems for Pseudo-Differential Operators without the Transmission Property

by

S. Rempel and B.-W. Schulze
Institut für Mathematik der Akademie der Wissenschaften der DDR,
1086 Berlin, Mohrenstrasse 39

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0. Introduction

The method of treating boundary value problems applied in the paper of Boutet de Monvel [B1] is formally analogous to that for pseudo-differential operators on manifolds without boundary. The usual symbolic calculus is replaced by a calculus for pairs (σ_X, σ_Y), where σ_X denotes the interior and σ_Y the so-called boundary symbol. Certain applications (e.g. the study of boundary problems with discontinuous boundary conditions or the complex powers of operators connected with boundary problems) lead in a natural way to symbols which have not the transmission property with respect to the boundary. Among these symbols for instance are those with non-integer orders. Thus one can ask whether the transmission property is necessary for the construction of an operator calculus with a corresponding symbolic level (interior and boundary symbols). A theory of boundary problems for operators with general symbols was developed by Višik and Eskin (see e.g. [V1], [E1]), but this approach seems not to be available to generate an algebra in connection with a symbolic calculus. In [R1] the authors give a theory of boundary problems generalizing Boutet de Monvel's theory to the case of symbols without the transmission property. Let us denote by $\mathcal{W}$ the class of operators

$$\mathcal{A} = \begin{pmatrix} A+B+W & K \\ T & Q \end{pmatrix} : \begin{matrix} H^s(X,E) \\ \oplus \\ H^{t+\lambda+1/2}(Y,J) \end{matrix} \longrightarrow \begin{matrix} H^t(X,F) \\ \oplus \\ H^{s-\gamma-1/2}(Y,G) \end{matrix} \qquad (0.1)$$

described in [R1] (cf. also Section 1 below) and let $\mathcal{G}$

be the subclass of Boutet de Monvel consisting of matrices of the sort (0.1), where the left upper corner has the form A+B, where A is a pseudo-differential operator of order s-t on the manifold X with boundary Y and B is a (singular) Green operator. E,F and J,G are vector bundles on X and Y, respectively. The fibre dimensions of J or G (or both) may vanish such that 'classical' boundary problems of the form

$$\mathcal{A} = \begin{pmatrix} A+B+W \\ \\ T \end{pmatrix} : H^{s}(X,E) \longrightarrow \begin{matrix} H^{t}(X,F) \\ \oplus \\ H^{s-\gamma-1/2}(Y,G) \end{matrix} \qquad (0.2)$$

are included as well as their parametrices (P+G+V,K) in the elliptic case.

In the present note we sketch some results of [R2 , II] on the resolvent construction of elliptic boundary problems in $\mathcal{B}$ of the form (0.2) following a similar approach as Seeley [S 1]. However, here we have some extra difficulties. E.g. the definition of general higher order operators on manifolds with boundary requires appropriate order reducing operators (cf. Section 1). In Section 2 we describe these reducing objects in detail (they are obtained in a similar way as the resolvents in [S1]). A second point is that the parameter dependence of the symbols is more complicated than usual. In our case a reasonable analogue of the resolvent approximation of [S1] is obtained (cf. Section 3). We suppose here that the boundary problem satisfies a so-called condition (ehp), i.e. that on boundary symbolic level there exists some elliptic homogeneous problem approximating the given problem for each fixed (x',ξ') on the cosphere bundle to the boundary. The precise definition will be given below, cf. also [R2, I]. More general cases will be treated in [R2, II]. Our results are related to to those of Grubb [G1], [G2] in the special cases considered there but the methods are different. They are also available to study operators of the form (0.1) depending on a parameter in a more general way.

1. Boundary problems for operators without the transmission property

In this section we introduce some definitions concerning the class $\mathfrak{D}$ of boundary problems. We first define a subclass of operators

$$\begin{matrix} H^s(Y,\mathcal{H}) \\ \oplus \\ H^s(Y,\mathbb{C}^{j_1}) \end{matrix} \longrightarrow \begin{matrix} H^{s'}(Y,\mathcal{H}) \\ \oplus \\ H^{s'}(Y,\mathbb{C}^{j_2}) \end{matrix} , \tag{1.1}$$

where

$$\mathcal{H} = H^0(\mathbb{R}_+) \tag{1.2}$$

denotes the space of square integrable functions on the x_n half axis $\mathbb{R}_+$ and $H^s(\ldots)$ denote the Sobolev spaces with values in $\mathcal{H}$ and $\mathbb{C}^j$, respectively, $s \in \mathbb{R}$. For convenience we only study scalar operators and trivial bundles E,F,J,G; the general case is an obvious generalization.

