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ELLIPTIC BOUNDARY PROBLEMS ON SPACES WITH CONIC POINTS.

by

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1. Introduction

For some time the general theory of elliptic boundary problems for linear operators has been well-established (see for example Chazarain & Piriou /1/, Grubb /3/). This note is an outline of work in progress on the properties of elliptic problems near characteristic boundary components. Typically, such problems arise from the analysis of a standard elliptic problem, say for the Laplace operator, on a space which is a Riemannian manifold (possibly with boundary) except for singular points of simple conic type. If one makes the analytically very restrictive assumption that the Riemann metric is of 'product-conic' type near the singular points then the method of separation of variables is available (see Cheeger /2/). In general the natural class of operators in which to work seems to be the totally characteristic operator ring, as defined in /5/; it is this approach to the analysis of conic points which is discussed below.

The main type of result described below is the almost-hypoellipticity for an elliptic boundary problem (as defined in Section 3) of this conic type. That is, any solution with homogeneous data has only a classical (graded conormal) singularity at the conic point. More specifically, suppose that $f: \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ is a C^∞ function such that

$$f(0)=0,\ df(0)=0,\ \frac{\partial^2 f}{\partial x^2}(0) \text{ has signature } +,+,..,+,-$$

and $df(x)\neq 0$ if $f(x)=0$, $x\neq 0$.

The domain $\Omega_f = \{f < 0\}$ is therefore a C^∞ submanifold with boundary of $\mathbb{R}^{n+1}$ except that it has a conic point at 0. Consider the problem

$$\Delta u = 0 \quad \text{in } \Omega_f,\ u\big|_{\partial\Omega_f} = 0$$

where $u \in \mathcal{D}'(\bar{\Omega}_f)$, i.e. u is the restriction to Ω_f of a distribution on $\mathbb{R}^{n+1}$. It follows from the main result of this paper, see Section 5, that u has an asymptotic expansion with C^∞ coefficients:

$$u \sim \sum g_{j,k,p}(r,\omega)\, r^{m_j+k} \log^p r, \quad g_{j,k,p} \in C^\infty(\bar{\mathbb{R}}_+ \times S^n)$$

where $m_j \to \infty$ as $j \to \infty$, (r,ω) are (any) polar coordinates in $\mathbb{R}^{n+1}$ and the sum over p is finite for each j,k. The m_j are singular values of a certain associated 'indicial' problem.

Particular examples of this phenomenon have been observed by many authors, especially in the product-conic case mentioned above. It should be noted that the methods of this paper allow one to discuss directly Fredholm properties and the finite order regularity of solutions for such elliptic problems, in terms of suitable weighted Sobolev norms near the conic points. These matters, together with generalizations to 'kinks' and corners of lower codimension, will appear elsewhere.

2. Polar coordinates

Let Y be a C^∞ manifold and suppose $Y \setminus \bar{p}$ carries a C^∞ Riemannian structure. We wish to impose conditions on the metric so that $\bar{p}$ is, metrically, a conic point in Y. This is most easily done by considering the standard cone in $\mathbb{R}^3$:

$$x^2 + y^2 = t^2, \quad t \geq 0. \tag{1}$$

The introduction of polar coordinates,

$$r^2 = 2(x^2 + y^2) \quad (r \geq 0), \quad \omega = r^{-1}(x + iy) \in S^1, \tag{2}$$

reduces this cone to the manifold with boundary

$$X = \overline{\mathbb{R}}_+ \times S^1. \tag{3}$$

On X the metric from Y, induced by the Euclidean metric on $\mathbb{R}^3$, takes the form

$$g = dr^2 + r^2 d\omega^2 \tag{4}$$

where $d\omega^2$ is a metric on S^1.

Generalizing this we shall consider that a manifold with boundary, X, has a metrically conic boundary, ∂X, if X carries a C^∞ 2-tensor

$$g \in C^\infty(X, \operatorname{Symm}_2 TX)$$

such that

(5) g is positive definite (Riemannian) on $X \setminus \partial X$.

(6) g has rank 1 at each point of ∂X.

(7) If $V \in C^\infty(X,TX)$ is tangent to ∂X then $g(V,V)$ vanishes to second order on ∂X and to precisely second order at each point $p \in \partial X$ where $V(p) \neq 0$.

