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# NON EMBEDDABLE CR-STRUCTURES (*)

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A CR-structure on a smooth manifold $\Omega$ is the datum of a closed (see [5], Ch. 1, Def. 1.1) vector subbundle T' of the complex cotangent bundle $\mathbb{C}T^*\Omega$, such that

(1) $$\mathbb{C}T^*\Omega = T' + \overline{T'} .$$

We shall call m the fiber dimension of T'. Note that, by (1), dim $\Omega \leq 2m$. (If dim $\Omega = 2m$ the structure is a complex one, a case in which we are not interested here). The structure T' is said to be locally integrable or, equivalently, the CR manifold $(\Omega, T')$ is said to be locally embeddable if every point of $\Omega$ has an open neighborhood over which T' is generated by m closed (or exact) one-forms. A function, or a distribution, f, such that df is a section of T' is said to be a CR function, or distribution. It ought perhaps to be said that CR stands for Cauchy-Riemann.

H. Lewy [3] (1956) has raised the question as to whether a strongly pseudoconvex CR structure, on a (2m-1)-dimensional manifold $\Omega$, is always locally embeddable. Pseudoconvexity is defined by means of the Levi form (see below, (8)). That the answer is no was shown by L. Nirenberg [4] (1972) when dim $\Omega = 3$, in which case the Levi form is a scalar (and m = 2) . (♦) Here we show that the CR-structures that have non degenerate Levi forms, with one eigenvalue of one sign and all others of the opposite sign, and which are not locally embeddable, are dense (in a sensemade precise below : see Theorem and remarks that follow).

Our view point will be strictly local. We shall hence forth suppose that $\Omega$ is an open neighborhood of the origin in an Euclidean space, specifically $\mathbb{R}^{2n+1}$. We shall limit ourselves to the case where

(2) $$n = m - 1 .$$

Thus the fiber dimension of $T' \cap \overline{T'}$ is one. We shall begin by assuming that there are m $C^\infty$ functions $Z^1, \ldots, Z^m$ in $\Omega$, complex valued, such that $dZ^1, \ldots, dZ^m$ span T' at

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(*) The present work is a generalization of some recent joint work, [2], with H. Jacobowitz (Rutgers University).

(♦) For a positive answer to the global embeddability question, when $\Omega$ is compact and has dimension $\geq 5$, see Boutet de Monvel [1].

each point of $\Omega$. After a contraction of $\Omega$ about the origin, possibly a modification of the coordinates in $\mathbb{R}^{2n+1}$, which we denote by $x^1,\dots,x^m, y^1,\dots,y^n$, and a $\mathbb{C}$-linear substitution on the $Z^j$'s, we may assume that

(3) $$Z^j = x^j + \sqrt{-1}\, y^j, \quad j = 1,\dots,m-1 \ (= n),$$

(4) $$Z^m = x^m + \sqrt{-1}\, \Phi(x,y),$$

with

(5) $$\Phi \ \underline{\text{real}}, \quad \Phi(0,0) = 0, \qquad d\Phi(0,0) = 0.$$

(see [5], Ch. I, p.20) .

Henceforth we write $z^j = x^j + iy^j$ $(j = 1,\dots,n)$. But notice that the mapping

(6) $$(x,y) \longmapsto Z(x,y) = (Z^1(x,y),\dots,Z^m(x,y))$$

defines a diffeomorphism on the (real) hypersurface $Z(\Omega)$ of $\mathbb{C}^m$ defined by the equation

(7) $$y^m = \Phi(x,y'), \quad y' = (y^1,\dots,y^{m-1}).$$

This justifies that we call (6) a (local) embedding.

Next we introduce the Levi form of the structure, at the origin (without attempting to give here an invariant definition) :

(8) $$Q(\zeta) = \sum_{j,k=1}^{n} \frac{\partial^2 \Phi}{\partial z^j \partial \bar{z}^k}(0,0)\zeta^j\bar{\zeta}^k \qquad (\zeta \in \mathbb{C}^n) .$$

Note that

(9) $$\Phi(x',0,y') = \mathrm{Re}\Big(\sum_{j,k=1}^{n} b_{jk} z^j z^k\Big) + Q(z) + O(|z|^3).$$

It is convenient to introduce the function

$$W = Z^m - \sqrt{-1} \sum_{j,k=1}^{n} b_{jk}\, z^j z^k ,$$

and to use the new coordinate $s = \mathrm{Re}W$ in the place of $x^m$. Instead of $Z^m$ (see (4)) we shall reason with

(10) $$W = s + i\varphi(z,s),$$

noting that

(11) $$\varphi(z,s) = Q(z) + O(|z|^3 + |s||z| + |s|^2).$$

Our basic hypothesis will be :

(12) <u>The Levi form</u> $Q$ <u>is non degenerate and it has exactly</u> $n - 1$ <u>eigenvalues of a given sign,</u> <u>and one of the opposite sign</u> (i.e. it has signature $n-2$).