On $\mathcal{H}$ we have a dilation

$$\varkappa_\lambda : \mathcal{H} \to \mathcal{H} \tag{1.3}$$

defined by $(\varkappa_\lambda u)(x_n) = u(\lambda x_n)$, $\lambda > 0$. Set

$$\mathcal{H}_k = \begin{matrix} \mathcal{H} \\ \oplus \\ \mathbb{C}^{j_k} \end{matrix} , \qquad k = 1,2,$$

with fixed numbers $j_1, j_2 \in \mathbb{Z}_+$. By $S^l(R^{n-1}, \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2))$ we denote the Hörmander class S^l of operator-valued symbols (operators in $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$), $l \in \mathbb{R}$. An element $\mathfrak{a}(x', \xi') \in S^l(R^{n-1}, \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2))$, $(x', \xi') \in R^{n-1} \times R^{n-1}$, is called positively homogeneous of degree m for $|\xi'| \geqq 1$, if

$$\mathfrak{a}(x', \lambda\xi') = \lambda^m \begin{pmatrix} \varkappa_\lambda & 0 \\ 0 & 1 \end{pmatrix} \mathfrak{a}(x', \xi') \begin{pmatrix} \varkappa_\lambda & 0 \\ 0 & 1 \end{pmatrix}^{-1} \tag{1.4}$$

for $\lambda \geqq 1$, $|\xi'| \geqq 1$. A symbol $\mathfrak{a}$ in S^l is called classical ($\in S^l_{cl}$) if it has an asymptotic expansion into homogeneous symbols of degree $l-j$, $j \in \mathbb{Z}_+$. The pseudo-differential operator connected with $\mathfrak{a}$ will be denoted by $Op'(\mathfrak{a})$. For given $\mathfrak{a} \in S^l$ we have a continuous map

$$Op'(\mathfrak{a}) : H^s_{comp}(R^{n-1}, \mathcal{H}_1) \rightarrow H^{s-l}_{loc}(R^{n-1}, \mathcal{H}_2), \tag{1.5}$$

$s \in \mathbb{R}$. Since Y is compact, we suppose once and for all that all symbols we consider in local coordinates are independent of x' for $|x'|$ sufficiently large. In that case the subscripts comp and loc can be dropped in (1.5).

Per def. the operators in the class $\mathcal{D}^l$ on $X = Y \times \overline{\mathbb{R}}_+$, $-l \in \mathbb{Z}_+$, are locally of the form

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} + Op'(\mathfrak{a}) , \tag{1.6}$$

where $A = r^+ Op(a) e^+$ is a pseudo-differential operator defined by a symbol of a class to be described below,

$$Op(a)u = (2\pi)^{-n} \int e^{ix\xi} a(x,\xi) \hat{u}(\xi) d\xi \tag{1.7}$$

and

$$e^+ : H^s(R^{n-1},\mathcal{H}) \longrightarrow H^s(R^{n-1},\tilde{\mathcal{H}}),$$

$$r^+ : H^s(R^{n-1},\tilde{\mathcal{H}}) \longrightarrow H^s(R^{n-1},\mathcal{H})$$

($\tilde{\mathcal{H}} = H^0(\mathbb{R})$) denotes the extension by zero and the restriction to $x_n = 0$, respectively. $\mathfrak{a}(x',\xi') \in S^l_{cl}(\mathbb{R}^{n-1}, \mathcal{L}(\mathcal{H}_1,\mathcal{H}_2))$ has the form $\mathfrak{a} \sim \sum_{i=0}^{\infty} \mathfrak{a}_{l-i}$, where

$$\mathfrak{a}_j(x',\xi') = \begin{pmatrix} op(b)_j + \omega\, op(m)_j & op(k)_j \\ op(t)_j & q_j \end{pmatrix} (x',\xi') \quad (1.8)$$

are homogeneous of degree j in the sense (1.4) and the entries in (1.8) are defined as follows.

$$op(b)_{l-i} \in S^{l-i}(R^{n-1}, \mathcal{L}(\mathcal{H},\mathcal{H})) \quad (1.9)$$

has values in the space of Hilbert Schmidt operators in $\mathcal{H}$,

$$op(k)_{l-i} : \mathbb{C}^{j_1} \to \mathcal{H} \;,\quad op(t)_{l-i} : \mathcal{H} \longrightarrow \mathbb{C}^{j_2} \quad (1.10)$$

are arbitrary finite-dimensional operators, $q_{l-i}(x',\xi') : \mathbb{C}^{j_1} \longrightarrow \mathbb{C}^{j_2}$ is a pseudo-differential symbol in $S^{l-i}_{cl}(\mathbb{R}^{n-1})$. The notations in (1.9), (1.10) indicate that these objects can be described by functions $b(x',\xi',\vartheta,\tau)$ on $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}$ and $k(x',\xi',\vartheta)$, $t(x',\xi',\tau)$ on $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R}$, respectively, of corresponding homogeneities in the variables (ξ',ϑ,τ), (ξ',ϑ) and (ξ',τ), respectively (cf. [R1], [R2]), belonging to $F_n(H^0(\mathbb{R}_+))_\vartheta \otimes F_n(H^0(\mathbb{R}_-))_\tau$, $F_n(H^0(\mathbb{R}_+))_\vartheta$ and $F_n(H^0(R_-))_\tau$ (F_n denotes the Fourier transform with respect to x_n on $\mathbb{R}$). Finally $\omega\, op(m)_j = 0$ for $-j > 0$ and $\omega\, op(m)_o = \omega\, op(m)$ (only occurring for $l = 0$) is given by

$$u \rightarrow \omega(\zeta(\xi')x_n)M^{-1}(m(x',s)\tilde{u}(s)), \quad (1.11)$$