Clearly g in (4) satisfies these conditions. In fact if $r \geq 0$, $y^1,\dots,y^n$ are coordinates in X near $\bar{p} \in \partial X$ then g satisfying (5), (6), (7) takes the form:

$$(8) \qquad g = h_{00}dr^2 + 2r \sum_{j=1}^{n} h_{0j}dr.dy^j + r^2 \sum_{j=1}^{n} h_{ij}dy^i . dy^j$$

where the coefficients h_{ab} are C^∞ functions of r,y. Moreover,

$$(9) \qquad h > 0 .$$

The form (8) follows easily from (6), (7). For example the vector fields D_{y^k} are tangent to ∂X, leading to the factor r^2. Similarly setting $V = D_{y^j} + rD_r$ in (7) shows that the cross terms $dr.dy^j$ must have coefficients vanishing at $r=0$.

Next recall (from /5/) that on any manifold with boundary there is a natural subring:

$$\mathrm{Diff}_b(X) \subset \mathrm{Diff}(X)$$

of the (filtered) ring of all differential operators with C^∞ coefficients on X. Namely, $\mathrm{Diff}_b(X)$ is the sub-C^∞-module locally generated by the vector fields tangent to ∂X. In the local coordinates r,y this means

$$\mathrm{Diff}_b^k(X) \ni P = \sum_{|\alpha|+q \leq k} p_{\alpha,q}(r,y)\, r^q D_r^q D_y^\alpha .$$

(10) Lemma The Laplace-Beltrami operator, Δ, of a metric satisfying (5), (6), (7) on X is such that

(11) $\quad r^2\Delta \in \mathrm{Diff}_b^2(X).$

Notice this means that, although its coefficients are not C^∞, Δ can be written as a C^∞ combination of vector fields D_r, $r^{-1}D_{y^j}$. We shall outline a more general computation than is needed to prove this Lemma.

In /5/ the notation $\tilde{T}X$ (compressed tangent bundle) was used for the C^∞ vector bundle of vector fields tangent to ∂X. Thus, $\tilde{T}X$ is locally spanned by rD_r, D_{y^j}. In view of (8) the metric g defines a uniformly degenerate metric on $\tilde{T}X$: If $r \in C^\infty(X)$ is a global defining function for ∂X, $r > 0$ in $\overset{\circ}{X}$, then

$$V \longmapsto r^{-1}|V|_g$$

is a C^∞ metric on the fibres of $\tilde{T}X$. The dual metric $r|\cdot|_g$ on sections of the dual bundle $\tilde{T}^*X$ (compressed cotangent bundle) is therefore also C^∞ and non-degenerate. In fact the symbol of any $P \in \mathrm{Diff}_b^k(X)$ can be regarded as a C^∞ function on $\tilde{T}^*X$ and then:

$$(12) \qquad \sigma(r^2\Delta) = r^2|\cdot|_g^2$$

shows that

(13) <u>Corollary</u> $r^2\Delta \in \mathrm{Diff}_b^2(X)$ is elliptic.

In general, let $\tilde{\Lambda}^p = \Lambda^p \tilde{T}^*X$ be the p-fold exterior powers of the compressed cotangent bundle. A C^∞ section ω of $\tilde{\Lambda}^p$ in local coordinates is of the form:

$$(14) \qquad \omega = \sum_{|\alpha|=p} a_\alpha(r,y)\left(\frac{dr}{r}\right)^{\alpha_0} \wedge \cdots \wedge dy^{n\alpha_n},$$

using the canonical identification $\tilde{T}^*X\big|_{\partial X} = T^*X\big|_{\partial X}$.

In particular,

$$d: C^\infty(X, \tilde{\Lambda}^p) \longrightarrow C^\infty(X, \tilde{\Lambda}^{p+1})$$

is a well-defined C^∞ differential operator. From (14) it follows trivially that it is totally characteristic.

$$(15) \qquad d \in \mathrm{Diff}_b^1(X, \tilde{\Lambda}^p, \tilde{\Lambda}^{p+1}) \quad \forall\, p.$$