We shall assume that one eigenvalue of $Q$ is $> 0$ and $n - 1$ are $< 0$. After a linear substitution on the $z^j$'s we may assume that

(13) $$Q(z) = |z^1|^2 - |z''|^2 ,$$

whre $z'' = (z^2,\dots,z^n)$. By (11) we see that, in a suitable neighborhood of the origin, $U \subset \Omega$ ,

(14) $$\varphi(z,s) \leqslant 2|z^1|^2 - \frac{1}{2}|z''|^2 + C|s|\,(|z| + |s|),$$

The orthogonal $T'^{\perp}$ of $T'$, for the natural duality between vectors and covectors, is generated over $\Omega$ by the following n vector fields :

(15) $$L_j = \frac{\partial}{\partial \bar{z}^j} - i\lambda_j(z,s)\frac{\partial}{\partial s} \quad , \; j = 1,\dots,n ,$$

where the coefficients $\lambda_j$ are computed by writing that $L_j W = 0$ :

(16) $$\lambda_j = \left(1 + i\,\frac{\partial\varphi}{\partial s}\right)^{-1}\frac{\partial\varphi}{\partial \bar{z}^j} \quad , \; j = 1,\dots,n .$$

(Incidentally the fact that $T'$ is closed is equivalent to the property that the commutation bracket of any two smooth sections of $T'^{\perp}$ is a section of $T'^{\perp}$ ).

We may now state our result :

Theorem : Suppose (13) holds. Then there is a function $g \in C^{\infty}(\Omega)$, vanishing to infinite order at the origin, such that the following is true :

(17) There is an open neighborhood U of the origin in $\Omega$ such that, for every $j = 1,\ldots,n$,

$$\lambda_j^{\#} = \lambda_j(1 + g/z^1)$$

is a $C^{\infty}$ function in U ;

(18) the vector fields $L_j^{\#} = \frac{\partial}{\partial z^j} - i\lambda_j^{\#}\frac{\partial}{\partial s}$ in U $(j = 1,\ldots,n)$ commute pairwise ;

(19) given any open neighborhood $V \subset U$ of the origin, any solution $h \in C^1(V)$ of the equations

(20) $$L_j^{\#} h = 0,\ j = 1,\ldots,n,$$

has the property that $dh|_0$ is a linear comnination of $dz^1,\ldots,dz^n$ .

The meaning of this theorem is, roughly, the following :

Let T' be a CR structure on a manifold $\Omega$ of dimension 2n+1. Suppose that $T' \cap \overline{T}'$ is a line bundle (i.e., the structure has " codimension one"). Suppose that, in the neighborhood of a point $\omega_0$ of $\Omega$ , the CR structure T' is embedabble, and has a non degenerate Levi form whose signature is equal to n - 2. Then there is another CR structure $T'^{\#}$ in the neighborhood of $\omega_0$, tangent at $\omega_0$ to T' to infinite order, which is *not* locally embeddable (at $\omega_0$).

Proof of Theorem : Inspired by Nirenberg [4] we select a sequence of compact subsets $K_\nu$ $(\nu = 1,2,\ldots)$ in the upper half-plane Im w > 0 (w = s + it will denote the variable in $\mathbb{C}^1$) having various properties :

(21) as $\nu \to +\infty$ , $K_\nu$ converges to $\{0\}$ ;

(22) the projections of the $K_\nu$ into the real axis are pairwise disjoint ;

(23) there is a number $\varepsilon > 0$ such that

$$K_\nu \subset \Gamma^{\varepsilon} = \{s + it\ ; |s| < \varepsilon t\}.$$

We shall furthermore assume that the interior $\mathring{K}_\nu$ of $K_\nu$ is not empty, whatever $\nu$ .