$u \in \mathcal{H} = H^0(\mathbb{R}_+)$. Here

$$Mu(s) = \tilde{u}(s) = \int_0^\infty t^{s-1} u(t)\, dt$$

denotes the Mellin transform which is an isomorphism

$$M : H^0(\mathbb{R}_+) \to H^0(\{\mathrm{Re}\, s = 1/2\}),$$

$m(x',s) \in C^\infty(\mathbb{R}^{n-1}_{x'} \times \{\mathrm{Re}\, s = 1/2\})$ is supposed to decrease as $|s| \to \infty$ on $\mathrm{Re}\, s = 1/2$ (cf. [R1]), $\omega \in C^\infty(\mathbb{R}_+)$, $\omega = 1$ near the origin, $\omega = 0$ outside $[0,1]$, $\zeta(\xi') \in C^\infty(\mathbb{R}^{n-1})$, $\zeta(\xi') > 0$ for all $\xi' \in \mathbb{R}^{n-1}$, $\zeta(\xi') = |\xi'|$ for $|\xi'| \geq 1$.

Next we define the symbols $a(x,\xi)$ which are involved in (1.6), (1.7). A function $a(x,\xi) \in C^\infty(U \times \mathbb{R}^n)$ ($U = U(\delta, \delta_1) = \{-\delta < x_n < \delta_1\} \times \mathbb{R}^{n-1}_{x'}$, $\delta > 0$, $0 < \delta_1 \leq \infty$) belongs to $\mathcal{T}(\mathcal{V}^\ell)$, $-l \in \mathbb{Z}_+$ (U is indicated as subscript if necessary: $\mathcal{T}(\mathcal{V}^\ell) = \mathcal{T}_U(\mathcal{V}^\ell)$), if for all $x \in K \Subset U$

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \leq c \langle \xi' \rangle^{\ell - |\beta|} , \tag{1.12}$$

($\langle \xi' \rangle = (1 + |\xi'|^2)^{1/2}$), $\nu \in \mathbb{R}$, with constants $c = c(\alpha, \beta, K)$, and if further there exists a sequence of functions

$$c^\varepsilon_{\beta,j}(x,\xi') \in C^\infty(U \times \mathbb{R}^{n-1}), \quad \varepsilon = \pm 1,$$

$j \in \mathbb{Z}_+$, $j \geq -l$, where $c^\varepsilon_{0,0}$ is independent of ξ', such that for each $M \in \mathbb{Z}_+$, $M \geq -\ell$

$$|D_x^\alpha \{ D_\xi^\beta a(x,\xi) - \sum_{j=-\ell}^{M} c^\varepsilon_{\beta,j}(x,\xi') \nu^{-j} \}| \leq c\, |\nu|^{-(M+1)} \tag{1.13}$$

for $|\nu| \geq 1$, $\nu \to \pm\infty$ with constants $c = c(\alpha, \beta, K, K')$,

$x \in K$, $\xi' \in K' \subset\subset \mathbb{R}^{n-1}$. This definition is motivated by the following typical example. Let

$$\lambda_m(\xi) = (\zeta(\xi') - i\nu)^m \ , \ m \in \mathbb{R} \ , \qquad (1.14)$$

where we take the branch of the logarithm which is real for positive arguments. Let further $a_t(x,\xi)$ be in $S^t_{cl}(\mathbb{R}^n)$. Then

$$a(x,\xi) = a_t(x,\xi) \ \lambda_m(\xi)$$

belongs to $\mathcal{T}(\mathcal{W}^\ell)$ when $-l = -(m+t) \in \mathbb{Z}_+$. In particular $S^l_{cl}(\mathbb{R}^n) \subsetneqq \mathcal{T}(\mathcal{W}^\ell)$ when $-l \in \mathbb{Z}_+$. Further examples and basic properties of the class $\mathcal{T}(\mathcal{W}^\ell)$ are given in [R2,II]. By $\tilde{\mathcal{T}}(\mathcal{W}^\ell)$ we denote the subclass of those $a \in \mathcal{T}(\mathcal{W}^\ell)$ which belong to $S^l_{cl}(U(\delta_2,\infty))$ for some δ_2, $0 < \delta_2 < 1$, $-l \in \mathbb{Z}_+$. Asymptotic expansions in $\tilde{\mathcal{T}}(\mathcal{W}^\ell)$ are understood in the $\mathcal{T}(\mathcal{W}^\ell)$ manner for $\delta < x_n < 1$ and in the S^l manner for $\delta_2 < x_n < \infty$. This is no contradiction over $\delta_2 < x_n < 1$ since an S^l asymptotic expansion implies that in the sense of $\mathcal{T}(\mathcal{W}^\ell)$.