Now, the adjoint δ of d is defined by

$$(16) \qquad \int \langle d\omega, \nu \rangle_g \, |dg| = \int \langle \omega, \delta\nu \rangle_g \, |dg|$$

where, to ensure convergence, it is certainly enough to require $\omega \in \dot{C}_c^\infty(X, \tilde{\Lambda}^p)$, $\nu \in \dot{C}_c^\infty(X, \tilde{\Lambda}^{p+1})$, i.e. both vanish to all orders at the boundary. The singular inner product on $\tilde{\Lambda}^p$ is clearly of the form

$$(17) \qquad \langle \ , \ \rangle_g = r^{-2p} \langle \ , \ \rangle_g^{\sim}$$

in terms of a smooth and non-degenerate inner product $\langle \ , \ \rangle_g^{\sim}$. Similarly, as is usual in polar coordinates,

$$(18) \qquad |dg| = r^n |\tilde{dg}|$$

in terms of a smooth non-vanishing density $|\tilde{dg}|$ on X. Thus, (16) can be written:

$$(19) \qquad \int \langle d\omega, \nu' \rangle_g^{\sim} \, |\tilde{dg}| = \int \langle \omega, \tilde{\delta}\nu' \rangle_g^{\sim} \, |\tilde{dg}|$$

if $\nu' = r^{n-2p-2}\nu$ and $\tilde{\delta}\nu' = r^{n-2p}\,\delta\nu$. The important point is that (19) shows $\tilde{\delta}$ to be the adjoint of d with respect to non-degenerate smooth structures on fibre bundles and base, so

(20) $\qquad \tilde{\delta} \in \mathrm{Diff}_b^1(X; \tilde{\Lambda}^{p+1}, \tilde{\Lambda}^p) \quad \forall\, p.$

Since $\delta = r^{-n+2p} \cdot \tilde{\delta} \cdot r^{n-2p-2}$ the elementary properties of totally characteristic operators show

$$r^2\delta \in \mathrm{Diff}_b^1(X; \tilde{\Lambda}^{p+1}, \tilde{\Lambda}^p).$$

Now, on the bundles $\tilde{\Lambda}^p$

$$\Delta = \delta d + d\delta,$$

so it follows easily that Lemma 10 holds, in the more general setting

(21) $\qquad r^2\Delta \in \mathrm{Diff}_b^2(X; \tilde{\Lambda}^p, \tilde{\Lambda}^p) \qquad \forall\, p,$

and Corollary 13 generalizes immediately too. Since Δ is the usual Laplace-Beltrami operator with only minor changes on the C^∞ structures of the bundles these observations allow the results discussed below to be applied to Hodge theory (cf. Cheeger /2/).

Finally in this section we shall make another important generalization. If in place of the cone (1) one considers the solid cone:

(22) $\qquad x^2 + y^2 \leq t^2 \qquad t \geq 0$

the introduction of polar coordinates as in (2), i.e. $r^2 = x^2 + y^2 + t^2$ etc., again reduces the metric to (4) where now $d\omega^2$ is a Riemannian metric on S^2 and

(23) $\qquad X = \overline{\mathbf{R}}_+ \times S$

where $S \subset S^2$ is a C^∞ submanifold with boundary.

To study these more general manifolds with conic points, for

which the local cross-section has boundary, we consider the following class of manifolds. Let X be a manifold with (interior) corner. Thus, each point of X has a neighbourhood diffeomorphic to either $\mathbb{R}^{n+1}$ or $\overline{\mathbb{R}}_+ \times \mathbb{R}^n$ or $\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \times \mathbb{R}^{n-1}$. The subset of points in the last two cases is $\partial X \subset X$ the topological boundary (boundary as a topological manifold), this consists of $\partial_1 X \cup \partial_2 X$ as locally X is a half-space or a quarter space; $\partial_2 X$ is the corner of X. We further require that the boundary of X has a decomposition:

$$(24) \qquad \partial X = \partial' X \cup \partial'' X, \quad \partial_2 X = \partial' X \cap \partial'' X$$

where $\partial' X$, $\partial'' X \hookrightarrow X$ are C^∞ submanifolds (of codimension 1) with boundary, $\partial(\partial' X) = \partial(\partial'' X) = \partial_2 X$.

With respect to this decomposition we can consider a metric tensor g on X which is non-degenerate throughout $X \setminus \partial'' X$ and which satisfies (5),(6),(7) if ∂X is replaced by $\partial'' X$. Thus, the case considered above corresponds to $\partial' X = \emptyset$ - the absence of 'real' boundary as opposed to blown-up conic points. It is then a straightforward matter to see that all the remarks above have immediate extensions to the new case provided $\widetilde{TX}$, $\mathrm{Diff}_b(X)$, $\widetilde{\Lambda}^p$ etc. are constructed as though X is to be extended across $\partial' X$ to a manifold $\overline{X}$ with boundary $\partial\overline{X}$ extending $\partial'' X$. In particular near $\partial'' X$ local coordinates will be taken with

$$(25) \qquad r \geq 0,\ y^1, \ldots, y^n \text{ and } y^1 \geq 0 \text{ if the base point } \bar{p} \in \partial_2 X.$$

(26) Remark It is not true that $r^2 A \in \mathrm{Diff}_b(\Lambda^p)$ for all p (see Cheeger /2, (2.5)/).