We note that, if $s + i\varphi(z,s) \in \Gamma^{\varepsilon}$, we derive from (14) :

$$(\varepsilon^{-1} - C(|z| + |s|))|s| + \frac{1}{2}|z''|^2 \leqslant 2|z^1|^2 ,$$

and therefore, by choosing $\varepsilon > 0$ small enough ,

(24) $$\varepsilon^{-1}|s| + |z''|^2 \leqslant 4|z^1|^2 , \quad (s,z) \in U, W \in \Gamma^{\varepsilon}.$$

According to (11) we have

(25) $$\frac{\partial\varphi}{\partial\bar{z}^j} = \pm z^j + O(|z|^2 + |s|).$$

We note that, by (16), we have :

(26) $$\lambda_j/z^1 = [\pm z^j + O(|z|^2 + |s|)]/z^1 .$$

We select, for each $\nu$, a function $f_\nu \in C^\infty(\mathbb{R}^2)$ having the following properties :

(27) $f_\nu \geqslant 0$ <u>everywhere</u>, $\operatorname{supp} f_\nu \subset K_\nu$, $f_\nu(w_\nu) > 0$ <u>for some</u> $w_\nu \in K_\nu$ ;

(28) $$f = \sum_{\nu=1}^{\infty} f_\nu \in C^\infty(\mathbb{R}^2) ;$$

(29) $$\lambda_j g/z^1 \in C^\infty(U),$$

where

$$g(f \circ W)/[1 + (f \circ W)(\log W_s)/z^1] .$$

Let us show that (29) can be achieved (in particular by taking U small enough). Recalling that $W = s + i\varphi(z,s)$, we see that $\log(1 + i\varphi_s)$ is well defined provided U is small; furthermore $\log(1 + i\varphi_s) = O(|z| + |s|)$, hence is $O(|z'|)$ on $\operatorname{supp}(f \circ W)$, by (23) and (24). Since f is flat at the origin, both $(f \circ W)(\log W_s)/z^1$ and $\lambda_j(f \circ W)/z_1$ (cf. (26)) are $C^\infty$ in U, and flat at the origin, whence easily (29).

By differentiating $L_jW = 0$ with respect to s and dividing by $W_s$ we get

(30) $$L_j(\log W_s) = i\lambda_{js} , \quad j = 1,\dots,n .$$

A straightforward computation yields

$$L_j g + i\lambda_{js} g^2/z^1 = \lambda_j \, h \, , \; j = 1,\dots,n, \tag{31}$$

where h is a certain function of (z,s). We have used the fact that $L_j(f \circ W) = L_j\overline{W}(\frac{\partial f}{\partial w} \circ W)$, and $L_j\overline{W} = L_j(W + \overline{W}) = 2L_j s = -2i\lambda_j$ :

$$\lambda_j = \frac{i}{2} L_j \overline{W} \, , \; j = 1,\dots,n \, . \tag{32}$$

Note that $L_k\lambda_j = L_j \lambda_k$ (hence $[L_j, L_k] = 0$). We have

$$[L_j^{\#}, L_k^{\#}] = [L_j - i\lambda_j \frac{g}{z^1}\frac{\partial}{\partial s} \, , \, L_k - i\lambda_k \frac{g}{z^1}\frac{\partial}{\partial s}] = -iq\frac{\partial}{\partial s} \, ,$$

where

$$\begin{aligned} z^1 q &= L_j(\lambda_k g) - L_k(\lambda_j g) - i\lambda_j \frac{g}{z^1}\frac{\partial}{\partial s}(\lambda_k g) \\ &\quad + i\,\lambda_k \frac{g}{z^1}\frac{\partial}{\partial s}(\lambda_j g) \\ &= \lambda_k(L_j g + i\,\frac{g^2}{z^1}\lambda_{js}) - \lambda_j(L_k g + i\,\frac{g^2}{z^1}\lambda_{ks}) \\ &= 0 \quad \text{according to (31).} \end{aligned}$$

This proves (18).

Finally suppose that $h \in C^1(V)$ is a solution of (20). In particular it is a solution of $L_1^{\#}h = 0$ on the plane $z^2 = \dots = z^n = 0$. We shall prove below that this implies $h_s(0,0) = 0$ . Because of the special form of the equations (20) (see (18)) this implies $\partial_{\bar{z}}h(0,0) = 0$, whence (19).