With $a \in \mathcal{T}(\mathcal{W}^\ell)$ we can connect a family of operators

$$op_+(a)(x',\xi') = r^+ op(a)(x',\xi')e^+ \ : \ \mathcal{H} \to \mathcal{H} \qquad (1.15)$$

which is defined by

$$op(a)(x',\xi')u = (2\pi)^{-1} \int e^{ix_n\nu} \ a(x',0,\xi',\nu)(e^+u)^\wedge(\nu)d\nu.$$

(1.15) is called the (complete) boundary symbol connected with a. We call a classical if $op_+(a) \in S^l_{cl}(\mathbb{R}^{n-1}, \mathcal{L}(\mathcal{H},\mathcal{H}))$.

With the objects just described we have locally defined the operator (1.6), and

$$\mathcal{A} \; : H^s(Y,\mathcal{H}_1) \to H^{s-1}(Y,\mathcal{H}_2) \tag{1.16}$$

is continuous, $s \in \mathbb{R}$. The left upper corner in (1.16) is an operator

$$H^{s,o}(X) \to H^{s-1,o}(X), \; X = Y \times \overline{\mathbb{R}}_+ \tag{1.17}$$

where $H^{s,o}(X) = H^s(Y,\mathcal{H})$.

<u>1. Definition</u>. Let $\mathcal{A}$ be defined by (1.6). Then $a(x,\xi)$ is called the complete interior symbol and $(op_+(a) \oplus 0)(x',\xi') + \mathfrak{a}(x',\xi')$ the complete boundary symbol of $\mathcal{A}$. The principal interior symbol $\sigma_X(\mathcal{A})$ of $\mathcal{A}$ is defined as the equivalence class of a modulo $\mathcal{T}(\mathcal{W}^{1-1})$, the principal boundary symbol $\sigma_Y(\mathcal{A})(x',\xi')$ of $\mathcal{A}$ is defined as the homogeneous principal part in the sense of (1.4) of the complete boundary symbol.

If X is an arbitrary manifold with boundary Y we have to define the class $\mathcal{W}^{\ell}$ $(-1 \in \mathbb{Z}_+)$ as the set of operators

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{A}' \tag{1.18}$$

where the pseudo-differential operator A is given in the usual way by local symbols $a \in \widetilde{\mathcal{T}}(\mathcal{W}^{\ell})$ and a partition of unity, and

$$\mathcal{A}' = \begin{pmatrix} \varphi & 0 \\ 0 & 1 \end{pmatrix} Op'(\mathfrak{a}) \quad ,$$

where $\varphi \in C^\infty(X)$ is a function which is 1 near Y and 0 outside some neighbourhood of Y.

Since near Y we have some non-standard convention about the interior symbols, the Sobolev spaces have to be modified near Y, too, when we describe the regularity properties of boundary problems with general symbols (cf. [R2, II]). The major effects of course come from a neighbourhood of Y. Thus for brevity we restrict the considerations here and in Section 3 to the case of operators on $X = Y \times \overline{\mathbb{R}}_+$ and consider interior symbols which are independent of x_n.

Let $H^{r,s}(\mathbb{R}^n)$ be the space of those distributions u in $\mathcal{S}'(\mathbb{R}^n)$ for which

$$\|u\|^2_{r,s} = \int |\langle \xi' \rangle^r \langle \xi \rangle^s \hat{u}(\xi)|^2 d\xi < \infty .$$

Further let $H^{r,s}(\mathbb{R}^n_+)$ be the space of restrictions to $\mathbb{R}^n_+$ of elements in $H^{r,s}(\mathbb{R}^n)$, $r,s \in \mathbb{R}$. The spaces $H^{r,s}(Y \times \mathbb{R}_+)$ can be defined in a similar way.

The pseudo-differential operator over $\mathbb{R}^n_+$ connected with (1.14) defines an isomorphism

$$Op_+(\lambda_m) : H^{r,s}(\mathbb{R}^n_+) \longrightarrow H^{r,s-m}(\mathbb{R}^n_+) \tag{1.19}$$

for all $r,s,m \in \mathbb{R}$.

<u>2. Definition</u>. The class $\mathcal{D}^{\ell;s,t}(X)$, $s,t \in \mathbb{R}$, $-l \in \mathbb{Z}_+$, $X = Y \times \overline{\mathbb{R}}_+$, consists of those operators

$$\mathcal{A} = \begin{pmatrix} \underline{A} & T \\ K & R \end{pmatrix} : \begin{matrix} H^{r,s}(X,E) \\ \oplus \\ H^{r+t}(Y,J) \end{matrix} \longrightarrow \begin{matrix} H^{r-l,t}(X,F) \\ \oplus \\ H^{r+s}(Y,G) \end{matrix} \tag{1.20}$$

$(r \in \mathbb{R})$ which are locally of the form

$$\mathcal{A} = \mathcal{L}_1 \mathcal{A}_0 \mathcal{L}_2 \tag{1.21}$$

with $\mathcal{A}_0 \in \mathcal{W}^\ell$, $\mathcal{A}_0 : \begin{matrix} H^{r,o} \\ \oplus \\ H^r \end{matrix} \longrightarrow \begin{matrix} H^{r-1,o} \\ \oplus \\ H^{r-1} \end{matrix}$ and order reducing operators of the form

$$\mathcal{L}_1 = \begin{pmatrix} Op_+(\lambda_{-t}) & 0 \\ 0 & Op'(\zeta(\xi')^{-1-s}) \end{pmatrix}, \mathcal{L}_2 = \begin{pmatrix} Op_+(\lambda_s) & 0 \\ 0 & Op'(\zeta(\xi')^t) \end{pmatrix}$$

$$\mathcal{L}_1 : \begin{matrix} H^{r-1,o} \\ \oplus \\ H^{r-1} \end{matrix} \longrightarrow \begin{matrix} H^{r-1,t} \\ \oplus \\ H^{r+s} \end{matrix}, \quad \mathcal{L}_2 : \begin{matrix} H^{r,s} \\ \oplus \\ H^{r+t} \end{matrix} \longrightarrow \begin{matrix} H^{r,0} \\ \oplus \\ H^r \end{matrix}.$$