3. Totally characteristic problems

Following the analysis in the previous section it is natural, for regularity questions, to drop the assumption on order and other geometric properties of

$$P = r^2\Delta \tag{27}$$

and treat a more general case. Let X be a manifold with corner and with boundary split as in (24). We shall consider

$$P \in \mathrm{Diff}_b^m(X;E,F) \tag{28}$$

a totally characteristic operator with respect to $\partial^n X$, from sections of a complex vector bundle E to sections of another bundle F. Naturally we demand that P be elliptic:

$$\sigma(P) \in C^\infty(\tilde{T}^*X;\pi^*E,\pi^*F) \text{ is invertible on } \tilde{T}^*X \setminus 0. \tag{29}$$

Here, π^*E, π^*F are the vector bundles obtained by pulling back E,F from X to $\tilde{T}^*X$.

We recall from /5/ that the space $A(X,E) \subset \mathcal{D}'(X,E)$ of almost regular or conormal sections of E can be characterized simply as the space of Lagrangian sections of E associated to the conormal bundle $N^*(\partial X)$. Moreover, A is the residual space for the calculus of totally characteristic pseudodifferential operators on X and elliptic operators are A-hypoelliptic. Thus, the results of /5/ immediately give:

(30) **Proposition** If $\partial' X = \emptyset$ and $P \in \mathrm{Diff}_b(X;E,F)$ is elliptic then

$$Pu \in A(X,F) \Rightarrow u \in A(X,E). \tag{31}$$

This result will be considerably strenghtened in the next two sections, but first we shall discuss the analogue of (31) when $\partial' X \neq \emptyset$. As noted in Section 2 the two parts, $\partial' X$ and $\partial'' X$, of the boundary should be regarded somewhat differently. In particular for an elliptic boundary problem we should expect to impose boundary conditions in the usual sense only at $\partial' X$ to get smoothness of solutions there. The space $A(X)$ of conormal distributions is therefore defined by taking an extension $\bar{X}$ of X across $\partial' X$ so $\bar{X}$ is a manifold with boundary $\partial \bar{X}$ extending $\partial'' X$, then setting

$$(32) \qquad A(X,E) = A(\bar{X},\bar{E})\big|_X \subset \mathcal{D}'(X,E)$$

the space of distributional sections of E.

Now, the symbol of P, (29), restricts to a section over $\tilde{T}^*X\big|_{\partial' X}$ of the homomorphism bundle from π^*E to π^*F, and is polynomial of degree m in the fibres of $\tilde{T}^*X\big|_{\partial' X}$. In particular along the affine fibration

$$\iota^*:\tilde{T}^*X\big|_{\partial' X} \longrightarrow \tilde{T}^*\partial' X, \quad \iota:\partial' X \hookrightarrow X$$

$\sigma(P)$ is a polynomial of degree m without singularities, i.e. invertible except at the zero section. If a choice $\nu^*:\partial' X \longrightarrow N^*\partial' X$ of non-vanishing section of the conormal bundle of $\partial' X$ is made this defines an isomorphism of π^*E to π^*F, hence a normalized symbol:

$$(33) \qquad \mathcal{P} = \pi^*(\sigma(P)\big|_{\nu^*})^{-1} \cdot \sigma(P) \in C^\infty(\tilde{T}^*X\big|_{\partial' X};\pi^*E,\pi^*F),$$

which is invertible except at 0. Thus, for each $\eta \in \tilde{T}^*\partial' X \setminus 0$, $\mathcal{P}$ is a polynomial along $\iota^{*-1}(\eta)$ without real singular points. The splitting

of the zeros of $\det \mathcal{P}$, according to the signs of the imaginary parts, provides a natural splitting

$$\pi_b^* E^{(m)} \simeq E_+^{(m)} \oplus E_-^{(m)}, \tag{34}$$

where $\pi_b^* E^{(m)}$ is the lift to $\tilde{T}^* \partial' X$ of the vector bundle $E^{(m)}$ over $\partial' X$, defined in turn by its local C^∞ sections

$$C^\infty(X,E) \,/\, x^m C^\infty(X,E)$$

where x is a defining function for $\partial' X$ in X. Thus, $E^{(m)}$ is the bundle of Cauchy data at $\partial' X$ for P. In fact, $E_+^{(m)}$ then corresponds to Cauchy data of exponentially decreasing solutions of $Pu=0$, $E_-^{(m)}$ to exponentially increasing solutions.