The proof is reduced to the case where n = 1. We content ourselves with sketching the argument, which is essentially the same as that given, with full details, in [2]. Let us write x, y, z = x + iy, rather than $x^1$, $y^1$, $z^1$, and $L = \frac{\partial}{\partial z} - i\lambda(z,s)\frac{\partial}{\partial s}$ rather than $L_1$. We have

$$\varphi(z,s) = |z|^2 + O(|z|^3 + |s||z| + |s|^2).$$

By the implicit function theorem there is a $C^\infty$ function, in a neighborhood of zero, $s \longmapsto z(s)$, with $z(0) = 0$, such that, if we set $\varphi_0(s) = \varphi(z(s),s)$, we have

(33) $$\varphi(z,s) - \varphi_o(s) > c_o|z - z(s)|^2 \qquad (c_o > 0).$$

Furthermore $\varphi_o(0) = 0$. We may therefore assume that the intersection of the cone $\Gamma^\varepsilon$ (see (23)) with a small open disk centered at the origin, in the $w = s + it$ plane, is entirely contained in the region

(34) $$t > \varphi_o(s) .$$

We may and shall assume that all the compact sets $K_\nu$ are contained in the open set (34), and we shall denote by $\mathcal{R}_o$ the complement of $\overline{\bigcup_\nu K_\nu}$ in (34), by $\mathcal{R}$ the set of points $(z,s) \in \Omega$ such that $W = s + i\varphi(z,s) \in \mathcal{R}_o$. Notice that we have, in $\mathcal{R}$ :

(35) $$Lh = 0.$$

Because of (33), when $w = s + it \in \mathcal{R}_o$, the equation $\varphi(z,s) = t$ defines a smooth closed curve in the z-plane, $\gamma(w)$, winding aroun $z(s)$. We can use the parameter $\theta = \mathrm{Arg}(z - z(s))$ on $\gamma(w)$. This defines a smooth map

(36) $$S^1 \times \mathcal{R}_o \ni (\theta,w) \longmapsto (Z,s) \in \mathcal{R}$$

By virtue of (35) we have $dh = A\, dw + B\, dz$ in $\mathcal{R}$, hence

$$\frac{\partial}{\partial \bar{w}}\left(h\frac{\partial z}{\partial \theta}\right) = \frac{\partial}{\partial \theta}\left(h\frac{\partial z}{\partial \bar{w}}\right) \qquad \text{since} \qquad \frac{\partial W}{\partial \bar{w}} = \frac{\partial W}{\partial \theta} = 0 .$$

This implies that the integral $I(w) = \int_{\gamma(w)} h\, dZ$ is a holomorphic function of $w$ in $\mathcal{R}_o$. Since $\gamma(w)$ contracts to the point $z(s)$ when $t = \varphi_o(s)$, we have $I(w) = 0$ on this curve, therefore everywhere in $\mathcal{R}_o$ (note that there is "enough room" for the zeros to propagate around the sets $K_\nu$, thanks to (22)). We select then a smooth closed curve $c_\nu$ in $\mathcal{R}_o$ winding around $K_\nu$, such that no point of any set $K_{\nu'}$, $\nu' \neq \nu$, lies inside or on $c_\nu$. We derive

(37) $$\int_{c_\nu} \int_\gamma h(z,s)dz \wedge dw = 0 .$$

The 2-chain $\Sigma_\nu = \{(z,s); (W,z) \in c_\nu \times \gamma\}$ is a kind of torus whose inside we call $\mathcal{T}_\nu$. Stokes theorem implies

(38) $$\iiint_{\mathcal{T}_\nu} dh \wedge dz \wedge dw = 0 .$$

But $dh = A\, dW + B\, dZ + Lh\, d\bar{Z}$, hence (38) reads

$$\iiint_{\mathcal{C}_\nu} (\lambda/z^1)\, g\, h_s\, d\bar{z} \wedge dz \wedge dw = 0 \tag{39}$$

since $Lh = i\lambda g\, h_s/z^1$ . Near the origin, on supp g(cf.(24))

$$\lambda/z^1 \sim 1 \ , \ g \sim f \circ W \ .$$

If $h_s(0,0)$ were $\neq 0$, in (39) the argument of the integrand would have a well-defined limit as $\nu \to +\infty$ : (39) could not hold true for $\nu$ large enough. □

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## REFERENCES


[1] Boutet de Monvel, L. - Intégration des équations de Cauchy-Riemann induites formelles, Séminaire Goulaouic-Schwartz 1974-75, Exposé n° 9, Ecole Polytechnique, Palaiseau (France).

[2] Jacobowitz, H. & Treves, F. - Homogeneous solvability, to appear.

[3] Lewy, H. - On the local character of the solutions of an atypical linear differential equation in three variables and a related theorem for regular functions of two complex variables, Ann. of Math. 64 (1956), 514-522.

[4] Nirenberg, L. - On a question of Hans Lewy, Uspekhi, vol. in honor of Petrowski, English transl. (1973), 251-262.

[5] Treves, F. - Approximation and representation of functions and distributions annihilated by a system of complex vector fields, Lecture Notes, Ecole Polytechnique, Palaiseau (France) 1981.


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