In particular we have defined in this way the action of an operator in $\mathbb{R}^n_+$ connected with an arbitrary symbol $a_m \in S^m_{cl}$, $m = s-t$. The interior symbol of $\mathcal{A}_0$ in this case is $a_0 = \lambda_t \ a_m \ \lambda_{-s}$. The question how this definition depends on the choice of the order reducing symbols λ_t, λ_s is discussed in details in [R2, II].

If X is an arbitrary manifold with boundary Y the class $\mathcal{W}^{\ell;s,t}$ (X) is defined as the set of operators which are locally equal $\tilde{\mathcal{L}}_1 \tilde{\mathcal{A}}_0 \tilde{\mathcal{L}}_2$, where $\tilde{\mathcal{A}}_0$ is of the form (1.18) with interior symbol in $\tilde{\sigma}(\mathcal{W}^\ell)$ and $\tilde{\mathcal{L}}_1, \tilde{\mathcal{L}}_2$ are defined by means of $Op_+(\tilde{\lambda}_{-t})$ and $Op_+(\tilde{\lambda}_s)$, respectively, where $\tilde{\lambda}_m$ is defined by (2.1) in Section 2.

The definition of the class $\mathcal{W}^{\ell;s,t}$ can be generalized in an obvious way when the Sobolev spaces over Y are of the form

$$\bigoplus_{i=1}^{j_1} H^{t+\lambda_i+\frac{1}{2}}(Y, J_i), \quad \bigoplus_{i=1}^{j_2} H^{s-\gamma_i-\frac{1}{2}}(Y, G_i),$$

where $J = \oplus J_i$, $G = \oplus G_i$. This will be used in Section 3.

2. Order Reducing operators on manifolds with boundary

In the previous section in Definition 2 we have used the local order reducing operators (1.19). The resolvent construction in Section 3 requires certain global order reducing operators depending on a parameter in a suitable way. Let X be a closed compact manifold with boundary Y. Let $\alpha \in C^{\infty}(\mathbb{R}_+)$, $0 \leqq \alpha \leqq 1$, $\alpha(x_n) = 0$ for $x_n < \varepsilon$, $= 1$ for $x_n > 1 - \varepsilon$ for some $1/2 > \varepsilon > 0$. Consider the following function in $\overline{\mathbb{R}^n_+} \times \mathbb{R}^n$

$$\tilde{\lambda}_m(x,\xi,q) = \left(\zeta_1(\xi',q) - i\nu\right)^{m\alpha(x_n)} \left(\zeta_1(\xi) + q\right)^{m(1-\alpha(x_n))}, \quad (2.1)$$

where $m \in \mathbb{R}$ is fixed, $q \in \mathbb{C}$ is a parameter with $\operatorname{Re} q > 0$, $\zeta_1(\xi',q) \overset{\text{def}}{=} \zeta_1(\xi',\operatorname{Re} q)$ and $\zeta_1(\xi)$ is a function in $C^{\infty}(\mathbb{R}^n)$ which is strictly positive and $\zeta_1(\xi) = |\xi|$ for $|\xi| \geqq 1$. We then have a continuous map

$$\operatorname{Op}_+(\tilde{\lambda}_m)(q) : H^s(\mathbb{R}^n_+) \longrightarrow H^{s-m}(\mathbb{R}^n_+) \quad (2.2)$$

for each $s \in \mathbb{R}$. Given a smooth complex vector bundle E on X with the expressions $\tilde{\lambda}_m \cdot id_{\pi^*E}$, $\pi : T^*X \rightarrow X$, we can connect a global operator

$$L_m(q) : H^s(X,E) \longrightarrow H^{s-m}(X,E) \quad (2.3)$$

in the usual way. We suppose here without loss of generality that the coordinate diffeomorphisms belonging to boundary neighbourhoods are independent of x_n. Locally the complete symbol l of $L_m(q)$ has an asymptotic expansion

$$l \sim \sum_{j=0}^{\infty} l_{m-j} \quad (2.4)$$

($l_m = \tilde{\lambda}_m$) defined in the following sense. $\tilde{\ell} = \ell\tilde{\lambda}_{-m}$ belongs to $\tilde{\mathcal{T}}(\mathcal{W}^{\circ})$ and has an asymptotic expansion $\tilde{\ell} \sim \sum \tilde{\ell}_{-j}$ in the sense of the class $\tilde{\mathcal{T}}(\mathcal{W}^{\circ})$. Then we set $\ell_{m-j} = \tilde{\ell}_{-j}\tilde{\lambda}_m$. The $\ell_{m-j}(x,\xi',\nu)$ are minus functions with respect to ν, cf. [E1].