By a boundary problem for P we mean a totally characteristic pseudo-differential operator

$$B \in \Psi_b^k(\partial' X ; E^{(m)}, G) \tag{35}$$

(or more generally of some filtered order in the sense of Friedrichs-Lax with respect to a filtration of G) such that B has microlocally constant rank, i.e. near each $\eta \in \tilde{T}^* \partial' X$ there exist elliptic pseudodifferential operators $A \in \Psi_b(\partial' X ; C^{mN}, E^{(m)})$, $A' \in \Psi_b(\partial' X ; G, C^M)$, where N,M are the ranks of E,G, with

$$A'.B.A = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{modulo } \Psi_b^{-\infty} \quad \text{near } \eta . \tag{36}$$

(37) Definition (P,B) is said to be an elliptic boundary problem if B has microlocally constant rank equal everywhere to the rank of $E_+^{(m)}$, with $\sigma(B) \mid E_+^{(m)}$ injective. (This is the Lopatinskii-Schapiro condition.)

(38) Proposition If (P,B) is an elliptic boundary problem and $u \in \mathcal{D}'(X,E)$ then $Pu \in A(X,F)$, $Bu \in A(\partial' X, G)$ implies $u \in A(X,E)$.

To prove this we use the method of multi-layered potentials as formalized by Calderon. One should first note that the following slight generalization of Peetre's Theorem on partial hypoellipticity holds:

(39) $Pu \in A(X,F)$ implies that u is C^∞ up to $\partial' X$ with boundary data extendible across $\partial_2 X \subset \partial' X$.

Then if one takes an extension $\bar{X}$ for X, as above, and an elliptic extension $\bar{P}$ of P, still totally characterisitc with respect to $\partial\bar{X}$ (which extends $\partial'' X$), the calculus of /5/ provides a parametrix

(40) $$Q \in \Psi_b^{-m}(\bar{X};F,E)$$

such that $Id - P.Q$, $Id - Q.P \in \bar{\Psi}_b^{-\infty}$, so map all distributions into A. Then any $u \in \mathcal{D}'(X,E)$ can be written

(41) $$u = Q.Pu + Ru \qquad Ru \in A(\bar{X},E).$$

Morover, Q has rational symbol as a totally characteristic operator.

Now, P defines a map

(42) $$\hat{P}: C^\infty(\partial' X, E^{(m)}) \longrightarrow C^\infty(\partial' X, F_{(m)}).$$

Here $F_{(m)}$ is the vector bundle over $\partial' X$ defined by its global distributional sections:

(43) $$\mathcal{D}'(\partial' X, F_{(m)}) = \left\{ f \in \dot{\mathcal{D}}'(X, \partial' X; F);\ x^m f = 0 \right\}$$

where $\dot{\mathcal{D}}'(X,\partial' X,F)$ is the space of distributional sections supported by $\partial' X$.

The rationality of the symbol of Q means that the map C_+ in

$$
\begin{array}{ccccc}
C_c^\infty(\partial' X, E^{(m)}) & \xrightarrow{\dot{P}} & C^\infty(\partial' X, F_{(m)}) & \longrightarrow & \dot{\mathcal{X}}'(X,F) \\
& \searrow^{C_+} & & & \downarrow \\
& & C^\infty(\partial' X, E^{(m)}) & \longleftarrow & \dot{\mathcal{U}}'(X,E)
\end{array}
\tag{44}
$$

where the last horizontal map is the evaluation of Cauchy data, using (39), is well-defined.

(45) Lemma $C_+ \in \Psi_b^0(\partial' X, E^{(m)}, E^{(m)})$ has mocrolocally constant rank and symbol the projection of $\pi^* E$ onto E_+ along E_-. Moreover, C_+ is independent of the choice of Q, as a parametrix for an extension of P, up to terms in $\Psi_b^{-\infty}$.