<u>1. Theorem</u>. Let X be a smooth compact manifold with boundary Y and E a smooth complex vector bundle over X. Then the operator (2.3), $m,s \in \mathbb{R}$, defines an isomorphism for all $q \in \mathbb{C}$, $\operatorname{Re} q > 0$, $|q| > c_0$ with a constant c_0.

In [B1] order reducing operators have been constructed for $m \in \mathbb{Z}$ in the class $\mathcal{J}$. The operators (2.3) are different from those of [B1] also for $m \in \mathbb{Z}$, since we have here not homogeneous symbols near Y. Also the method for proving the existence is different from that in [B1]. In the proof of Theorem 1 we proceed basically as in Seeley's resolvent construction [S1] by successive solving the equations $b_{-m}l_m = 1$,

$$b_{-m-p}\ell_m + \sum \frac{1}{\alpha!}\partial_\xi^\alpha b_{-m-j} D_x^\alpha \ell_{m-k} = 0 \quad ,$$

$p > 0$, where the sum is taken for $j < p$, $j + k + |\alpha| = p$. The reason for the applicability of this straightforward method in the case of manifolds with boundary is that we only deal with minus functions near Y. Thus we also have the isomorphism (2.3) for all $s \in \mathbb{R}$ (not only for $s > -1/2$ as in [B1]).

<u>2. Theorem</u>. There exists a pseudo-differential operator B(q) with the principal symbol b_{-m} such that for Re $q > 0$

$$\| B(q)\, L_m(q) - I \|_{s,s} \leq c|q|^{-1} ,$$
$$\| L_m(q)\, B(q) - I \|_{s,s} \leq c|q|^{-1}$$

with a constant c only depending on s ; $\| \cdot \|_s$ denotes the norm of operators in $\mathcal{L}(H^s(X,E))$.

3. Corollary. The operator (2.3) is invertible for $|q| > c_0$ with a constant $c_0 > 0$ and for $m > 0$ we have

$$\| L_m^{-1}(q) \|_{s,s-r} \leq c_{s,r} |q|^{-m+r}$$

for each $r \in \mathbb{R}$, $0 \leq r \leq m$, $s \in \mathbb{R}$.

3. The resolvent of elliptic boundary problems under the condition (ehp)

An operator $\mathcal{A} = \mathcal{L}_1 \mathcal{A}_0 \mathcal{L}_2$ of the class $\mathcal{W}^{0;s,t}(X)$ (cf. Section 1, Definition 2) is called elliptic if (in local coordinates) the interior symbol of $\mathcal{A}_0$ has an inverse in the class $\tilde{\mathcal{T}}(\mathcal{W}^0)$ for $|\xi|$ sufficiently large and if the principal boundary symbol $\sigma_Y(\mathcal{A}_0)(x',\xi')$ is bijective for $|\xi'| \geq 1$.

In this section we consider the resolvents of elliptic boundary problems of the class $\mathcal{W}$. They are obtained again by an approximation argument and described on boundary symbolic level.

For simplicity here we restrict the consideration to operators of the form

$$\mathcal{A} = \begin{pmatrix} \underline{A} \\ T \end{pmatrix} : H^m(X) \to \begin{matrix} H^0(X) \\ \oplus \\ \bigoplus_{i=1}^{r} H^{m-\gamma_i-1/2}(Y) \end{matrix} , \qquad (3.1)$$

$X = Y \times \overline{\mathbb{R}}_+$, $m > 0$, where T is a column of trace operators

$T_1,\dots,T_r$ and

$$\underline{A} = \underline{A}_o L_m, \ \underline{A}_o = A_o + B_o + W_o$$

$\underline{A}_o$ is of the class $\mathcal{D}^0$ and A_o is the pseudo-differential operator, B_o the Green operator and W_o the Mellin operator (cf. the left upper corner of (1.6)).

L_m denotes an operator of the sort mentioned in Section 2, Theorem 1 for a fixed $q = q_o$, $|q_o|$ sufficiently large.

We assume that the complete interior symbol a of $\mathcal{A}$ belongs to S^m_{cl} in local coordinates and that the symbols of the trace operators T_i are in $S^{\gamma_i}_{cl}$ near the boundary, $\gamma_i \in \mathbb{R}$, $m-\gamma_i-1/2 > 0$, $i = 1,\dots,r$. Let $\tilde{\Gamma} \subset \mathbb{C}$ be a closed cone containing $\mathbb{R}_+$ and consider the operator family

$$\mathcal{A} - \begin{pmatrix} h \\ 0 \end{pmatrix} = \begin{pmatrix} \underline{A} - h \\ T \end{pmatrix} : H^m(X) \longrightarrow \begin{matrix} H^0(X) \\ \oplus \\ \bigoplus_{i=1}^{r} H^{m-\gamma_i-1/2}(Y) \end{matrix} , \qquad (3.2)$$

$h \in \tilde{\Gamma}$. By Γ we denote the closed cone in the complex q-plane defined by $q \in \Gamma \iff q^m = h \in \tilde{\Gamma}$ ($|\arg h|$ is assumed to be small enough for $h \in \tilde{\Gamma}$). Let a_m be the homogeneous principal symbol of a of order m.