The standard method of elliptic regularity now applies. From

$$Pu = f \in A(X,F) \tag{46}$$

it follows that

$$C_+ u^{(m)} \in A(\partial' X, E^{(m)}), \tag{47}$$

where $u^{(m)}$ is the Cauchy data of u. The ellipticity condition in Definition 37 shows that near any point $\eta \in \tilde{T}^* \partial' X$ one can use (36) to find $B' \in \Psi_b^0(\partial' X, E^{(m)}, C^{mN})$, elliptic at η, such that

$$B' u^{(m)} \in A(\partial' X, C^{mN}). \tag{48}$$

Thus, $WF_b u^{(m)} = \emptyset$, i.e. $u^{(m)} \in A$. Then it follows readily from (46) that $u \in A(X,E)$ as claimed.

4. Mellin transform

Let $u \in \mathcal{E}'(\overline{\mathbb{R}_+})$ be a distribution on the open half line $(0, \infty)$ which is zero near infinity and extendible across 0. Then, there exists $\bar{u} \in \mathcal{E}'(\overline{\mathbb{R}_+})$ extending u, i.e. $\bar{u} \in \mathcal{E}'(\mathbb{R})$ with

(49) $\quad \bar{u} = u$ in $\mathbb{R}_+$, $\bar{u} = 0$ in $\mathbb{R}_-$.

Of course, this choice of zero extension $\bar{u}$ is unique only up to an arbitrary element of $\mathcal{E}'(\mathbb{R}, \{0\})$, the space of Dirac distributions supported at 0:

$$(50) \qquad 0 \longrightarrow \mathcal{E}'(\mathbb{R}, \{0\}) \hookrightarrow \dot{\mathcal{E}}'(\overline{\mathbb{R}_+}) \xrightarrow{\;|\,\mathbb{R}_+\;} \mathcal{E}'(\overline{\mathbb{R}_+}) \longrightarrow 0$$

is exact.

Now, such a distribution $\bar{u}$ is, for some k, in the space $\dot{C}_c^{-k}(\overline{\mathbb{R}_+})$ of finite sums of at most k-fold derivatives of continuous functions, so

(51) $\quad \langle u, \varphi \rangle$ is defined for all $\varphi \in \dot{C}^k(\overline{\mathbb{R}_+})$,

the space of k times continuously differentiable functions on $\mathbb{R}$ supported in $\overline{\mathbb{R}_+}$. The Mellin transform:

$$(52) \qquad u_M(s) = \int_0^\infty x^{-is-1}\, \bar{u}(x)\, dx = \langle \bar{u}, x_+^{is-1} \rangle$$

is therefore well-defined and holomorphic for $\mathrm{Im}(s) > k+1$. Notice that

$$\langle v, x_+^{is-1} \rangle = 0 \quad \text{if } v \in \mathcal{E}'(\mathbb{R}, \{0\}) \quad \mathrm{Im}(s) \gg 0,$$

so that u_M only depends on u itself.

(53) Lemma (Paley, Wiener, Schwartz) For $u \in \mathcal{E}'(\overline{\mathbb{R}}_+)$, $u_M = 0$ iff $u = 0$. If $v : \{\mathrm{Im}(s) > k\} \to \mathbb{C}$ is holomorphic then $v = u_M$ for some $u \in \mathcal{E}'(\overline{\mathbb{R}}_+)$ iff v is of exponential type

$$(54) \qquad |v(s)| \leq C \exp(A|\mathrm{Im}(s)|)\,(1+|s|)^m$$

in $\mathrm{Im}(s) > k'$ for some C, A, m, k'.

Proof Reduce to the Fourier transform by setting $t = -\log(x)$.

We are particularly interested in the Mellin transforms of conormal distributions, $u \in A(\overline{\mathbb{R}}_+)$.

(55) Proposition If v is holomorphic in a half-space $\mathrm{Im}(s) > k$ then $v = u_M$ for some $u \in A_c(\overline{\mathbb{R}}_+)$ iff it is of exponential type and is rapidly decreasing as $|\mathrm{Re}(s)| \to \infty$:

$$(56) \qquad |v(s)| \leq C_m \exp(A|\mathrm{Im}(s)|)\,(1+|s|)^m \qquad \mathrm{Im}(s) > k'$$

for all m.

In terms of the Mellin transform we next introduce a large space of 'classical' conormal distributions.