__1. Condition.__

$$\sigma_X(\mathcal{A})(x,\xi) - h = a_m(x,\xi) - q^m \qquad (3.3)$$

is invertible for all $(x,\xi,q) \in \overline{\mathbb{R}^n_+} \times \mathbb{R}^n \times \Gamma$.

__2. Condition.__ The operator

$$\sigma_Y(\mathcal{A})(x',\xi') - \begin{pmatrix} q^m \\ 0 \end{pmatrix} : H^m(\mathbb{R}_+) \longrightarrow \begin{matrix} H^0(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^r \end{matrix} \qquad (3.4)$$

is invertible for all $(x',\xi',q) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \Gamma$.

Remember that the operator (3.1) is given in the form $\mathcal{A} = \mathcal{A}_0 L_m$. For such compositions we do not have a convenient symbolic calculus. Therefore we have to pass to operators of order zero. In our case it is reasonable to do that by means of the parameter depending order reducing operators of Section 2.

Set

$$\mathcal{A}_0(q) = \begin{pmatrix} (\underline{A} - q^m)\, L_{-m}(q) \\ \\ T\, L_{-m}(q) \end{pmatrix} : H^0(X) \longrightarrow \begin{matrix} H^0(X) \\ \oplus \\ \bigoplus_{i=1}^{r} H^{m-\gamma_i-1/2}(Y) \end{matrix} , \qquad (3.5)$$

$$N(q) = \begin{pmatrix} 1 & & & 0 \\ & q^{m_1} & & \\ & & \ddots & \\ 0 & & & q^{m_r} \end{pmatrix}, \quad \mathcal{A}(q) = N(q)\, \mathcal{A}_0(q) , \qquad (3.6)$$

where $m_i \in \mathbb{R}$ are certain fixed exponents. Formally q is considered as an extra covariable and the symbolic calculus for operators of the form (3.5) refers to this parameter dependence. The details of such a symbolic calculus are elaborated in [R2,II].

E.g. the complete symbol of the pseudo-differential part of $(\underline{A} - q^m)\, L_{-m}(q)$ which we denote by $a(x,\xi,q)$ satisfies near Y the following estimates (cf. (1.12))

$$|D_x^\alpha D_\xi^\beta \; a(x,\xi,q)| \leq \begin{cases} c & \text{for } |\beta| = 0, \\ c\,|q|^{-1} \langle \xi' \rangle^{1-|\beta|} & \text{for } |\beta| > 0 \end{cases} , \qquad (3.7)$$

$\nu \in \mathbb{R}$, with constants $c = c(\alpha,\beta,K)$, $x \in K$, K compact, $|q| \geq c_0$. Outside some neighbourhood of Y in (3.7) we have to replace $\langle \xi' \rangle$ by $\langle \xi \rangle$.

Moreover there is a reasonable notion of classical symbols (the homogeneity now refers to (ξ,q)) and of asymptotic expansions. The same is true for operator-valued symbols. Since we assume that the interior symbol is independent of x_n, everything reduces to the operator-valued case (the boundary symbols) and for $\mathcal{A}(q)$ in particular we have a homogeneous (in (ξ',q), cf. (1.4)) principal boundary symbol

$$\mathfrak{a}(x',\xi',q) = \begin{pmatrix} \underline{a}(x',\xi',q) \\ \\ op(t)(x',\xi',q) \end{pmatrix} : H^0(\mathbb{R}_+) \longrightarrow \begin{matrix} H^0(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^r \end{matrix} \quad , \qquad (3.8)$$

$(x',\xi') \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$, $|\xi'| \geqq 1$, $|q| \geqq c_0$, $q \in \Gamma$.

<u>3. Condition (ehp)</u>. For a suitable choice of the numbers $m_i \in \mathbb{R}$ there exists an operator

$$\mathfrak{a}^*(x',\xi',q) = \begin{pmatrix} \underline{a}^*(x',\xi',q) \\ \\ op(t)^*(x',\xi',q) \end{pmatrix} : H^0(\mathbb{R}_+) \longrightarrow \begin{matrix} H^0(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^r \end{matrix}$$

$q \in \Gamma \setminus 0$, which is bijective and homogeneous in the sense

$$\mathfrak{a}^*(x',\xi',\lambda q) = \begin{pmatrix} \varkappa_\lambda & 0 \\ 0 & 1 \end{pmatrix} \mathfrak{a}(x',\xi',q)\,\varkappa_\lambda^{-1}, \quad \lambda > 0,$$

(cf. (1.3)) satisfying

$$\| \mathfrak{a}(x',\xi',q) - \mathfrak{a}^*(x',\xi',q) \| \longrightarrow 0$$

as $|q| \to \infty$, $(x',\xi') \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$.

The existence of $\mathfrak{a}_0$ means roughly speaking that there is an elliptic homogeneous problem (ehp) (the homogeneity refers to the (q,x_n) variables) approximating the given boundary symbol for large $|q|$. The choice of the numbers m_i in classical examples is discussed in [R2, I], Section 2.4.

<u>4. Theorem.</u> Under the conditions 1,2,3 the operator (3.2) is invertible for $h \in \tilde{\Gamma}$, $|h| \geq c$ with a constant $c > 0$.

The inverse has the form

$$(\mathcal{A} - \binom{h}{0})^{-1} = L_{-m}(q)\, \mathcal{A}(q)^{-1}\, N(q) = L_{-m}(q) \mathcal{A}_0(q)^{-1} ,$$

$q^m = h$, and $\mathcal{A}(q)^{-1}$ can be constructed on boundary symbolic level.

First we obtain an operator family $\mathcal{B}(q)$ by successive solving the equations

$$b_0 a_0 = 1 ,$$

$$b_{-l} a_0 + \sum \frac{1}{\alpha!} \partial_{\xi'}^{\alpha} b_{-j} D_{x'}^{\alpha} a_{-k} = 0 ,$$

$l > 0$, with the sum over $j < l$, $j + k + |\alpha| = l$. Then $\mathcal{B}(q) = \mathcal{B}_K(q)$ is associated with the b_{-j}, $0 \leq j \leq K$ and a partition of unity on Y.

<u>5. Theorem.</u> Under the conditions of Theorem 4 for each $k \in \mathbb{Z}$ there exists a $K \in \mathbb{Z}$ such that $\mathcal{B}(q)$ satisfies

$$\| \mathcal{B}(q)\ \mathcal{A}(q) - I_1 \|^{1}_{l,l+k} \leq c|q|^{-1} \tag{3.10}$$

$$\| \mathcal{A}(q)\ \mathcal{B}(q) - I_2 \|^{2}_{l,l+k} \leq c|q|^{-1} \tag{3.11}$$

for every $l \in \mathbb{R}$ with a constant $c = c_{k,l}$.

Here $\| \cdot \|^{1,2}_{l,l'}$ denotes the norm of operators

$$H^{l}(Y,H^{0}(\mathbb{R}_+)) \to H^{l'}(Y,H^{0}(\mathbb{R}_+))$$

and

$$\begin{matrix} H^{l}(Y,H^{0}(\mathbb{R}_+)) & & H^{l'}(Y,H^{0}(\mathbb{R}_+)) \\ \oplus & \longrightarrow & \oplus \\ \bigoplus_{i=0}^{r} H^{l-m-\gamma_j-1/2}(Y) & & \bigoplus_{i=0}^{r} H^{l'-m-\gamma_j-1/2}(Y) \end{matrix} ,$$

respectively. $I_{1,2}$ are the identities in the corresponding spaces. The invertibility of $\mathcal{A}(q)$ for large $|q|$ then follows by a Neumann series argument.

Set

$$\mathcal{A}'(q) = N(q)\,(\mathcal{A} - (\begin{smallmatrix} q^m \\ 0 \end{smallmatrix}))$$

6. Corollary. For every $l \in \mathbb{R}$, $0 \leqq p \leqq m$,

$$\| \mathcal{A}'(q)^{-1} \|_{\ell,(\ell,p)} = O(|q|^{-m+p}),$$

$\| \cdot \|_{\ell,(\ell',p)}$ denotes the norm of operators $H^{l,o}(X) \oplus \bigoplus_{i=1}^{r} H^{l-m-\gamma_i-1/2}(Y) \rightarrow H^{l',p}(X)$.

7. Corollary. With $\mathcal{B}'(q) = L_{-m}(q)\,\mathcal{B}(q)$, $\mathcal{B}(q)$ as in Theorem 5, we have

$$\| \mathcal{B}'(q) - \mathcal{A}'(q)^{-1} \|_{1,(1+k,p)} = O(|q|^{-1-m+p})$$

$0 \leqq p \leqq m$, $l \in \mathbb{R}$.

Note that

$$\mathcal{A}'(h^{\frac{1}{m}})^{-1} = (\underline{P}(h),\ K'(h)),$$

where $\underline{P}(h)$ is the inverse of

$$\underline{A}|_{\ker T} : \ \ker T \rightarrow H^o(X).$$

Usually in the resolvent construction one is mainly interested in $\underline{P}(h)$. The corresponding estimates follow immediately from the above assertions by omitting the potential parts. At the same time we also got growth estimates for K(h), the potential part of the inverse of (3.2).

When X is an arbitrary compact manifold with boundary Y the construction is similar but more technical. The formulation of the results refers to a modification of the Sobolev spaces consisting roughly speaking of those distributions which belong to the ordinary Sobolev spaces in the interior of X (outside some neighbourhood of Y) and near Y to the classes indicated above (cf. [R2, II]). Denoting these spaces by $\tilde{H}^1(X)$ and $\tilde{H}^{1,s}(X) = L_{-s}(H^1(X))$, respectively, the results in the general case have the same form as above with the new spaces instead of the old ones.

The $\tilde{H}$ spaces reflect a typical loss of regularity in normal direction to Y, compared with the parametrix and resolvent constructions for operators with the transmission property.

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