(57) Definition $A_{gr}(\overline{\mathbb{R}}_+) \subset A(\overline{\mathbb{R}}_+)$, the subspace of graded conormal distributions, consists of those $u \in A(\overline{\mathbb{R}}_+)$ for which $(\rho u) = u'$ has, for each $\rho \in C_c^\infty(\mathbb{R})$, Mellin transform u'_M extending meromorphically with

(58) Finitely many poles in any half-plane $\mathrm{Im}(s) > r$

(59) Estimates (56) in $\mathrm{Im}(s) > r$, $|\mathrm{Re}(s)| \geq d(r)$ for all m, r.

These graded conormal distributions are precisely those with

complete asymptotic expansions:

$$(60) \qquad u \sim \sum_{k,p} c_{k,p} x_+^{m_k} (\log(x))^p$$

where $m_k \to \infty$ with k, the sum over p is finite for each k and (60) has the usual meaning that the difference between u and a suitable truncation of the sum is in any preassigned space $\dot{C}^j(\overline{\mathbb{R}_+})$. Of course, the elements of A_{gr} are just the usual 'classical' or 'polyhomogeneous' conormal distributions except that there are no integrality conditions on the m_k and arbitrary finite powers of log are permitted. We further remark that A_{gr} is coordinate free. It can be extended to higher dimensions in at least two distinct ways.

By the Mellin transform of $u \in A(Z)$, where $Z = \overline{\mathbb{R}_+} \times \mathbb{R}^n$, we just mean the partial Mellin transform:

$$(61) \qquad u_M(s,y) = \int_0^\infty x^{-is-1} u(x,y)\, dx.$$

Then Proposition 55 extends easily, with C^∞ dependence on y. We shall define

$$A_{gr}(Z) \subset A(Z)$$

as consisting of those $u \in A(Z)$ for which $(\rho u)_M(s,y)$ extends meromorphically, as a function of s, with values in $C^\infty(\mathbb{R}^n)$, and with the obvious estimates in any $K \subset\subset \mathbb{R}^n_y$:

$$(62) \qquad \left|D_y^\alpha u_M(s,y)\right| \leq C_m \exp(A|\mathrm{Im}(s)|)\,(1+|s|)^m \qquad \forall\ m,$$

in $\mathrm{Im}(s) > r$, $|\mathrm{Re}(s)| \geq d(r)$. In particular this means that the position of the poles in s is independent of y. For boundary problems with 'kinks'

or corners of less than maximum codimension it is necessary to use a more general notion of classicality, such as $C^\infty(\mathbb{R}^n_y; A_{gr}(\mathbb{R}))$.

(63) Proposition $A_{gr}(Z) \subset A(Z)$ is coordinate free, so $A_{gr}(X) \subset A(X)$ is defined for any C^∞ manifold with boundary, and similarly for sections of any vector bundle.

In order to prove, inductively, that a distribution $u \in A(X)$ lies in $A_{gr}(X)$ we introduce relative spaces. First consider some order filtration such as:

$$(64) \qquad A^{(k)}(X) = \left\{ u \in A(X);\ P(x^k u) \in L^2_{loc}(X) \ \forall\ P \in \mathrm{Diff}_b(X) \right\}.$$

So,

$$(65) \qquad A(X) = \bigcup_k A^{(k)}(X).$$

Then define

$$(66) \qquad A_r^{(k)}(X) \subset A^{(k)}(X)$$

as consisting of those u such that, after localization and in any coordinate system, u_M is meromorphic in $\mathrm{Im}(s) > r$ and satisfies (58) and (59) for that value of r. Clearly,

$$(67) \qquad A_k^{(k)}(X) = A^{(k)}(X),$$

$$(68) \qquad A_{-\infty}^{(k)}(X) = A^{(k)}(X) \cap A_{gr}(X) = A_{gr}^{(k)}(X).$$

(69) Proposition $A_{gr}^{(k)}(X)$ is a $\mathrm{Diff}_b(X)$-module ($Pu \in A_r^{(k)}(X)$ if $P \in \mathrm{Diff}_b(X)$, $u \in A_r^{(k)}(X)$) and $xu \in A_{r-1}^{(k-1)}(X)$ if $x \in C^\infty(X)$, $x=0$ on ∂M, $u \in A_r^{(k)}(X)$.

5. Grading

To prove that any solution to the differential problems considered in Section 3 we first recall some invariance properties of conormal distributions. Any local coordinates (25) in X based at $\bar{p} \in \partial X$ induce an isomorphism between a neighbourhood of $\bar{p}$ in X and a neighbourhood of $0 \in N_p\partial X$ ($N_p\partial''X$ when X has a real boundary) in $N\partial X = T_{\partial X}X/T\partial X$, the normal bundle of ∂X. Of course then,

$$(70) \qquad A_r^{(p)}(X) \cong A_r^{(p)}(N_+\partial X) \quad \forall\ p,r \quad \text{(in coordinates)}$$

and there is a certain degree of invariance, since

$$(71) \qquad A^{(p)}(X)/A^{(p-1)}(X) \cong A^{(p)}(N_+\partial X)/A^{(p-1)}(N_+\partial X)$$

is coordinate free; this is just a reinterpretation of the symbol of a Lagrangian (conormal) distribution.

If $P \in \mathrm{Diff}_b^k(X)$ is totally characteristic with respect to ∂X then there is a well-defined operator

$$(72) \qquad P_0 \in \mathrm{Diff}_b^k(N_+\partial X) \text{ totally characteristic with respect to } 0_{N\partial X}$$

with constant coeficients in this sense, i.e. P_0 is invariant under the $\mathbb{R}_+$ action on the fibres $N_+\partial X$. In coordinates,

$$(73) \qquad P_0 = \sum_{|\alpha|+k \le m} p_{\alpha,k}(0,y)\ (rD_r)^k D_y^\alpha$$

if P is given by

$$(74) \qquad P = \sum_{|\alpha|+k \le m} p_{\alpha,k}(r,y)\ (rD_r)^k D_y^\alpha .$$

Now, for any $u \in A_r^{(k)}(X)$ if u' is the coordinate image of u in $A(N_+\partial X)$ then the coordinate image of Pu is

(75) $\quad P_0u' + xP'u' = P_0u' + xA_r^{(k)}(\partial X)$

where, $P' \in \mathrm{Diff}_b(X)$.

(76) **Theorem** Suppose X is a C^∞ manifold with compact boundary and $P \in \mathrm{Diff}_b^m(X;E,F)$ is elliptic then for any $u \in \mathcal{E}'(X,E)$,

(77) $\quad Pu \in A_{gr}(X,F) \Rightarrow u \in A_{gr}(X,E)$.

The proof of this rests on (75). From Proposition 30 we already know that $u \in A^{(k)}(X,E)$ for some k. Proceeding by induction we can suppose that

(78) $\quad u \in A_r^{(k)}(X,E)$.

Then, by (75), in local coordinates,

(79) $\quad Pu' = P_0u' + xP'u' = P_0u' \mod A_{r-1}^{(k)}(X,F)$.

Thus, it suffices to prove the inductive result for P_0 instead of P.

(80) **Proposition** If $u \in A_r^{(k)}(N_+\partial X;E)$ and $P_0u \in A_{r-1}^{(k)}(N_+\partial X,F)$ then $u \in A_{r-1}^{(k)}(N_+\partial X,E)$.

This in turn is a variant of well-known properties of elliptic operators depending (elliptically) on a complex parameter.

(81) **Proposition** If P_0 is of the form (73) and elliptic then

(82) $\quad P_0(s) = \sum_{|\alpha|+k \le m} p_{\alpha,k}(0,y)\, s^k D_y^\alpha \in \mathrm{Diff}(\partial X;E,F)$

is elliptic for each $s \in C$ and defines an isomorphism on Sobolev spaces:

(83) $\quad P_0(s): H^{k+m}(\partial X,E) \longrightarrow H^k(\partial X,F)$

for all except a discrete set $\mathrm{spec}(P_0) \subset \mathbb{C}$ lying inside a set

$$\mathrm{spec}(P_0) \subset \{s \in \mathbb{C};\ |\mathrm{Re}(s)| \geq a|\mathrm{Im}(s)| + b\} \tag{84}$$

for some constants a,b. Moreover, $P_0(s)^{-1}$ only has finite poles at $\mathrm{spec}(P_0)$ and in any set $|\mathrm{Im}(s)| \geq T$, $|\mathrm{Re}(s)| \gg T$, for each k in (83),

$$\| P_0(s)^{-1} \| \leq C_h(1+|s|)^h \quad \text{for some } h. \tag{85}$$

The exponents occurring in the asymptotic expansion of u come from amongst the set $\mathrm{spec}(P_0) + \mathbb{N}$. Finally we remark that the proof of the corresponding result for elliptic boundary problems is similar, with Proposition 81 for example, suitably extended to elliptic boundary problems depending on a parameter.

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